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## ON AN INTEGRAL INEQUALITY

J. PEČARIĆ AND T. PEJKOVIĆ

Faculty of Textile Technology  
University of Zagreb  
Pierottijeva 6, 10000 Zagreb  
Croatia.

*EMail:* [pecaric@mahazu.hazu.hr](mailto:pecaric@mahazu.hazu.hr)

*EMail:* [pejkovic@student.math.hr](mailto:pejkovic@student.math.hr)

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## Abstract

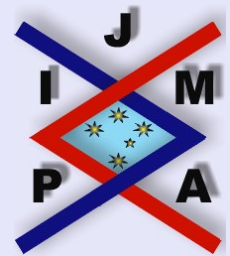
In this article we give different sufficient conditions for inequality  $\left(\int_a^b f(x)^\alpha dx\right)^\beta \geq \int_a^b f(x)^\gamma dx$  to hold.

*2000 Mathematics Subject Classification:* 26D15.

*Key words:* Integral inequality, Inequalities between means.

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# 1. Introduction

In this paper we wish to investigate some sufficient conditions for the following inequality:

$$(1.1) \quad \left( \int_a^b f(x)^\alpha dx \right)^\beta \geq \int_a^b f(x)^\gamma dx.$$

This is a generalization of the inequalities that appear in the papers [4, 5, 6, 7, 8, 9].

F. Qi in [7] considered inequality (1.1) for  $\alpha = n + 2$ ,  $\beta = 1/(n + 1)$ ,  $\gamma = 1$ ,  $n \in \mathbb{N}$ . He proved that under conditions

$$f \in C^n([a, b]); \quad f^{(i)}(a) \geq 0, \quad 0 \leq i \leq n - 1; \quad f^{(n)}(x) \geq n!, \quad x \in [a, b]$$

the inequality is valid.

Later, S. Mazouzi and F. Qi gave what appeared to be a simpler proof of the inequality under the same conditions (Corollary 3.6 in [1]). Unfortunately their proof was incorrect. Namely, they made a false substitution and arrived at the condition  $f(x) \geq (n + 1)(x - a)^n$  which is not true, e.g. for function  $f(x) = x - a$ , whereas this function obviously satisfies the conditions of the theorem if  $n = 1$ .

K.-W. Yu and F. Qi ([9]) and N. Towghi ([8]) gave other conditions for the inequality (1.1) to hold under this special choice of constants  $\alpha, \beta, \gamma$ .

T.K. Pogány in [6], by avoiding the assumption of differentiability used in [7, 8, 9], and instead using the inequalities due to Hölder, Nehari (Lemma 2.4) and Barnes, Godunova and Levin (Lemma 2.5) established some inequalities which are a special case of (1.1) when  $\alpha = 1$  or  $\gamma = 1$ .



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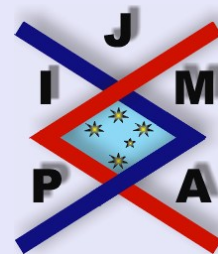
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To obtain some conditions for the inequality (1.1) we will first proceed similarly to T.K. Pogány ([6]) and in the second part of this article we will be using a method from the paper [4].



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## 2. Conditions Based on Inequalities Between Means

We want to transform inequality (1.1) to a form more suitable for us. It can easily be seen that if  $f(x) \geq 0$ , for all  $x \in [a, b]$  and  $\gamma > 0$ , inequality (1.1) is equivalent to

$$(2.1) \quad \left[ \left( \frac{\int_a^b f(x)^\alpha dx}{b-a} \right)^{\frac{1}{\alpha}} \right]^{\frac{\alpha\beta}{\gamma}} (b-a)^{\frac{\beta-1}{\gamma}} \geq \left( \frac{\int_a^b f(x)^\gamma dx}{b-a} \right)^{\frac{1}{\gamma}}.$$

**Definition 2.1.** Let  $f$  be a nonnegative and integrable function on the segment  $[a, b]$ . The  $r$ -mean (or the  $r$ -th power mean) of  $f$  is defined as

$$M^{[r]}(f) := \begin{cases} \left( \frac{\int_a^b f(x)^r dx}{b-a} \right)^{\frac{1}{r}} & (r \neq 0, +\infty, -\infty), \\ \exp \left( \frac{\int_a^b \ln f(x) dx}{b-a} \right) & (r = 0), \\ m & (r = -\infty), \\ M & (r = +\infty). \end{cases}$$

where  $m = \inf f(x)$  and  $M = \sup f(x)$  for  $x \in [a, b]$ .

According to the previous definition inequality (2.1) can be written as

$$(2.2) \quad (M^{[\alpha]}(f))^{\frac{\alpha\beta}{\gamma}} (b-a)^{\frac{\beta-1}{\gamma}} \geq M^{[\gamma]}(f)$$

We will be using the following inequalities:



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**Lemma 2.1 (power mean inequality, [2]).** *If  $f$  is a nonnegative function on  $[a, b]$  and  $-\infty \leq r < s \leq +\infty$ , then*

$$M^{[r]}(f) \leq M^{[s]}(f).$$

**Lemma 2.2 (Berwald inequality, [3, 5]).** *If  $f$  is a nonnegative concave function on  $[a, b]$ , then for  $0 < r < s$  we have*

$$M^{[s]}(f) \leq \frac{(r+1)^{1/r}}{(s+1)^{1/s}} M^{[r]}(f).$$

**Lemma 2.3 (Thunsdorff inequality, [3]).** *If  $f$  is a nonnegative convex function on  $[a, b]$  with  $f(a) = 0$ , then for  $0 < r < s$  we have*

$$M^{[s]}(f) \geq \frac{(r+1)^{1/r}}{(s+1)^{1/s}} M^{[r]}(f).$$

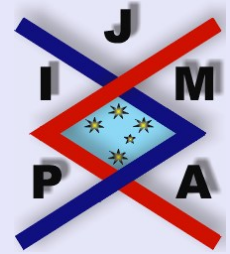
**Lemma 2.4 (Nehari inequality, [2]).** *Let  $f, g$  be nonnegative concave functions on  $[a, b]$ . Then, for  $p, q > 0$  such that  $p^{-1} + q^{-1} = 1$ , we have*

$$\left( \int_a^b f(x)^p dx \right)^{\frac{1}{p}} \left( \int_a^b g(x)^q dx \right)^{\frac{1}{q}} \leq N(p, q) \int_a^b f(x)g(x) dx,$$

where  $N(p, q) = \frac{6}{(1+p)^{1/p}(1+q)^{1/q}}$ .

**Lemma 2.5 (Barnes-Godunova-Levin inequality, [3, 2]).** *Let  $f, g$  be nonnegative concave functions on  $[a, b]$ . Then, for  $p, q > 1$  we have*

$$\left( \int_a^b f(x)^p dx \right)^{\frac{1}{p}} \left( \int_a^b g(x)^q dx \right)^{\frac{1}{q}} \leq B(p, q) \int_a^b f(x)g(x) dx,$$



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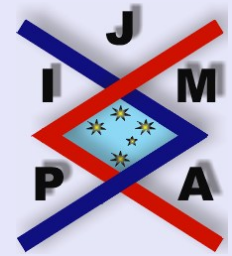
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where  $B(p, q) = \frac{6(b-a)^{1/p+1/q-1}}{(1+p)^{1/p}(1+q)^{1/q}}$ .

Let us first state our results in a clear table. Each result is an independent set of conditions that guarantee the inequality (1.1) is valid.

Result	Conditions on constants $\alpha, \beta, \gamma, a, b$	Conditions on function $f$ (holds for all $x \in [a, b]$ )	Lemma for the proof
1.	$\alpha \geq \gamma > 0, \alpha\beta > \gamma$	$f(x) \geq (b-a)^{\frac{-\beta+1}{\alpha\beta-\gamma}}$	Lemma 2.1
2.	$\alpha \leq \gamma > 0, \alpha\beta < 0$	$0 \leq f(x) \leq (b-a)^{\frac{-\beta+1}{\alpha\beta-\gamma}}$	Lemma 2.1
3 (i).	$\alpha \geq \gamma > 0, \alpha\beta \geq \gamma,$ $(b-a)^{\frac{\beta-1}{\gamma}} \geq 1$	$f(x) \geq 1$	Lemma 2.1
3 (ii).	$\alpha \geq \gamma > 0, \alpha\beta \leq \gamma,$ $(b-a)^{\frac{\beta-1}{\gamma}} \geq 1$	$0 \leq f(x) \leq 1$	Lemma 2.1
4 (i).	$0 < \alpha \leq \gamma, \alpha\beta \geq \gamma$ $(b-a)^{\frac{\beta-1}{\gamma}} \geq \frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}}$	$f$ concave $f(x) \geq 1$	Lemma 2.2
4 (ii).	$0 < \alpha \leq \gamma, \alpha\beta \leq \gamma$ $(b-a)^{\frac{\beta-1}{\gamma}} \geq \frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}}$	$f$ concave $0 \leq f(x) \leq 1$	Lemma 2.2
5.	$0 < \alpha \leq \gamma, \alpha\beta > \gamma$	$f$ concave, $f(x) \geq$ $(b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left( \frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}} \right)^{\frac{\gamma}{\alpha\beta-\gamma}}$	Lemma 2.2
6.	$0 < \gamma \leq \alpha, \beta < 0$	$f$ concave, $0 \leq f(x) \leq$ $(b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left( \frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}} \right)^{\frac{\alpha\beta}{\alpha\beta-\gamma}}$	Lemma 2.2



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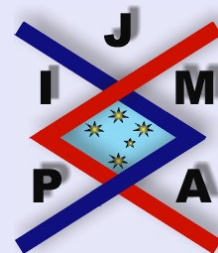
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Result	Conditions on constants $\alpha, \beta, \gamma, a, b$	Conditions on function $f$ (holds for all $x \in [a, b]$ )	Lemma for the proof
7.	$0 < \gamma \leq \alpha, \alpha\beta > \gamma$	$f$ convex, $f(a) = 0, f(x) \geq \frac{(b-a)^{\frac{\alpha-\gamma}{\alpha(\alpha\beta-\gamma)}}}{(b^{\alpha+1}-a^{\alpha+1})^{1/\alpha}} \cdot \frac{(\alpha+1)^{\frac{\beta}{\alpha\beta-\gamma}}}{(\gamma+1)^{\frac{1}{\alpha\beta-\gamma}}} x$	Lemma 2.3
8.	$0 < \alpha \leq \gamma, \beta < 0$	$f$ convex, $f(a) = 0, 0 \leq f(x) \leq (b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left( \frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}} \right)^{\frac{\alpha\beta}{\alpha\beta-\gamma}}$	Lemma 2.3
9.	$0 < \gamma < \alpha, \beta < 0$	$f$ concave, $0 \leq f(x) \leq (b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \times \frac{\frac{\alpha\beta}{6\gamma(\gamma-\alpha\beta)}}{\left(\frac{2\alpha-\gamma}{\alpha-\gamma}\right)^{\frac{\beta(\alpha-\gamma)}{\gamma(\gamma-\alpha\beta)}} \left(\frac{\alpha+\gamma}{\gamma}\right)^{\frac{\beta}{\gamma-\alpha\beta}}}$	Lemma 2.4

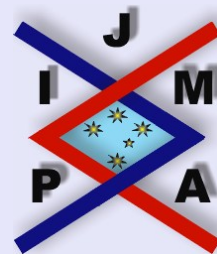
**Remark 2.1.** Observe that in the results 4(i). and 5. it is enough for the condition on  $f$  to hold in endpoints of segment  $[a, b]$  (ie., for  $f(a)$  and  $f(b)$ ).

**Remark 2.2.** There is only one result in the table obtained with the help of Lemma 2.4 and none with the Lemma 2.5 because the constants in the conditions are quite complicated.

**Remark 2.3.** If we make the substitution  $\gamma \mapsto 1, \beta \mapsto \frac{1}{\beta}$  in Result 1, Theorem 2.1 in [6] is acquired.

We will prove only a few results after which the method of proving the others will become clear.





*Proof of Result 1.* Lemma 2.1 implies that

$$(2.3) \quad M^{[\alpha]}(f) \geq M^{[\gamma]}(f).$$

Also

$$M^{[\alpha]}(f) \geq M^{[-\infty]}(f) \geq (b-a)^{\frac{-\beta+1}{\alpha\beta-\gamma}},$$

so by raising this inequality to a power, we get

$$(2.4) \quad (M^{[\alpha]}(f))^{\frac{\alpha\beta-\gamma}{\gamma}} \geq (b-a)^{\frac{-\beta+1}{\gamma}}.$$

Multiplying (2.3) and (2.4) we get (2.2).  $\square$

*Proof of Result 3 (i).*  $M^{[\alpha]}(f) \geq 1$  because  $f(x) \geq 1$ , so from  $\frac{\alpha\beta}{\gamma} \geq 1$  and Lemma 2.1 it follows

$$(2.5) \quad (M^{[\alpha]}(f))^{\frac{\alpha\beta}{\gamma}} \geq M^{[\alpha]}(f) \geq M^{[\gamma]}(f).$$

By multiplication of (2.5) and the condition  $(b-a)^{\frac{\beta-1}{\gamma}} \geq 1$  we get (2.2).  $\square$

*Proof of Result 5.* From

$$M^{[\alpha]}(f) \geq M^{[-\infty]}(f) \geq (b-a)^{\frac{1-\beta}{\alpha\beta-\gamma}} \left( \frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}} \right)^{\frac{\gamma}{\alpha\beta-\gamma}}$$

and  $\frac{\alpha\beta-\gamma}{\gamma} > 0$  we obtain

$$(2.6) \quad (M^{[\alpha]}(f))^{\frac{\alpha\beta-\gamma}{\gamma}} (b-a)^{\frac{\beta-1}{\gamma}} \geq \frac{(\alpha+1)^{1/\alpha}}{(\gamma+1)^{1/\gamma}}.$$

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According to Lemma 2.2:

$$(2.7) \quad M^{[\alpha]}(f) \frac{(\alpha + 1)^{1/\alpha}}{(\gamma + 1)^{1/\gamma}} \geq M^{[\gamma]}(f).$$

From (2.6) and (2.7), by multiplication, we arrive at (2.2). □

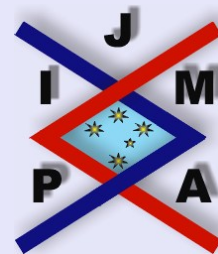
*Proof of Result 7.* Since

$$f(x) \geq \frac{(b - a)^{\frac{\alpha - \gamma}{\alpha(\alpha\beta - \gamma)}} \cdot (\alpha + 1)^{\frac{\beta}{\alpha\beta - \gamma}}}{(b^{\alpha+1} - a^{\alpha+1})^{1/\alpha}} \cdot \frac{1}{(\gamma + 1)^{\frac{1}{\alpha\beta - \gamma}}} x$$

by integration it follows that

$$(2.8) \quad M^{[\alpha]}(f) \geq (b - a)^{\frac{1 - \beta}{\alpha\beta - \gamma}} \left( \frac{(\alpha + 1)^{1/\alpha}}{(\gamma + 1)^{1/\gamma}} \right)^{\frac{\gamma}{\alpha\beta - \gamma}}.$$

However, Lemma 2.3 implies inequality (2.7). Thus, from (2.8) and (2.7) we finally find that inequality (2.2) is valid. □




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### 3. Conditions Associated with the Functions with Bounded Derivative

In this section we will prove inequality (1.1) under different assumptions including the differentiability of  $f$  and boundedness of its derivative.

J. Pečarić and W. Janous proved in [4] the following theorem.

**Theorem 3.1.** *Let  $1 < p \leq 2$  and  $r \geq 3$ . The differentiable function  $f : [0, c] \rightarrow \mathbb{R}$  satisfies  $f(0) = 0$  and  $0 \leq f'(x) \leq M$  for all  $0 \leq x \leq c$ ,  $c$  subject to*

$$(3.1) \quad 0 < c \leq \left( \frac{p(p-1)2^{2-p}M^{p-r}}{r-1} \right)^{\frac{1}{r-2p+1}}.$$

Then

$$\left( \int_0^c f(x) dx \right)^p \geq \int_0^c f(x)^r dx.$$

(If  $f'(x) \geq M$  the reverse inequality holds true under the condition that the second inequality in (3.1) is reversed.)

**Remark 3.1.** *The emphasized words were left out of [4].*

The following generalization will be proved:

**Theorem 3.2.** *Let  $\alpha > 0$ ,  $1 < \beta \leq 2$  and  $\gamma \geq 2\alpha + 1$ . The differentiable function  $f : [0, c] \rightarrow \mathbb{R}$  satisfies  $f(0) = 0$  and  $0 \leq f'(x) \leq M$  for all  $0 \leq x \leq c$ ,  $c$  subject to*

$$(3.2) \quad 0 < c \leq \left( \frac{\beta(\beta-1)(\alpha+1)^{2-\beta}M^{\alpha\beta-\gamma}}{\gamma-\alpha} \right)^{\frac{1}{\gamma-\alpha\beta-\beta+1}}.$$



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Then

$$\left( \int_0^c f(x)^\alpha dx \right)^\beta \geq \int_0^c f(x)^\gamma dx.$$

**Remark 3.2.** For  $\alpha = 1$ ,  $\beta = p$ ,  $\gamma = r$ , we get Theorem 3.1.

*Proof.* From  $f(0) = 0$  and  $0 \leq f'(x) \leq M$  we obtain

$$0 \leq f(x)^\alpha \leq M^\alpha x^\alpha \quad \text{and} \quad 0 \leq \int_0^x f(t)^\alpha dt \leq \frac{M^\alpha x^{\alpha+1}}{\alpha+1} \quad \text{for} \quad 0 \leq x \leq c.$$

Now we define

$$F(x) := \left( \int_0^x f(t)^\alpha dt \right)^\beta - \int_0^x f(t)^\gamma dt.$$

Then  $F(0) = 0$  and  $F'(x) = f(x)^\alpha g(x)$ , where

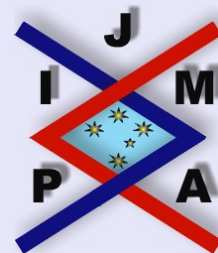
$$g(x) := \beta \left( \int_0^x f(t)^\alpha dt \right)^{\beta-1} - f(x)^{\gamma-\alpha}.$$

Clearly,  $g(0) = 0$  and  $g'(x) = f(x)^\alpha h(x)$ , where

$$h(x) := \beta(\beta-1) \left( \int_0^x f(t)^\alpha dt \right)^{\beta-2} - (\gamma-\alpha) f(x)^{\gamma-2\alpha-1} f'(x).$$

From the conditions of the theorem we have

$$\begin{aligned} h(x) &\geq \beta(\beta-1) \left( \frac{M^\alpha x^{\alpha+1}}{\alpha+1} \right)^{\beta-2} - (\gamma-\alpha)(Mx)^{\gamma-2\alpha-1} M \\ &= M^{\gamma-2\alpha} x^{(\alpha+1)(\beta-2)} (\beta(\beta-1)(\alpha+1)^{2-\beta} M^{\alpha\beta-\gamma} - (\gamma-\alpha)x^{\gamma-\alpha\beta-\beta+1}) \end{aligned}$$



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Thus, since (3.2) is equivalent to

$$\beta(\beta - 1)(\alpha + 1)^{2-\beta} M^{\alpha\beta-\gamma} \geq (\gamma - \alpha)x^{\gamma-\alpha\beta-\beta+1}, \quad x \in [0, c],$$

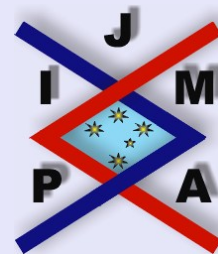
we have  $h(x) \geq 0$ ,  $g'(x) \geq 0$ ,  $g(x) \geq 0$ ,  $F'(x) \geq 0$  and finally  $F(x) \geq 0$ . So  $F(c) \geq 0$ .  $\square$

Substituting  $c = a - b$  and translating function  $f$   $a$  units to the right ( $f(x) \mapsto f(x - a)$ ) we obtain the following theorem.

**Theorem 3.3.** *Let  $\alpha > 0$ ,  $1 < \beta \leq 2$  and  $\gamma \geq 2\alpha + 1$ . The differentiable function  $f : [a, b] \rightarrow \mathbb{R}$  satisfies  $f(a) = 0$  and  $0 \leq f'(x) \leq M$  for all  $a \leq x \leq b$ , where*

$$(3.3) \quad 0 < b - a \leq \left( \frac{\beta(\beta - 1)(\alpha + 1)^{2-\beta} M^{\alpha\beta-\gamma}}{\gamma - \alpha} \right)^{\frac{1}{\gamma - \alpha\beta - \beta + 1}}.$$

Then the inequality (1.1) holds.



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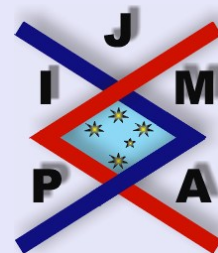
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