

Journal of Inequalities in Pure and Applied Mathematics

SOME INEQUALITIES INVOLVING THE GAMMA FUNCTION

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©2000 Victoria University
ISSN (electronic): 1443-5756
142-06



volume 7, issue 5, article 179,
2006.

*Received 15 May, 2006;
accepted 13 July, 2006.*

Communicated by: J. Sándor

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

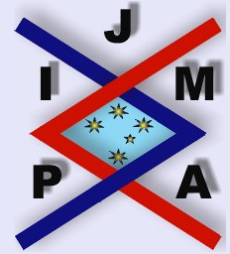
In this short paper, as a complement of the double inequality on the Euler gamma function, obtained by József Sándor in the paper [A note on certain inequalities for the gamma function, *J. Ineq. Pure Appl. Math.*, 6(3) (2005), Art. 61], several inequalities involving the Euler gamma function are established by using the same method of J. Sándor that is used in [2].

2000 Mathematics Subject Classification: 33B15.

Key words: Euler gamma function, Digamma function.

Contents

1	Introduction and Lemma	3
2	Main Results	5
	References	



Some Inequalities Involving the Gamma Function

Lazhar Bougoffa

Title Page

Contents



Go Back

Close

Quit

Page 2 of 7

1. Introduction and Lemma

In [1], C. Alsina and M.S. Tomas studied a very interesting inequality involving the gamma function and proved the following double inequality

$$(1.1) \quad \frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1, \quad x \in [0, 1], \quad n \in \mathbb{N},$$

by using a geometrical method [1]. In view of the interest in this type of inequalities, J. Sándor [2] extended this result to a more general case, and obtained the following inequality

$$(1.2) \quad \frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \leq 1, \quad x \in [0, 1], \quad a \geq 1.$$

The method used in [2] to obtain these results is based on the following lemma.

Lemma 1.1. *For all $x > 0$, and all $a \geq 1$ one has*

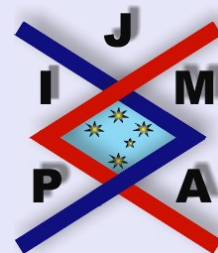
$$(1.3) \quad \psi(1+ax) \geq \psi(1+x),$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, $x > 0$ is the digamma function and has the following series representation

$$(1.4) \quad \psi(x) = -\gamma + (x-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(x+k)}.$$

Proof. See [2].

□



Some Inequalities Involving the
Gamma Function

Lazhar Bougoffa

Title Page

Contents



Go Back

Close

Quit

Page 3 of 7

In [3], A.McD. Mercer continued to create new inequalities on this subject and other special functions and obtained the following inequalities

$$(1.5) \quad \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} < \frac{\Gamma(1+y)^a}{\Gamma(1+ay)}, \quad 0 < a < 1,$$

$$(1.6) \quad \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} > \frac{\Gamma(1+y)^a}{\Gamma(1+ay)}, \quad a < 0 \text{ or } a > 1,$$

where $y > x > 0$, $1 + ax > 0$, and $1 + bx > 0$.

This paper is a continuation of the above papers. As in [2], our goal is to prove several inequalities involving the gamma function, using the same method of J. Sándor that is used in [2]. Here, the essential lemma is the following

Lemma 1.2. *For all $x > 0$, and all $a \geq b$ we have*

$$(1.7) \quad \psi(1+ax) \geq \psi(1+bx),$$

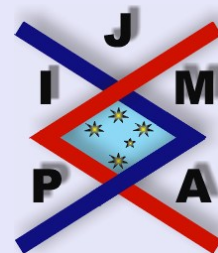
in which $1 + ax > 0$ and $1 + bx > 0$.

Proof. By the above series representation of ψ , observe that:

$$\psi(1+ax) - \psi(1+bx) = \sum_{k=0}^{\infty} \left[\frac{ax}{(k+1)(ax+k+1)} - \frac{bx}{(k+1)(bx+k+1)} \right],$$

$$\psi(1+ax) - \psi(1+bx) = (a-b)x \sum_{k=0}^{\infty} \frac{1}{(ax+k+1)(bx+k+1)} \geq 0,$$

by $a \geq b$, $1 + ax > 0$, $1 + bx > 0$, $x > 0$ and $k > 0$. Thus the inequality (1.7) is proved. The equality in (1.7) holds only if $a = b$. \square



Some Inequalities Involving the Gamma Function

Lazhar Bougoffa

Title Page

Contents



Go Back

Close

Quit

Page 4 of 7

2. Main Results

Now we are in a position to give the following theorem.

Theorem 2.1. *Let f be a function defined by*

$$f(x) = \frac{\Gamma(1 + bx)^a}{\Gamma(1 + ax)^b}, \quad \forall x \geq 0,$$

in which $1 + ax > 0$ and $1 + bx > 0$, then for all $a \geq b > 0$ or $0 > a \geq b$ ($a > 0$ and $b < 0$), f is decreasing (increasing) respectively on $[0, \infty)$.

Proof. Let g be a function defined by

$$g(x) = \log f(x) = a \log \Gamma(1 + bx) - b \log \Gamma(1 + ax),$$

then

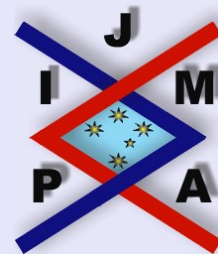
$$g'(x) = ab[\psi(1 + bx) - \psi(1 + ax)].$$

By Lemma 1.2, we get $g'(x) \leq 0$ if $a \geq b > 0$ or $0 > a \geq b$ ($g'(x) \geq 0$ if $0 > a \geq b$), i.e., g is decreasing on $[0, \infty)$ (increasing on $[0, \infty)$) respectively. Hence f is decreasing on $[0, \infty)$ if $a \geq b > 0$ or $0 > a \geq b$ (increasing if $a > 0$ and $b < 0$) respectively.

The proof is complete. \square

Corollary 2.2. *For all $x \in [0, 1]$, and all $a \geq b > 0$ or $0 > a \geq b$, we have*

$$(2.1) \quad \frac{\Gamma(1 + b)^a}{\Gamma(1 + a)^b} \leq \frac{\Gamma(1 + bx)^a}{\Gamma(1 + ax)^b} \leq 1.$$



Some Inequalities Involving the
Gamma Function

Lazhar Bougoffa

Title Page

Contents



Go Back

Close

Quit

Page 5 of 7

Proof. To prove (2.1), applying Theorem 2.1, and taking account of $\Gamma(1) = 1$ we get $f(1) \leq f(x) \leq f(0)$ for all $x \in [0, 1]$, and we omit (2.1). \square

Corollary 2.3. For all $x \in [0, 1]$, and all $a > 0$ and $b < 0$, we have

$$(2.2) \quad 1 \leq \frac{\Gamma(1 + bx)^a}{\Gamma(1 + ax)^b} \leq \frac{\Gamma(1 + b)^a}{\Gamma(1 + a)^b}.$$

Proof. Applying Theorem 2.1, we get $f(0) \leq f(x) \leq f(1)$ for all $x \in [0, 1]$, and we omit (2.2). \square

Now we consider the simplest cases of Corollary 2.2 to obtain the known results of C. Alsina and M.S. Tomas [1] and J. Sándor [2].

Remark 1. Taking $a = n$ and $b = 1$ ($a \geq 1$ and $b = 1$), in Corollary 2.2, we obtain (1.1) ((1.2)) respectively.

Also we conclude different generalizations of (1.5)–(1.6) which are obtained by A.McD. Mercer [3].

Corollary 2.4. For all $x \in [0, 1]$, and all $a \geq b > 0$ or $0 > a \geq b$, we have

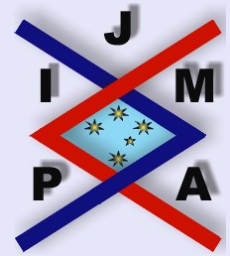
$$(2.3) \quad \frac{\Gamma(1 + bx)^a}{\Gamma(1 + ax)^b} < \frac{\Gamma(1 + by)^a}{\Gamma(1 + ay)^b},$$

where $0 < y < x \leq 1$.

Corollary 2.5. For all $x \in [0, 1]$, and all $a > 0$ and $b < 0$, we have

$$(2.4) \quad \frac{\Gamma(1 + by)^a}{\Gamma(1 + ay)^b} < \frac{\Gamma(1 + bx)^a}{\Gamma(1 + ax)^b},$$

where $0 < y < x \leq 1$.



Some Inequalities Involving the
Gamma Function

Lazhar Bougoffa

Title Page

Contents



Go Back

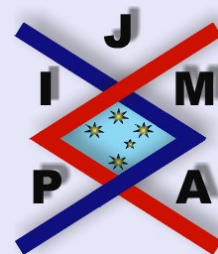
Close

Quit

Page 6 of 7

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Some Inequalities Involving the Gamma Function

Lazhar Bougoffa

Title Page

Contents



Go Back

Close

Quit

Page 7 of 7