ON SOME NEW FRACTIONAL INTEGRAL INEQUALITIES

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Abstract: In this paper, using the Riemann-Liouville fractional integral, we establish some new integral inequalities for the Chebyshev functional in the case of two synchronous functions.

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1. Introduction

Let us consider the functional [1]:

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.1}$$

where f and g are two integrable functions which are synchronous on $[a,b]$ $\Big($i.e. $(f(x)-f(y))(g(x)-g(y)) \geq 0$, for any $x, y \in [a,b]\Big)$.

Many researchers have given considerable attention to (1.1) and a number of inequalities have appeared in the literature, see [3, 4, 5].

The main purpose of this paper is to establish some inequalities for the functional (1.1) using fractional integrals.

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2. Description of Fractional Calculus

We will give the necessary notation and basic definitions below. For more details, one can consult [2, 6].

Definition 2.1. *A real valued function $f(t)$, $t \geq 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0,\infty[)$.*

Definition 2.2. *A function $f(t), t \geq 0$ is said to be in the space $C_\mu^n, n \in \mathbb{R}$, if $f^{(n)} \in C_\mu$.*

Definition 2.3. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu, (\mu \geq -1)$ is defined as*

$$(2.1)\qquad J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \qquad \alpha > 0, t > 0,$$
$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

For the convenience of establishing the results, we give the semigroup property:

$$(2.2)\qquad J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \qquad \alpha \geq 0, \beta \geq 0,$$

which implies the commutative property:

$$(2.3)\qquad J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t).$$

From (2.1), when $f(t) = t^\mu$ we get another expression that will be used later:

$$(2.4)\qquad J^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\alpha+\mu}, \qquad \alpha > 0; \quad \mu > -1, t > 0.$$

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3. Main Results

Theorem 3.1. *Let f and g be two synchronous functions on $[0,\infty[$. Then for all $t>0, \alpha>0$, we have:*

$$J^{\alpha}(fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^{\alpha}} J^{\alpha}f(t) J^{\alpha}g(t). \tag{3.1}$$

Proof. Since the functions f and g are synchronous on $[0,\infty[$, then for all $\tau \geq 0, \rho \geq 0$, we have

$$\Big(f(\tau)-f(\rho)\Big)\Big(g(\tau)-g(\rho)\Big) \geq 0. \tag{3.2}$$

Therefore

$$f(\tau)g(\tau)+f(\rho)g(\rho) \geq f(\tau)g(\rho)+f(\rho)g(\tau). \tag{3.3}$$

Now, multiplying both sides of (3.3) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$, $\tau \in (0,t)$, we get

$$\begin{aligned} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\tau) + \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)g(\rho) \\ \geq \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)g(\rho) + \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\rho)g(\tau). \end{aligned} \tag{3.4}$$

Then integrating (3.4) over $(0,t)$, we obtain:

$$\begin{aligned} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)g(\tau)d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\rho)g(\rho)d\tau \\ \geq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)g(\rho)d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\rho)g(\tau)d\tau. \end{aligned} \tag{3.5}$$

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Consequently,

$$J^{\alpha}(fg)(t)+f\left(\rho\right)g\left(\rho\right)\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-\tau\right)^{\alpha-1}d\tau \geq \frac{g\left(\rho\right)}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-\tau\right)^{\alpha-1}f\left(\tau\right)d\tau+\frac{f\left(\rho\right)}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-\tau\right)^{\alpha-1}g\left(\tau\right)d\tau. \tag{3.6}$$

So we have

$$J^{\alpha}(fg)(t)+f\left(\rho\right)g\left(\rho\right)J^{\alpha}(1)\geq g\left(\rho\right)J^{\alpha}(f)(t)+f\left(\rho\right)J^{\alpha}(g)(t). \tag{3.7}$$

Multiplying both sides of (3.7) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}$, $\rho\in(0,t)$, we obtain:

$$\frac{\left(t-\rho\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}J^{\alpha}(fg)(t)+\frac{\left(t-\rho\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}f\left(\rho\right)g\left(\rho\right)J^{\alpha}(1) \geq \frac{\left(t-\rho\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}g\left(\rho\right)J^{\alpha}f(t)+\frac{\left(t-\rho\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}f\left(\rho\right)J^{\alpha}g(t). \tag{3.8}$$

Now integrating (3.8) over $(0,t)$, we get:

$$J^{\alpha}(fg)(t)\int_{0}^{t}\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}d\rho+\frac{J^{\alpha}(1)}{\Gamma(\alpha)}\int_{0}^{t}f(\rho)g(\rho)(t-\rho)^{\alpha-1}d\rho \geq \frac{J^{\alpha}f(t)}{\Gamma(\alpha)}\int_{0}^{t}(t-\rho)^{\alpha-1}g(\rho)d\rho+\frac{J^{\alpha}g(t)}{\Gamma(\alpha)}\int_{0}^{t}(t-\rho)^{\alpha-1}f(\rho)d\rho. \tag{3.9}$$

Hence

$$J^{\alpha}(fg)(t)\geq\frac{1}{J^{\alpha}(1)}J^{\alpha}f(t)J^{\alpha}g(t), \tag{3.10}$$

and this ends the proof. □

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The second result is:

Theorem 3.2. *Let f and g be two synchronous functions on $[0,\infty[$. Then for all $t>0$, $\alpha>0$, $\beta>0$, we have:*

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}(fg)(t)+\frac{t^{\beta}}{\Gamma(\beta+1)}J^{\alpha}(fg)(t)\geq J^{\alpha}f(t)J^{\beta}g(t)+J^{\beta}f(t)J^{\alpha}g(t). \tag{3.11}$$

Proof. Using similar arguments as in the proof of Theorem 3.1, we can write

$$\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}J^{\alpha}(fg)(t)+J^{\alpha}(1)\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}f(\rho)g(\rho)\geq\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}g(\rho)J^{\alpha}f(t)+\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}f(\rho)J^{\alpha}g(t). \tag{3.12}$$

By integrating (3.12) over $(0,t)$, we obtain

$$J^{\alpha}(fg)(t)\int_0^t\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}d\rho+\frac{J^{\alpha}(1)}{\Gamma(\beta)}\int_0^t f(\rho)g(\rho)(t-\rho)^{\beta-1}d\rho\geq\frac{J^{\alpha}f(t)}{\Gamma(\beta)}\int_0^t(t-\rho)^{\beta-1}g(\rho)d\rho+\frac{J^{\alpha}g(t)}{\Gamma(\beta)}\int_0^t(t-\rho)^{\beta-1}f(\rho)d\rho, \tag{3.13}$$

and this ends the proof. □

Remark 1. The inequalities (3.1) and (3.11) are reversed if the functions are asynchronous on $[0,\infty[$ (i.e. $(f(x)-f(y))(g(x)-g(y))\leq 0$, for any $x,y\in[0,\infty[$).

Remark 2. Applying Theorem 3.2 for $\alpha=\beta$, we obtain Theorem 3.1.

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The third result is:

Theorem 3.3. *Let* $(f_i)_{i=1,\dots,n}$ *be* n *positive increasing functions on* $[0,\infty[$. *Then for any* $t>0, \alpha>0$, *we have*

$$J^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) \geq (J^{\alpha}(1))^{1-n} \prod_{i=1}^{n} J^{\alpha} f_i(t). \tag{3.14}$$

Proof. We prove this theorem by induction.

Clearly, for $n=1$, we have $J^{\alpha}(f_1)(t) \geq J^{\alpha}(f_1)(t)$, for all $t>0, \alpha>0$.

For $n=2$, applying (3.1), we obtain:

$$J^{\alpha}(f_1 f_2)(t) \geq (J^{\alpha}(1))^{-1} J^{\alpha}(f_1)(t) J^{\alpha}(f_2)(t), \qquad \text{for all } t>0, \alpha>0.$$

Now, suppose that (induction hypothesis)

$$J^{\alpha}\left(\prod_{i=1}^{n-1} f_i\right)(t) \geq (J^{\alpha}(1))^{2-n} \prod_{i=1}^{n-1} J^{\alpha} f_i(t), \qquad t>0, \alpha>0. \tag{3.15}$$

Since $(f_i)_{i=1,\dots,n}$ are positive increasing functions, then $\left(\prod_{i=1}^{n-1} f_i\right)(t)$ is an increasing function. Hence we can apply Theorem 3.1 to the functions $\prod_{i=1}^{n-1} f_i = g$, $f_n = f$. We obtain:

$$J^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) = J^{\alpha}(fg)(t) \geq (J^{\alpha}(1))^{-1} J^{\alpha}\left(\prod_{i=1}^{n-1} f_i\right)(t) J^{\alpha}(f_n)(t). \tag{3.16}$$

Taking into account the hypothesis (3.15), we obtain:

$$J^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) \geq (J^{\alpha}(1))^{-1}\left((J^{\alpha}(1))^{2-n}\left(\prod_{i=1}^{n-1} J^{\alpha} f_i\right)(t)\right) J^{\alpha}(f_n)(t), \tag{3.17}$$

and this ends the proof. □

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We further have:

Theorem 3.4. *Let f and g be two functions defined on $[0,+\infty[$, such that f is increasing, g is differentiable and there exists a real number $m := \inf_{t\geq 0} g'(t)$. Then the inequality*

$$J^{\alpha}(fg)(t) \geq (J^{\alpha}(1))^{-1} J^{\alpha}f(t)J^{\alpha}g(t) - \frac{mt}{\alpha+1}J^{\alpha}f(t) + mJ^{\alpha}(tf(t)) \tag{3.18}$$

is valid for all $t>0, \alpha>0$.

Proof. We consider the function $h(t) := g(t) - mt$. It is clear that h is differentiable and it is increasing on $[0,+\infty[$. Then using Theorem 3.1, we can write:

$$\begin{aligned} & J^{\alpha}\Big((g-mt)f(t)\Big) \\ & \geq (J^{\alpha}(1))^{-1} J^{\alpha}f(t)\Big(J^{\alpha}g(t) - mJ^{\alpha}(t)\Big) \\ & \geq (J^{\alpha}(1))^{-1} J^{\alpha}f(t)J^{\alpha}g(t) - \frac{m(J^{\alpha}(1))^{-1}t^{\alpha+1}}{\Gamma(\alpha+2)}J^{\alpha}f(t) \\ & \geq (J^{\alpha}(1))^{-1} J^{\alpha}f(t)J^{\alpha}g(t) - \frac{m\Gamma(\alpha+1)t}{\Gamma(\alpha+2)}J^{\alpha}f(t) \\ & \geq (J^{\alpha}(1))^{-1} J^{\alpha}f(t)J^{\alpha}g(t) - \frac{mt}{\alpha+1}J^{\alpha}f(t). \end{aligned} \tag{3.19}$$

Hence

$$\begin{aligned} J^{\alpha}(fg)(t) \geq {} & (J^{\alpha}(1))^{-1} J^{\alpha}f(t)J^{\alpha}g(t) \\ & - \frac{mt}{\alpha+1}J^{\alpha}f(t) + mJ^{\alpha}(tf(t)), \quad t>0, \alpha>0. \end{aligned} \tag{3.20}$$

Theorem 3.4 is thus proved. □

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Corollary 3.5. *Let f and g be two functions defined on $[0, +\infty[$.*

(A) *Suppose that f is decreasing, g is differentiable and there exists a real number $M := \sup_{t\geq 0} g'(t)$. Then for all $t > 0$, $\alpha > 0$, we have:*

$$J^{\alpha}(fg)(t) \geq \left(J^{\alpha}(1)\right)^{-1} J^{\alpha}f(t) J^{\alpha}g(t) - \frac{Mt}{\alpha+1} J^{\alpha}f(t) + MJ^{\alpha}\left(tf(t)\right). \tag{3.21}$$

(B) *Suppose that f and g are differentiable and there exist $m_1 := \inf_{t\geq 0} f'(x)$, $m_2 := \inf_{t\geq 0} g'(t)$. Then we have*

$$\begin{aligned} J^{\alpha}(fg)(t) - m_1 J^{\alpha} t g(t) - m_2 J^{\alpha} t f(t) + m_1 m_2 J^{\alpha} t^2 \\ \geq \left(J^{\alpha}(1)\right)^{-1} \Big(J^{\alpha}f(t) J^{\alpha}g(t) - m_1 J^{\alpha} t J^{\alpha} g(t) \\ - m_2 J^{\alpha} t J^{\alpha} f(t) + m_1 m_2 (J^{\alpha} t)^2 \Big). \end{aligned} \tag{3.22}$$

(C) *Suppose that f and g are differentiable and there exist $M_1 := \sup_{t\geq 0} f'(t)$, $M_2 := \sup_{t\geq 0} g'(t)$. Then the inequality*

$$\begin{aligned} J^{\alpha}(fg)(t) - M_1 J^{\alpha} t g(t) - M_2 J^{\alpha} t f(t) + M_1 M_2 J^{\alpha} t^2 \\ \geq \left(J^{\alpha}(1)\right)^{-1} \Big(J^{\alpha}f(t) J^{\alpha}g(t) - M_1 J^{\alpha} t J^{\alpha} g(t) \\ - M_2 J^{\alpha} t J^{\alpha} f(t) + M_1 M_2 (J^{\alpha} t)^2 \Big). \end{aligned} \tag{3.23}$$

is valid.

Proof.
(A): Apply Theorem 3.1 to the functions f and $G(t) := g(t) - m_2 t$.

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(B): Apply Theorem 3.1 to the functions F and G, where: $F(t) := f(t) - m_1 t,\ G(t) := g(t) - m_2 t$.

To prove (C), we apply Theorem 3.1 to the functions

$$F(t) := f(t) - M_1 t,\ G(t) := g(t) - M_2 t.$$

□

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