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LOWER BOUNDS FOR EIGENVALUES OF SCHATTEN-VON NEUMANN OPERATORS

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ABSTRACT. Let S_p be the Schatten-von Neumann ideal of compact operators equipped with the norm $N_p(\cdot)$. For an $A \in S_p$ $(1 < p < \infty)$, the inequality

$$\left[\sum_{k=1}^{\infty} |\operatorname{Re}\lambda_k(A)|^p\right]^{\frac{1}{p}} + b_p \left[\sum_{k=1}^{\infty} |\operatorname{Im}\lambda_k(A)|^p\right]^{\frac{1}{p}} \geq N_p(A_R) - b_p N_p(A_I) \quad (b_p = \text{const.} > 0)$$

is derived, where $\lambda_j(A)$ $(j = 1, 2, \dots)$ are the eigenvalues of A, $A_I = (A - A^*)/2i$ and $A_R = (A + A^*)/2$. The suggested approach is based on some relations between the real and imaginary Hermitian components of quasinilpotent operators.

Key words and phrases: Schatten-von Neumann ideals, Inequalities for eigenvalues.

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1. STATEMENT OF THE MAIN RESULT

Let S_p $(1 \leq p < \infty)$ be the Schatten-von Neumann ideal of compact operators in a separable Hilbert space H equipped with the norm

$$N_p(A) := [\operatorname{Trace}(A^*A)^{p/2}]^{1/p} < \infty \quad (A \in S_p),$$

cf. [4, 6]. Let $\lambda_j(A)$ $(j = 1, 2, \dots)$ be the eigenvalues of $A \in S_p$ taken with their multiplicities. In addition, $\sigma(A)$ denotes the spectrum of A, $A_I = (A - A^*)/2i$ and $A_R = (A + A^*)/2$ are the Hermitian components of A.

Recall the classical inequalities

$$\sum_{k=1}^{j} |\lambda_k(A)|^p \leq \sum_{k=1}^{j} s_k^p(A) \qquad (p \geq 1, \; j = 1, 2, \dots)$$

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cf. [6, Corollary II.3.1] and

$$\sum_{k=1}^{j} |\operatorname{Im} \lambda_k(A)| \leq \sum_{k=1}^{j} s_k(A_I) \qquad (j = 1, 2, \dots)$$

(see [6, Theorem II.6.1]). These results give us the upper bounds for sums of the eigenvalues of compact operators. In the present paper we derive lower inequalities for the eigenvalues. Our results supplement the very interesting recent investigations of the Schatten-von Neumann operators, cf. [1, 2, 8, 9, 11, 12, 13, 14].

Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive numbers defined by

$$c_n = c_{n-1} + \sqrt{c_{n-1}^2 + 1} \ \ (n = 2, 3, \dots), \quad c_1 = 1. \tag{1.1}$$

To formulate our main result, for a $p \in [2^n, 2^{n+1}]$ $(n = 1, 2, \dots)$, put

$$b_p = c_n^t c_{n+1}^{1-t} \quad \text{with} \quad t = 2 - 2^{-n} p. \tag{1.2}$$

For instance, $b_2 = c_1 = 1$, $b_3 = \sqrt{c_1 c_2} = \sqrt{1 + \sqrt{2}} \leq 1.554$, $b_4 = c_2 \leq 2.415$,

$$b_5 = c_2^{3/4} c_3^{1/4} \leq 2.900; \quad b_6 = (c_2 c_3)^{1/2} \leq 3.485; \quad b_7 = c_2^{1/4} c_3^{1/4} \leq 4.185$$

and $b_8 = c_3 \leq 5.027$. In the case $1 < p < 2$, we use the relation

$$b_p = b_{p/(p-1)} \tag{1.3}$$

proved below.

The aim of this paper is to prove the following

Theorem 1.1. *Let* $A \in S_p$ $(1 < p < \infty)$. *Then*

$$\left[\sum_{k=1}^{\infty} |\operatorname{Re} \lambda_k(A)|^p\right]^{\frac{1}{p}} + b_p \left[\sum_{k=1}^{\infty} |\operatorname{Im} \lambda_k(A)|^p\right]^{\frac{1}{p}} \geq N_p(A_R) - b_p N_p(A_I). \tag{1.4}$$

The proof of this theorem is presented in the next section. Clearly, inequality (1.4) is effective only if its right-hand part is positive.

Replacing in (1.4) A by iA we get

Corollary 1.2. *Let* $A \in S_p$ $(1 < p < \infty)$. *Then*

$$\left[\sum_{k=1}^{\infty} |\operatorname{Im} \lambda_k(A)|^p\right]^{\frac{1}{p}} + b_p \left[\sum_{k=1}^{\infty} |\operatorname{Re} \lambda_k(A)|^p\right]^{\frac{1}{p}} \geq N_p(A_I) - b_p N_p(A_R).$$

Note that if A is self-adjoint, then inequality (1.4) is attained, since

$$\left[\sum_{k=1}^{\infty} |\operatorname{Re} \lambda_k(A)|^p\right]^{\frac{1}{p}} = N_p(A_R) = N_p(A).$$

Moreover, if $A \in S_2$ is a quasinilpotent operator, then from Theorem 1.1, it follows that $N_2(A_R) \leq N_2(A_I)$. However, as it is well known, $N_2(A_R) = N_2(A_I)$, cf. [5, Lemma 6.5.1]. So in the case of a quasinilpotent Hilbert-Schmidt operator, inequality (1.4) is also attained.

Let $\{e_k\}$ be an orthonormal basis in H, and $F \in S_p$ with $p \geq 2$. Then by Theorem 4.7 from [3, p. 82],

$$N_p(F) \geq \left(\sum_{k=1}^{\infty} \|F e_k\|^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{\infty} \left[\sum_{j=1}^{\infty} |f_{jk}|^2\right]^{\frac{p}{2}}\right)^{\frac{1}{p}}.$$

Here $\|\cdot\|$ is the norm in H and f_{jk} are the entries of F in $\{e_k\}$. Moreover,

$$N_p(F) \le \left[\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|f_{jk}|^{p'}\right)^{\frac{p}{p'}}\right]^{\frac{1}{p}}, \quad \frac{1}{p}+\frac{1}{p'}=1,$$

cf. [10, p. 298]. Let a_{jk} be the entries of A in $\{e_k\}$. Then the previous inequalities yield the relations

$$N_p(A_R) \ge m_p(A_R) := \left[\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\left|\frac{a_{jk}+\overline{a}_{kj}}{2}\right|^2\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}$$

and

$$N_p(A_I) \le M_p(A_I) := \left[\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\left|\frac{a_{jk}-\overline{a}_{kj}}{2}\right|^{p'}\right)^{\frac{p}{p'}}\right]^{\frac{1}{p}}.$$

Now Theorem 1.1 implies:

Corollary 1.3. *Let* $A \in S_p$ $(2 \le p < \infty)$. *Then*

$$\left[\sum_{k=1}^{\infty}|\operatorname{Re}\lambda_k(A)|^p\right]^{\frac{1}{p}} + b_p\left[\sum_{k=1}^{\infty}|\operatorname{Im}\lambda_k(A)|^p\right]^{\frac{1}{p}} \ge m_p(A_R) - b_p M_p(A_I).$$

Furthermore, from (1.1) it follows that $c_{n+1} \ge 2c_n \ge 2^n$. Therefore,

$$c_{n+1} \le c_n\left(1+\sqrt{1+2^{-(n-1)2}}\right).$$

Hence,

$$c_n \le \prod_{k=1}^{n-1}\left(1+\sqrt{1+4^{-(k-1)}}\right) \quad (n=2,3,\dots). \tag{1.5}$$

Since

$$\sqrt{1+x} \le 1+\frac{x}{2}, \qquad x \in (0,1),$$

$1+x \le e^x$ $(x \ge 0)$, and

$$\sum_{k=1}^{\infty}\frac{1}{4^k} = \frac{1}{3},$$

from inequality (1.5) it follows that

$$c_{n+1} \le 2^n \prod_{k=1}^{n}(1+4^{-k}) \le 2^{n+1}\frac{e^{1/3}}{2}.$$

Hence it follows that

$$b_p \le \frac{pe^{1/3}}{2} \quad (2 \le p < \infty). \tag{1.6}$$

Indeed, by (1.2) for $p = t2^n + (1-t)2^{n+1}$ $(n=1,2,\dots;\ 0 \le t \le 1)$ we have

$$b_p = c_n^t c_{n+1}^{1-t} \le 2^{nt}2^{(1-t)(n+1)} \cdot \frac{e^{1/3}}{2} = 2^{n-t}\cdot\frac{e^{1/3}}{2}.$$

However, $2^{n-t} \le p = t2^n + (1-t)2^{n+1}$ $(0 \le t \le 1)$. So (1.6) is valid.

2. PROOF OF THEOREM 1.1

First let us prove the following lemma.

Lemma 2.1. *Let V be a quasinilpotent operator, $V_R = (V+V^*)/2$ and $V_I = (V-V^*)/2i$ its real and imaginary parts, respectively. Assume that $V_I \in S_{2^n}$ for an integer $n \geq 2$. Then $N_{2^n}(V_R) \leq c_n N_{2^n}(V_I)$.*

Proof. To apply the mathematical induction method assume that for $p = 2^n$ there is a constant d_p, such that $N_p(W_R) \leq d_p N_p(W_I)$ for any quasinilpotent operator $W \in S_p$. Then replacing W by Wi we have $N_p(W_I) \leq d_p N_p(W_R)$. Now let $V \in S_{2p}$. Then $V^2 \in S_p$ and therefore,

$$N_p((V^2)_R) \leq d_p N_p((V^2)_I).$$

Here

$$(V^2)_R = \frac{V^2 + (V^2)^*}{2}, \qquad (V^2)_I = \frac{V^2 - (V^2)^*}{2i}.$$

However,

$$(V^2)_R = (V_R)^2 - (V_I)^2, \qquad (V^2)_I = V_I V_R + V_R V_I$$

and thus

$$N_p(V_R^2 - V_I^2) \leq d_p N_p(V_R V_I + V_I V_R) \leq 2 d_p N_{2p}(V_R) N_{2p}(V_I).$$

Take into account that

$$N_p((V_R)^2) = N_{2p}^2(V_R), N_p((V_I)^2) = N_{2p}^2(V_I).$$

So

$$N_{2p}^2(V_R) - N_{2p}^2(V_I) - 2 d_p N_{2p}(V_R) N_{2p}(V_I) \leq 0.$$

Solving this inequality with respect to $N_{2p}(V_R)$, we get

$$N_{2p}(V_R) \leq N_{2p}(V_I)\left[d_p + \sqrt{d_p^2 + 1}\right] = N_{2p}(V_I) d_{2p}$$

with

$$d_{2p} = d_p + \sqrt{d_p^2 + 1}.$$

In addition, $d_2 = 1$ according to Lemma 6.5.1 from [5]. We thus have the required result with $c_n = d_{2^n}$. □

We will say that a linear mapping T is *a linear transformer* if it is defined on a set of linear operators and its values are linear operators. A linear transformer $T : S_p \to S_r$ $(1 \leq p, r < \infty)$ is bounded if its norm

$$N_{p \to r}(T) := \sup_{A \in S_p} \frac{N_r(TA)}{N_p(A)}$$

is finite. Below we give some examples of transformers. To prove relation (1.3) we need Theorem III.6.3 from [7]. To formulate that theorem we recall some notions from [7, Section I.3]. A set π of projections in H is called a *chain of projections* if for all $P_1, P_2 \in \pi$ either $P_1 < P_2$ or $P_2 < P_1$. This means that either $P_1 H \subset P_2 H$ or $P_2 H \subset P_1 H$. A chain of projections is *continuous* if it does not have gaps. A continuous chain of projections π is called a complete one if the zero and the unit operators belong to π.

Let us introduce *the integral with respect to a chain of projections* π, cf. [7, Sections 1.4 and I.5]. To this end for a partition

$$0 = P_0 < P_1 < \cdots < P_n = I, \qquad P_k \in \pi,\ k = 1, \ldots, n$$

and an operator $R \in S_p$ put

$$T_n = \sum_{k=1}^{n} P_k R \Delta P_k \qquad (\Delta P_k = P_k - P_{k-1}).$$

If there is a limit $T_n \to T$ as $n \to \infty$ in the operator norm, we write

$$T = \int_\pi PRdP.$$

This limit is called the integral of R with respect to a chain of projections π. By Theorem III.4.1 from [7], this integral converges for any $R \in S_p, 1 < p < \infty$. Due to Theorem I.6.1 [7], any Volterra operator V with $V_I \in S_p$ can be represented as

$$V = 2i \int_\pi PV_I dP.$$

Hence,

$$V_R = F_\pi(iV_I),$$

where

$$F_\pi(R) := \int_\pi PRdP + \left(\int_\pi PRdP\right)^* \quad (R \in S_p, 1 < p < \infty). \tag{2.1}$$

A transformer of this form is called a transformer of the triangular truncation with respect to π.

Theorem III.6.3 from [7] *asserts the following:* Let π be a complete continuous chain of projections in H. Let $F_\pi(R)$ be a transformer of the triangular truncation with respect to π defined by (2.1). Then the norm $N_{p\to p}(F_\pi)$ is logarithmically convex. Moreover, the relation

$$N_{p\to p}(F_\pi) = N_{q\to q}(F_\pi) \text{ with } \frac{1}{p} + \frac{1}{q} = 1 \quad (p \geq 2) \tag{2.2}$$

is valid.

Lemma 2.2. *Let V be a quasinilpotent operator, and for a $p \in [2^n, 2^{n+1}]$, $n = 1, 2, \ldots$, let $V_I \in S_p$. Then*

$$N_p(V_R) \leq b_p N_p(V_I). \tag{2.3}$$

Proof. By Lemma 2.1, we have

$$N_{2^n \to 2^n}(F_\pi) \leq c_n = b_{2^n}.$$

Put

$$p = t2^n + (1-t)2^{n+1} \quad (0 \leq t \leq 1).$$

Since the norm of F_π is logarithmically convex and $F_\pi(iV_I) = V_R$, we can write

$$N_{p\to p}(F_\pi) \leq b_{2^n}^t b_{2^{n+1}}^{1-t} \quad (t = 2 - 2^{-n}p).$$

So

$$\frac{N_p(V_R)}{N_p(V_I)} \leq b_p.$$

This proves the lemma. □

Furthermore, taking in (2.1) $R = iV_I$, by the previous lemma and the equalities (2.2) and $F_\pi(iV_I) = V_R$, we get

$$N_q(V_R) \leq b_q N_q(V_I) \quad (q \in (1, 2))$$

with $b_q = b_p, q = p/(p-1)$. So we arrive at

Corollary 2.3. *Let $V \in S_p$ be a quasinilpotent operator with $p \in (1, 2)$. Then (2.3) holds with (1.3) taken into account.*

Proof of Theorem 1.1. As it is well known, cf. [6] for any compact operator A, there are a normal operator D and a quasinilpotent operator V, such that

$$A = D + V \quad \text{and } \sigma(D) = \sigma(A). \tag{2.4}$$

Relation (2.4) is called the triangular representation of A; V and D are called the nilpotent part and diagonal one of A, respectively. Clearly, by the triangular inequality,

$$N_p(V_R) = N_p(A_R - D_R) \geq N_p(A_R) - N_p(D_R)$$

and $N_p(A_I - D_I) \leq N_p(A_I) + N_p(D_I)$. This and the previous lemma imply that

$$N_p(A_R) - N_p(D_R) \leq b_p N_p(A_I - D_I) \leq b_p(N_p(A_I) + N_p(D_I)).$$

Hence, $N_p(A_R) - b_p N_p(A_I) \leq b_p N_p(D_I) + N_p(D_R)$. By (2.4),

$$N_p^p(D_R) = \sum_{k=1}^{\infty} |\operatorname{Re} \lambda_k(A)|^p \text{ and } N_p^p(D_I) = \sum_{k=1}^{\infty} |\operatorname{Im} \lambda_k(A)|^p.$$

So relation (1.4) is proved, as claimed. □

3. ADDITIONAL BOUNDS

By Lemma 6.5.2 [5], for an $A \in S_2$ we have

$$N_2^2(A) - \sum_{k=1}^{\infty} |\lambda_k(A)|^2 = 2N_2^2(A_I) - 2\sum_{k=1}^{\infty} (\operatorname{Im} \lambda_k(A))^2. \tag{3.1}$$

Hence,

$$N_2^2(A) - \sum_{k=1}^{\infty} |\lambda_k(A)|^2 = 2N_2^2(A_R) - 2\sum_{k=1}^{\infty} (\operatorname{Re} \lambda_k(A))^2$$

and therefore,

$$N_2^2(A_I) - \sum_{k=1}^{\infty} (\operatorname{Im} \lambda_k(A))^2 = N_2^2(A_R) - \sum_{k=1}^{\infty} (\operatorname{Re} \lambda_k(A))^2.$$

Or

$$\sum_{k=1}^{\infty} (\operatorname{Re} \lambda_k(A))^2 - \sum_{k=1}^{\infty} (\operatorname{Im} \lambda_k(A))^2 = N_2^2(A_R) - N_2^2(A_I) \quad (A \in S_2).$$

This equality improves Theorem 1.1 in the case $p = 2$. Moreover, from (3.1) it directly follows that

$$\begin{aligned} 2\sum_{k=1}^{\infty} (\operatorname{Im} \lambda_k(A))^2 &= 2N_2^2(A_I) - N_2^2(A) + \sum_{k=1}^{\infty} |\lambda_k(A)|^2 \\ &\geq 2N_2^2(A_I) - N_2^2(A) + \operatorname{Trace} A^2. \end{aligned}$$

Now replacing A by A^p we arrive at

Theorem 3.1. *Let $A \in S_{2p}$ $(1 \leq p < \infty)$. Then*

$$2\sum_{k=1}^{\infty} (\operatorname{Im}(\lambda_k^p(A)))^2 \geq 2N_2^2((A^p)_I) - N_{2p}^{2p}(A) + \operatorname{Trace} A^{2p}.$$

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