

TWO-DIMENSIONAL SUNOUCHI OPERATOR WITH RESPECT TO VILENKIN-LIKE SYSTEMS

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Abstract: In this paper two-dimensional Vilenkin-like systems will be investigated. We prove the Sunouchi operator is bounded from H^q to L^q for $(2/3 < q \leq 1)$. As a consequence, we prove the Sunouchi operator is L^s bounded for $1 < s < \infty$ and of weak type (H^s, L^1) .

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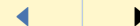
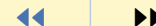
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1. Introduction

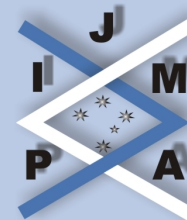
The operator U (called the Sunouchi operator) was first introduced and investigated by Sunouchi [1], [2] in Walsh-Fourier analysis. He showed a characterization for the L^p spaces for $p > 1$ by means of U , since this characterization fails to hold for $p = 1$. It was of interest to investigate the boundedness of U on a Hardy space. In [3] Simon showed that U is a sublinear bounded map from the dyadic Hardy space H^1 into L^1 .

The Vilenkin analogue of the Sunouchi operator was given by Gát [4], [5]. He investigated the boundedness of U from (Vilenkin) H^1 into L^1 and proved that if a Vilenkin group has an unbounded structure and H^1 is defined by means of the usual maximal function, then U is not bounded. Furthermore, when they considered a modified H^1 space (introduced by Simon [6]), then a necessary and sufficient condition could be given for a Vilenkin group that $U : H^1 \rightarrow L^1$ be bounded. All Vilenkin groups with bounded structure and certain groups without this boundedness property satisfy the condition given by Gát. Thus, in the so-called bounded case, the (H^1, L^1) -boundedness of U remains true also for Vilenkin system. In [7] Simon extended this result, by showing the (H^q, L^q) -boundedness of U for all $0 < q \leq 1$. Moreover, the equivalence

$$\|f\|_{H^q} \sim \|Uf\|_q \quad \left(\frac{1}{2} < q \leq 1 \right)$$

was also obtained for f with mean value zero.

In this paper we consider a two-dimensional case with respect to generalized Vilenkin-like systems.

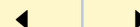
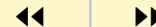


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2. Preliminaries and Notations

In this section, we introduce important definitions and notations. Furthermore, we formulate some known results with respect to Vilenkin-like systems, which play a basic role in further investigations. For details, see [8] by Vilenkin and [9] by Schipp, Wade, Simon and Pál.

Let $m := (m_k, k \in \mathbb{N})$ ($\mathbb{N} := \{0, 1, \dots, \}$) be a sequence of integers, each of them not less than 2. Denote by Z_{m_k} the m_k -th cyclic group ($k \in \mathbb{N}$). That is, Z_{m_k} can be represented by the set $\{0, 1, \dots, m_k - 1\}$, where the group operator is the mod m_k addition and every subset is open. The Harr measure on Z_{m_k} is given such that $\mu(\{j\}) = \frac{1}{m_k}$ ($j \in Z_{m_k}, k \in \mathbb{N}$).

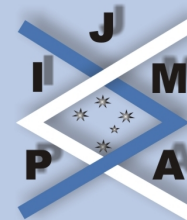
Let G_m denote the complete direct product of Z_{m_k} 's equipped with product topology and product measure μ , then G_m forms a compact Abelian group with Haar measure 1. The elements of G_m are sequences of the form $(x_0, x_1, \dots, x_k, \dots)$, where $x_k \in Z_{m_k}$ for every $k \in \mathbb{N}$ and the topology of the group G_m is completely determined by the sets

$$I_n(0) := \{(x_0, x_1, \dots, x_k, \dots) \in G_m : x_k = 0 \ (k = 0, \dots, n-1)\}$$

($I_0(0) := G_m$). Let $I_n(x) := I_n(0) + x$ ($n \in \mathbb{N}$); $M_0 := 1$ and $M_{k+1} := m_k M_k$ for $k \in \mathbb{N}$, the so-called generalized powers. Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbb{N}$. The sequence (n_0, n_1, \dots) is called the expansion of n with respect to m . We often use the following notations: $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$) and $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$.

Let $\hat{G}_m := \{\psi_n : n \in \mathbb{N}\}$ denote the character group of G_m . We enumerate the elements of \hat{G}_m as follows. For $k \in \mathbb{N}$ and $x \in G_m$ denote by r_k the k -th generalized Rademacher function:

$$r_k(x) := \exp\left(2\psi_k \frac{x_k}{m_k}\right) \quad (x \in G_m, \nu : \sqrt{-1}, k \in \mathbb{N}).$$

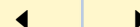
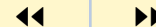


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It is known for $x \in G_m, n \in \mathbb{N}$ that

$$(2.1) \quad \sum_{i=0}^{m_n-1} r_n^i(x) = \begin{cases} 0, & \text{if } x_n \neq 0; \\ m_n, & \text{if } x_n = 0. \end{cases}$$

Now we define the ψ_n by

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k} \quad (n \in \mathbb{N}).$$

\hat{G}_m is a complete orthonormal system with respect to μ .

G. Gát introduced the so-called Vilenkin-like (or $\psi\alpha$) system. Let functions $\alpha_n, \alpha_j^k : G_m \rightarrow \mathcal{C}$ ($n, j, k \in \mathbb{N}$) satisfy:

- i) α_j^k is measurable with respect to Σ_j (i.e. α_j^k depends only on x_0, x_1, \dots, x_{j-1} , $j, k \in \mathbb{N}$);
- ii) $|\alpha_j^k| = \alpha_j^k(0) = \alpha_0^k = \alpha_j^0 = 1$ ($j, k \in \mathbb{N}$);
- iii) $\alpha_n := \prod_{j=0}^{\infty} \alpha_j^{n^{(j)}}$ ($n \in \mathbb{N}$).

Let $\chi_n := \psi_n \alpha_n$ ($n \in \mathbb{N}$). The system $\chi := \{\chi_n : n \in \mathbb{N}\}$ is called a Vilenkin-like (or $\psi\alpha$) system (see [10] and [13] for examples).

1. If $\alpha_j^k = 1$ for each $k, j \in \mathbb{N}$, then we have the "ordinary" Vilenkin systems.
2. If $m_j = 2$ for all $j \in \mathbb{N}$ and $\alpha_j^{n^{(j)}} = (\beta_j)^{n_j}$, where

$$\beta_j(x) = \exp\left(2\pi i \left(\frac{x_{j-1}}{2^2} + \dots + \frac{x_0}{2^{j+1}}\right)\right) \quad (n, j \in \mathbb{N}, x \in G_m),$$

then we have the character system of the group of 2-adic integers.



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3. If

$$\chi_n(x) := \exp \left(2\pi i \left(\sum_{j=0}^{\infty} \frac{n_j}{M_{j+1}} \sum_{j=0}^{\infty} x_j M_j \right) \right) \quad (x \in G_m, n \in \mathbb{N}),$$

then we have a Vilenkin-like system which is useful in the approximation of limit periodic almost even arithmetical functions.

In [10] Gát proved that a Vilenkin-like system is orthonormal and complete in $L^1(G_m)$. Define the Fourier coefficients, the Dirichlet kernels, and Fejér kernels with respect to the Vilenkin-like system χ as follows:

$$\hat{f}^\chi(n) = \hat{f}(n) := \int_{G_m} f \bar{\chi}_n, \quad \hat{f}^\chi(0) := \int_{G_m} f \quad (f \in L^1(G_m));$$

$$D_n^\chi(y, x) = D_n(y, x) := \sum_{k=0}^{n-1} \chi_n(y) \bar{\chi}_k(x);$$

$$K_n^\chi(y, x) = K_n(y, x) := \frac{1}{n} \sum_{k=0}^{n-1} D_n^\chi(y, x);$$

$$K_{h,H}^\chi(y, x) = K_{h,H}(y, x) := \sum_{j=h}^{h+H-1} D_j^\chi(y, x),$$

where the bar means complex conjugation.

In [10] Gát also proved the following expression of the Dirichlet kernel functions.

$$(2.2) \quad D_{M_n}^\chi(y, x) = D_{M_n}^\psi(y - x) = \begin{cases} M_n, & \text{if } y - x \in I_n \\ 0, & \text{if } y - x \in G_m \setminus I_n. \end{cases}$$



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Moreover,

$$D_n^x(y, x) = \alpha_n(y)\bar{\alpha}_n(x)D_n^\psi(y - x)$$

$$= \chi_n(y)\bar{\chi}_n(x) \left(\sum_{j=0}^{\infty} D_{M_j}(y - x) \sum_{k=m_j-n_j}^{m_j-1} r_j^k(y - x) \right)$$

$$(n \in \mathbb{P} := \mathbb{N} \setminus \{0\}, y, x \in G_m),$$

where the system ψ is the "ordinary" Vilenkin system.

If $\tilde{m} = (\tilde{m}_n, n \in \mathbb{N})$ is also a generating sequence then we consider the Vilenkin group $G_{\tilde{m}}$ as well. We write \tilde{M}_n instead of M_n . Let $G := G_m \times G_{\tilde{m}}$ and

$$\chi_{k, l}(x, y) = \chi_k(x)\chi_l(y) \quad (k, l \in \mathbb{N}, x \in G_m, y \in G_{\tilde{m}})$$

be the two-parameter Vilenkin groups and Vilenkin systems, respectively.

The symbol L^p ($0 < p \leq \infty$) will denote the usual Lebesgue space of complex-valued functions f defined on G with the norm (or quasinorm)

$$\|f\|_p := \left(\int_G |f|^p \right)^{\frac{1}{p}} \quad (0 < p < \infty), \quad \|f\|_\infty := \text{ess sup } |f|.$$

If $f \in L^1$, then $\hat{f}(k, l) := \int_G f \overline{\chi_{k, l}}(k, L \in \mathbb{N})$ is the usual Fourier coefficient of f . Let $S_{n, l}f$ ($n, l \in \mathbb{N}$) be the (n, l) -th rectangular partial sum of f :

$$S_{n, l}f := \sum_{k=0}^{n-1} \sum_{j=0}^{l-1} \hat{f}(k, j)\chi_{k, j}.$$

The so-called (martingale) maximal function of f is given by

$$f^*(x, y) = \sup_{n, l} M_n \tilde{M}_l \left| \int_{I_n(x)} \int_{I_l(y)} f \right| \quad (x \in G_m, y \in G_{\tilde{m}}).$$



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Furthermore, let f^\natural be the hybrid maximal function of f defined by

$$f^\natural(x, y) := \sup_n M_n \left| \int_{I_n(x)} f(t, y) dt \right| \quad (x \in G_m, y \in G_{\tilde{m}}).$$

Define the Hardy space $H^p(G_m \times G_{\tilde{m}})$ for $0 < p < \infty$ as the space of functions f for which

$$\|f\|_{H^p} := \|f^*\|_p < \infty.$$

Then $\|f\|_{H^p}$ is equivalent to $\|Qf\|_p$, where Qf is the quadratic variation of f :

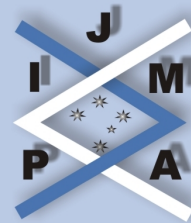
$$\begin{aligned} Qf &:= \left(\sum_{n=0}^{\infty} \sum_{l=0}^{\infty} |\Delta_{n,l} f|^2 \right)^{\frac{1}{2}} \\ &:= \left(\sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \left| S_{M_n, \tilde{M}_l} f - S_{M_n, \tilde{M}_{l-1}} f - S_{M_{n-1}, \tilde{M}_l} f + S_{M_{n-1}, \tilde{M}_{l-1}} f \right|^2 \right)^{\frac{1}{2}} \\ S_{M_n, \tilde{M}_{-1}} f &:= S_{M_{-1}, \tilde{M}_l} f := S_{M_{-1}, \tilde{M}_{-1}} f := 0 \quad (n, l \in \mathbb{N}). \end{aligned}$$

Let H^\natural be the set of functions f such that

$$\|f\|_{H^\natural} := \|f^\natural\|_1 < \infty.$$

In [11] Weisz defined the two-dimensional Sunouchi operator as follows:

$$Uf := \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |S_{2^n, 2^m} f - S_{2^n} \sigma_{2^m} f - \sigma_{2^n} S_{2^m} f + \sigma_{2^n} \sigma_{2^m} f|^2 \right)^{\frac{1}{2}}$$



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where σf is the Cesàro means of the Walsh Fourier series of $f \in L^1$. Now we extend the definition to the two-dimensional Vilenkin-like systems as follows:

$$Uf := \left(\sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \left| \sum_{j=1}^{M_{n+1}-1} \sum_{k=1}^{\tilde{M}_{s+1}-1} \frac{jk}{M_{n+1}\tilde{M}_{s+1}} \hat{f}(j, k) \chi_{j,k} \right|^2 \right)^{\frac{1}{2}} \quad (f \in L^1).$$

If $\alpha = (\alpha_n, n \in \mathbb{N})$, $\beta = (\beta_n, n \in \mathbb{N})$ are bounded sequences of complex numbers, then let

$$T_{\alpha, \beta} f := \sup_{n, l} \sum_{i=0}^{M_n-1} \sum_{j=0}^{\tilde{M}_l-1} \alpha_n \beta_k \hat{f}(n, k) \chi_{n,k}$$

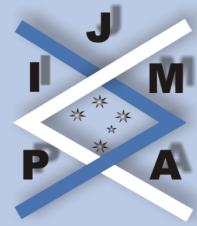
be defined at least on L^2 .

Moreover, let $\alpha_j := jM_l^{-1}$ ($l \in \mathbb{N}, j = M_l, \dots, M_{l+1} - 1$) and $\beta_k := k\tilde{M}_t^{-1}$ ($t \in \mathbb{N}, k = \tilde{M}_t, \dots, \tilde{M}_{t+1} - 1$) then

$$Uf = \left(\sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \left| \sum_{l=0}^n \sum_{t=0}^s M_l \tilde{M}_t \Delta_{l+1, t+1}(T_{\alpha, \beta} f) \right|^2 \right)^{\frac{1}{2}}.$$

In this paper we assume the sequences m, \tilde{m} are bounded. In the investigations of some operators defined on Hardy spaces, the concept of a q -atom is very useful. The function a is called a q -atom if either a is identically equal to 1 or there exist intervals $I_n(\tau) \subset G_m, I_L(\gamma) \subset G_{\tilde{m}}$ ($N, L \in \mathbb{N}, \tau \in G_m, \gamma \in G_{\tilde{m}}$) such that

- i) $a(x, y) = 0$ if $(x, y) \in G \setminus (I_N(\tau) \times I_L(\gamma))$,
- ii) $\|a\|_2 \leq \mu(I_N(\tau) \times I_L(\gamma))^{\frac{1}{2} - \frac{1}{q}}$,
- iii) $\int_{G_m} a(t, y) dt = \int_{G_{\tilde{m}}} a(x, u) du = 0$ if $x \in G_m, y \in G_{\tilde{m}}$.



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Lemma 2.1 ([1]). Let T be an operator defined at least on L_2 and assume that T is L_2 bounded. If there exists $\delta > 0$ such that for all q -atoms a with support $I_N(\tau) \times I_L(\gamma)$ and for all $r \in \mathbb{N}$, we have

$$\int_{G \setminus I_{N-r}(\tau) \times I_{L-r}(\gamma)} |Ta|^q \leq C_q 2^{-\delta r},$$

then T is bounded from H_q to L_q for all $0 < q \leq 1$.

Lemma 2.2. Let $\frac{2}{3} < q \leq 1$. Then there exist $\delta > 0$ and a constant C_q depending only on q such that for $N, L, r \in \mathbb{N}$

$$M_N^{1-\frac{q}{2}} \sum_{n=N+1}^{\infty} \int_{G_m \setminus I_{N-r}} \left(\int_{I_N} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k \chi_k(x) \bar{\chi}_k(t)}{M_n} \right|^2 dt \right)^{\frac{q}{2}} dx \leq C_q 2^{-\delta r}.$$

Proof. For $n \in \mathbb{N}, n \geq N$, we have

$$\begin{aligned} M_n K_{M_n}(x, t) &= \sum_{i=0}^{M_n-2} \chi_i(x) \bar{\chi}_i(t) \sum_{k=i+1}^{M_n-1} 1 \\ &= \sum_{i=0}^{M_n-2} (M_n - i - 1) \chi_i(x) \bar{\chi}_i(t) \\ &= (M_n - 1) D_{M_n-1}(x, t) - \sum_{i=0}^{M_n-1} i \chi_i(x) \bar{\chi}_i(t). \end{aligned}$$

This follows

$$\sum_{i=M_n}^{M_{n+1}-1} \frac{i \chi_i(x) \bar{\chi}_i(t)}{M_n} = m_n (D_{M_{n+1}}(x, t) - K_{M_{n+1}}(x, t))$$



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$$- (D_{M_n}(x, t) - K_{M_n}(x, t)) - \frac{D_{M_{n+1}}(x, t) - D_{M_n}(x, t)}{M_n}.$$

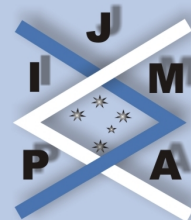
If $x \in G_m \setminus I_{N-r}$, $t \in I_N$, then there exists u ($0 \leq u \leq N - r - 1$) such that $x \in I_u \setminus I_{u+1}$. Since $x - t \in I_u \setminus I_{u+1}$, we have $D_{M_k}(x, t) = 0$ for all ($k \geq u + 1$). Suppose that $s > u$. From the definitions of the function α_n and the Fejér kernel, we have, if $x \in I_u(t) \setminus I_{u+1}(t)$,

$$\begin{aligned} K_{n^{(s)}, M_s}(x, t) &= \sum_{k=n^{(s)}}^{n^{(s)}+M_s-1} \left(\sum_{j=0}^{u-1} k_j M_j \right) \chi_k(x) \bar{\chi}_k(t) \\ &\quad + \sum_{k=n^{(s)}}^{n^{(s)}+M_s-1} M_u \sum_{p=m_u-k_u}^{m_u-1} r_t^p(x-t) \chi_k(x) \bar{\chi}_k(t) \\ &=: \sum^1 + \sum^2, \end{aligned}$$

where

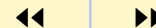
$$\begin{aligned} \sum^1 &= \sum_{k_{s-1}=0}^{m_{s-1}-1} \cdots \sum_{k_{u+1}=0}^{m_{u+1}-1} \sum_{k_{u-1}=0}^{m_{u-1}-1} \cdots \sum_{k_0=0}^{m_0-1} \left(\sum_{j=0}^{t-1} k_j M_j \right) \\ &\quad \cdot \prod_{l=u+1}^{\infty} r_l^{k_l}(x-t) \alpha_l^{k^{(l)}}(x) \bar{\alpha}_l^{k^{(l)}}(t) \sum_{k_u=0}^{m_u-1} r_u^{k_u}(x-t) \\ &= \sum_{k_u=0}^{m_u-1} r_u^{k_u}(x-t) \phi(x, t), \end{aligned}$$

and the function ϕ does not depend on k_t . Consequently, $\sum^1 = 0$ (see [12]).



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Since the sequence m is bounded, we have

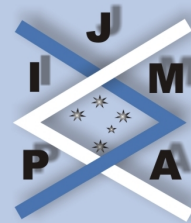
$$\begin{aligned} \int_{I_N} \left| \sum^2 \right|^2 dt &\leq CM_u^2 \sum_{p=0}^{m_u-1} \int_{I_N} \sum_{k, l=0; k_u=m_u=p}^{M_s-1} \chi_{n^{(s)+k}(t)} \bar{\chi}_{n^{(s)+l}(t)} \bar{\chi}_{n^{(s)+k}(x)} \chi_{n^{(s)+l}(x)} dt \\ &\leq CM_u^2 \frac{1}{M_N} M_s M_u. \end{aligned}$$

Recall that $k^{(u+1)} \neq l^{(u+1)}$ implies

$$\int_{I_N} \chi_{n^s+k}(x) \bar{\chi}_{n^{(s)+l}(x)} dx = 0.$$

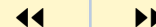
If $s \leq u$, then $|K_{n^{(s)}, M_s}(x, t)| \leq CM_u M_s$. Then

$$\begin{aligned} &M_N^{1-q/2} \sum_{n=N+1}^{\infty} \int_{G_m \setminus I_{N-r}} \left(\int_{I_N} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k \chi_k(x) \bar{\chi}_i(t)}{M_n} \right|^2 dt \right)^{\frac{q}{2}} dx \\ &\leq M_N^{1-q/2} \sum_{n=N+1}^{\infty} \int_{G_m \setminus I_{N-r}} \left(\int_{I_N} C(|D_{M_{n+1}}(x, t) - K_{M_{n+1}}(x, t)|^2 \right. \\ &\quad \left. + \left[|D_{M_n}(x, t) - K_{M_n}(x, t)| + \left| \frac{D_{M_{n+1}}(x, t) - D_{M_n}(x, t)}{M_n} \right| \right]^2 dt \right)^{\frac{q}{2}} dx \\ &= M_N^{1-q/2} \sum_{n=N+1}^{\infty} \int_{G_m \setminus I_{N-r}} \left(\int_{I_N} C(|K_{M_{n+1}}(x, t)|^2 + |K_{M_n}(x, t)|^2) dt \right)^{\frac{q}{2}} dx \end{aligned}$$



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$$\begin{aligned}
 &\leq C_q M_N^{1-q/2} \sum_{n=N+1}^{\infty} \sum_{u=0}^{N-r-1} \frac{1}{M_{n+1}} \sum_{s=0}^{n+1} \sum_{j=0}^{n_s-1} \int_{I_u \setminus I_{u+1}} \left(\int_{I_N} |K_{n^{(s+1)+j} M_s, M_s}(x, t)|^2 dt \right)^{\frac{q}{2}} dx \\
 &\quad + C_q M_N^{1-q/2} \sum_{n=N+1}^{\infty} \sum_{u=0}^{N-r-1} \frac{1}{M_n} \sum_{s=0}^n \sum_{j=0}^{n_s-1} \int_{I_u \setminus I_{u+1}} \left(\int_{I_N} |K_{n^{(s+1)+j} M_s, M_s}(x, t)|^2 dt \right)^{\frac{q}{2}} dx \\
 &\leq C_q M_N^{1-q/2} \sum_{n=N+1}^{\infty} \sum_{u=0}^{N-r-1} \frac{1}{M_{n+1}} \sum_{s=0}^{n+1} \sum_{j=0}^{n_s-1} \int_{I_u \setminus I_{u+1}} \left(\frac{M_u^3 M_s}{M_N} \right)^{\frac{q}{2}} dx \\
 &\quad + C_q M_N^{1-q/2} \sum_{n=N+1}^{\infty} \sum_{u=0}^{N-r-1} \frac{1}{M_n} \sum_{s=0}^n \sum_{j=0}^{n_s-1} \int_{I_u \setminus I_{u+1}} \left(\frac{M_u^3 M_s}{M_N} \right)^{\frac{q}{2}} dx \\
 &\leq C_q M_N^{1-q/2} \sum_{n=N+1}^{\infty} \sum_{u=0}^{N-r-1} M_u^{3q/2-1} M_n^{-q/2} M_N^{-q/2} \\
 &\leq C_q M_N^{1-q/2} M_{N-r-1}^{3q/2-1} M_N^{-q} = C_q (m_{N-r} \cdots m_{N-1})^{-(3q/2-1)} \leq C_q 2^{-\delta r} \\
 &\quad (\delta = 3q/2 - 1 > 0).
 \end{aligned}$$

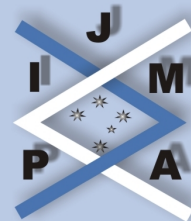
□

Theorem 2.3. Let $\frac{2}{3} < q \leq 1$. Then there exists a constant C_q such that

$$\|Uf\|_q \leq C_q \|f\|_{H^q} \quad (\forall f \in H^q(G_m \times G_{\tilde{m}})).$$

Proof. Let a be a q -atom. It can be assumed that the support of a is $I_N \times I_L$ for some $N, L \in \mathbb{N}$, that is

$$\|a\|_2 \leq (M_N P'_L)^{\frac{1}{q}-\frac{1}{2}} \text{ and } \int_{I_L} a(x, t) dt = \int_{I_N} a(u, y) du = 0 \quad \text{for all } x \in G_m, y \in G_{\tilde{m}}.$$



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This last property implies that

$$\hat{a}(i, j) = 0 \text{ if } i = 0, \dots, M_N - 1 \text{ or } j = 0, \dots, \tilde{M}_L - 1.$$

Let α and β as above. Then from the Cauchy inequality we have

$$\begin{aligned} & T_{\alpha, \beta} a(x, y) \\ & \leq \sum_{n=N+1}^{\infty} \sum_{j=L+1}^{\infty} \int_{I_N} \int_{J_L} |a(t, u)| \sum_{k=M_n}^{M_{n+1}-1} \frac{k}{M_n} \chi_k(x) \bar{\chi}_k(t) \sum_{l=M_j}^{M_{j+1}-1} \frac{l}{M_j} \chi_l(y) \bar{\chi}_l(u) dt du \\ (2.3) \quad & \leq \|a\|_2 \sum_{n=N+1}^{\infty} \sum_{j=L+1}^{\infty} \left(\int_{I_N} \int_{J_L} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k}{M_n} \chi_k(x) \bar{\chi}_k(t) \sum_{l=M_j}^{M_{j+1}-1} \frac{l}{M_j} \chi_l(y) \bar{\chi}_l(u) \right|^2 dt du \right)^{\frac{1}{2}}. \end{aligned}$$

First we will show $T_{\alpha, \beta}$ is q -quasi local. Let $r \in \mathbb{N}$ and define the sets X_i ($i = 1, 2, 3, 4$) as follows:

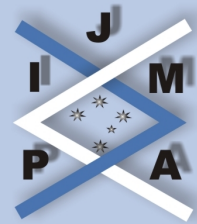
$$\begin{aligned} X_1 & := (G_m \setminus I_{N-r}) \times I_L, & X_2 & := (G_m \setminus I_{N-r}) \times (G_{\tilde{m}} \setminus I_L), \\ X_3 & := I_N \times (G_{\tilde{m}} \setminus I_{L-r}), & X_4 & := (G_m \setminus I_N) \times (G_{\tilde{m}} \setminus I_{L-r}). \end{aligned}$$

It is clear that

$$\int_{(G \setminus I_{N-r}) \times I_{L-r}} (T_{\alpha, \beta} a)^q \leq \sum_{i=1}^4 \int_{X_i} (T_{\alpha, \beta} a)^q.$$

To estimate the integral over X_1 , we have

$$\begin{aligned} & \int_{X_1} (T_{\alpha, \beta} a)^q(x, y) dx dy \\ & \leq |I_L|^{1-\frac{q}{2}} \sum_{n=N+1}^{\infty} \int_{G_m \setminus I_{N-r}} \left(\int_{I_L} \left(\int_{I_n} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k}{M_n} \chi_k(x) \bar{\chi}_k(t) \right| \right) \right) \end{aligned}$$



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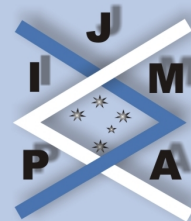
$$\begin{aligned} & \times \sup \int_{I_L} a(t, u) \left(\sum_{j=L+1}^l \sum_{l=M_j}^{M_{j+1}-1} \frac{l}{M_j} \chi_l(y) \bar{\chi}_l(u) |du| dt \right)^2 dy \Big)^{\frac{q}{2}} dx \\ & \leq |I_L|^{1-q/2} \sum_{n=N+1}^{\infty} \int_{G_m \setminus I_{N-r}} \left(\int_{I_N} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k}{M_n} \chi_k(x) \bar{\chi}_k(t) \right|^2 dt \right)^{\frac{q}{2}} dx \\ & \quad \times \left(\int_{I_N} \int_{J_L} |a(t, y)|^2 dy dt \right)^{\frac{q}{2}}. \end{aligned}$$

From the definition of q -atoms and Lemma 2.2, we have

$$\begin{aligned} & \int_{X_1} (T_{\alpha, \beta} a)^q(x, y) dx dy \\ & \leq \|a\|_2^q |I_L|^{1-\frac{q}{2}} \sum_{n=N+1}^{\infty} \int_{G_m \setminus I_{N-r}} \left(\int_{I_N} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k \chi_k(x) \bar{\chi}_k(t)}{M_n} \right|^2 dt \right)^{\frac{q}{2}} dx \\ & \leq C_q M_N^{1-\frac{q}{2}} \sum_{n=N+1}^{\infty} \int_{G_m \setminus I_{N-r}} \left(\int_{I_N} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k \chi_k(x) \bar{\chi}_k(t)}{M_n} \right|^2 dt \right)^{\frac{q}{2}} dx \\ (2.4) \quad & \leq C_q 2^{-\delta r}. \end{aligned}$$

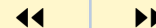
In a similar way, we have

$$(2.5) \quad \int_{X_3} (T_{\alpha, \beta} a)^q(x, y) dx dy \leq C_q 2^{-\delta r}.$$



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On the set X_2 , by inequality (2.3) we have

$$\begin{aligned}
 & \int_{X_3} (T_{\alpha, \beta} a)^q(x, y) dx dy \\
 & \leq \|a\|_2^q \sum_{n=N+1}^{\infty} \sum_{j=L+1}^{\infty} \int_{G_m \setminus I_{N-r}} \int_{G_{\tilde{m}} \setminus I_l} \\
 & \quad \left(\int_{J_N} \int_{J_L} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k \chi_k(x) \bar{\chi}_k(t)}{M_n} \sum_{l=M_{j-1}}^{M_j-1} \frac{l}{M_j} \chi_l(y) \bar{\chi}_l(u) \right|^2 dt du \right)^{\frac{q}{2}} dx dy \\
 & \leq (M_N P_L)^{1-\frac{q}{2}} \sum_{n=N+1}^{\infty} \sum_{j=L+1}^{\infty} \int_{G_m \setminus I_{N-r}} \int_{G_{\tilde{m}} \setminus I_l} \\
 & \quad \left(\int_{J_N} \int_{J_L} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k \chi_k(x) \bar{\chi}_k(t)}{M_n} \sum_{l=M_j}^{M_{j+1}-1} \frac{l}{M_j} \chi_l(y) \bar{\chi}_l(u) \right|^2 dt du \right)^{\frac{q}{2}} dx dy \\
 & \leq M_N^{1-\frac{q}{2}} \sum_{n=N+1}^{\infty} \int_{G_m \setminus I_{N-r}} \left(\int_{J_N} \left| \sum_{k=M_n}^{M_{n+1}-1} \frac{k \chi_k(x) \bar{\chi}_k(t)}{M_n} \right|^2 dt \right)^{\frac{q}{2}} dx \\
 & \leq C_q 2^{-\delta r} (\tilde{M}_L)^{1-\frac{q}{2}} \sum_{j=L+1}^{\infty} \int_{G_{\tilde{m}} \setminus J_L} \left(\int_{J_L} \sum_{l=M_j}^{M_{j+1}-1} \frac{l}{M_j} \chi_l(y) \bar{\chi}_l(u) |^2 du \right)^{\frac{q}{2}} dy \\
 & \leq C_q 2^{-\delta r}.
 \end{aligned}$$

An analogous estimate with X_4 instead of X_2 can be obtained using a similar ar-

gument and these prove that the operator $T_{\alpha,\beta}$ is q -quasi local. By Parseval's equality, it is clear that the operator $T_{\alpha,\beta}$ is L^2 bounded. Since

$$Uf = \left(\sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \left| \sum_{j=1}^{M_{n+1}-1} \sum_{k=1}^{\tilde{M}_{s+1}-1} \frac{jk}{M_{n+1}\tilde{M}_{s+1}} \hat{f}(j,k) \chi_{j,k} \right|^2 \right)^{\frac{1}{2}} \leq CQ(T_{\alpha,\beta}f),$$

where the operator Q is a two-dimensional quadratic variation of f . By Lemma 2.1, we have

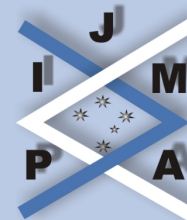
$$\|Uf\|_q \leq C_q \|Q(T_{\alpha,\beta}f)\|_q \leq C_q \|T_{\alpha,\beta}f\|_{H_q} \leq C_q \|f\|_{H_q}.$$

□

Applying known theorems on the interpolation of operators and a duality argument gives the following:

Theorem 2.4. *The operator U is $L^s \rightarrow L^s$ bounded and of weak type (H^{\natural}, L^1) , i.e., there exists a constant C such that for all $\delta > 0$ and $f \in H^{\natural}$ we have*

$$\mu\{(x, y) \in G : |Uf(x, y)| > \delta\} \leq C \frac{\|f\|_{H^{\natural}}}{\delta}.$$

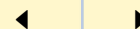
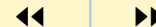


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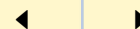
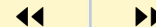


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