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A REFINEMENT OF VAN DER CORPUT'S INEQUALITY

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ABSTRACT. In this note, a refinement of van der Corput's inequality is given.

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1. INTRODUCTION

Let $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 < \sum_{n=1}^{\infty}(n+1)a_n < \infty$, and $S_n = \sum_{m=1}^{n} \frac{1}{m}$, the harmonic number. Then van der Corput's inequality [5] states that

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{\frac{1}{S_n}} < e^{1+\gamma}\sum_{n=1}^{\infty}(n+1)a_n, \tag{1.1}$$

where $\gamma = 0.57721566\ldots$ stands for Euler-Mascheroni's constant. The factor $e^{1+\gamma}$ in (1.1) is the best possible.

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In 2003, Hu in [3] gave a strengthened version of (1.1) by

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k^{1/k}\right)^{\frac{1}{S_n}}<e^{1+\gamma}\sum_{n=1}^{\infty}\left(n-\frac{\ln n}{4}\right)a_n. \tag{1.2}$$

Recently, Yang in [7] obtained a better result than Hu's inequality (1.2) as

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k^{1/k}\right)^{\frac{1}{S_n}}<e^{1+\gamma}\sum_{n=1}^{\infty}\left(n-\frac{\ln n}{3}\right)a_n. \tag{1.3}$$

Moreover, he also extended (1.1) in [7] as follows

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k^{1/(k+\beta)}\right)^{\frac{1}{S_n(\beta)}}<e^{1+\gamma(\beta)}\sum_{n=1}^{\infty}\left(n+\frac{1}{2}+\beta\right)a_n, \tag{1.4}$$

where $\beta\in(-1,\infty)$, $S_n(\beta)=\sum_{k=1}^{n}\frac{1}{k+\beta}$, and

$$\gamma(\beta)=\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k+\beta}-\ln(n+\beta)\right].$$

Applying $\beta=0$ in (1.4) leads to

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k^{1/k}\right)^{\frac{1}{S_n}}<e^{1+\gamma}\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)a_n, \tag{1.5}$$

which improved inequality (1.1) clearly, but is not more accurate than (1.2) and (1.3).

In [1], among other things, the authors established a sharper inequality than (1.1), (1.2), (1.3) and (1.5) as follows

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k^{1/k}\right)^{\frac{1}{S_n}}<e^{1+\gamma}\sum_{n=1}^{\infty}n\left(1-\frac{\ln n}{3n-1/4}\right)a_n. \tag{1.6}$$

The purpose of this note is to refine further inequality (1.6). Our main result is the following.

Theorem 1.1. *For $n\in\mathbb{N}$, let $S_n=\sum_{m=1}^{n}\frac{1}{m}$, the harmonic number. If $a_n\ge 0$ for $n\in\mathbb{N}$ and*

$$0<\sum_{n=1}^{\infty}n\left(1-\frac{\ln n}{2n+\ln n+11/6}\right)a_n<\infty,$$

then

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k^{1/k}\right)^{\frac{1}{S_n}}<e^{1+\gamma}\sum_{n=1}^{\infty}e^{-\frac{6(6n+1)\gamma-9}{(6n+1)(12n+11)}}n\left(1-\frac{\ln n}{2n+\ln n+11/6}\right)a_n, \tag{1.7}$$

where $\gamma=0.57721566\ldots$ is Euler-Mascheroni's constant.

Remark 1.2. Let

$$A_n=e^{1+\gamma}e^{-\frac{6(6n+1)\gamma-9}{(6n+1)(12n+11)}}n\left(1-\frac{\ln n}{2n+\ln n+11/6}\right)$$

for $n\in\mathbb{N}$. Numerical computation shows $A_1=4.40\ldots<e^{1+\gamma}\left(1-\frac{\ln 1}{3-1/4}\right)=4.84\ldots$ and $A_2=7.99\ldots<e^{1+\gamma}\left(2-\frac{2\ln 2}{6-1/4}\right)=8.51\ldots$. and $A_3=11.95\ldots<e^{1+\gamma}\left(3-\frac{3\ln 2}{9-1/4}\right)=$

$12.70\dots$. When $n \ge 4$, inequality $2n + \frac{11}{6} + \ln n < 3n - \frac{1}{4}$ is valid, which can be rearranged as

$$1 - \frac{\ln n}{2n + \ln n + 11/6} \le 1 - \frac{\ln n}{3n - 1/4}.$$

This implies that inequality (1.7) is a refinement of (1.6).

2. LEMMAS

In order to prove our main result, some lemmas are necessary.

Lemma 2.1 ([4]). *For* $n \in \mathbb{N}$,

$$\frac{1}{2n + 1/(1-\gamma) - 2} < S_n - \ln n - \gamma < \frac{1}{2n + 1/3}. \tag{2.1}$$

The constants $\frac{1}{1-\gamma} - 2$ *and* $\frac{1}{3}$ *in* (2.1) *are the best possible.*

Lemma 2.2 ([2, 6]). *If* $x > 0$, *then*

$$\left(1 + \frac{1}{x}\right)^x < e\left(1 - \frac{1}{2x + 11/6}\right). \tag{2.2}$$

Lemma 2.3. *For* $n \in \mathbb{N}$,

$$B_n \triangleq \left[\frac{(n+1)S_{n+1}}{nS_n}\right]^{nS_n} < e^{1+\gamma} e^{-\frac{6(6n+1)\gamma - 9}{(6n+1)(12n+11)}} n \left(1 - \frac{\ln n}{2n + \ln n + 11/6}\right). \tag{2.3}$$

Proof. By virtue of Lemma 2.2, it follows that

$$\begin{aligned}\left[\frac{(n+1)S_{n+1}}{nS_n}\right]^{\frac{nS_n}{S_n+1}} &= \left(1 + \frac{S_n + 1}{nS_n}\right)^{\frac{nS_n}{S_n+1}} \\ &< e\left[1 - \frac{S_n + 1}{2nS_n + 11(S_n+1)/6}\right] < e\left(1 - \frac{1}{2n + 11/6}\right).\end{aligned}$$

Applying Lemma 2.1 yields

$$\begin{aligned}B_n &< \left[e\left(1 - \frac{1}{2n + 11/6}\right)\right]^{S_n+1} \\ &< e^{1+\frac{1}{2n+1/3}+\gamma+\ln n}\left(1 - \frac{1}{2n + 11/6}\right)^{1+\frac{1}{2n+1/3}+\gamma+\ln n}.\end{aligned} \tag{2.4}$$

Taking advantage of inequalities $\left(1 - \frac{1}{x}\right)^{-x} > e$ for $x > 1$ and $e^{-x} \le \frac{1}{1+x}$ for $x > -1$ leads to

$$\left(1 - \frac{1}{2n + 11/6}\right)^{\ln n} \le \exp\left(-\frac{\ln n}{2n + 11/6}\right) \le \frac{2n + 11/6}{2n + 11/6 + \ln n} \tag{2.5}$$

and

$$\begin{aligned}&\left(1 - \frac{1}{2n + 11/6}\right)^{\frac{1}{2n+1/3}+1+\gamma} \exp\left(\frac{1}{2n + 1/3}\right) \\ &\quad< \exp\left[\frac{1}{2n + 1/3} - \frac{1 + \gamma}{2n + 11/6} - \frac{1}{(2n + 11/6)(2n + 1/3)}\right] \\ &\quad= \exp\left[-\frac{6(6n+1)\gamma - 9}{(6n+1)(12n+11)}\right].\end{aligned} \tag{2.6}$$

Combination of (2.4), (2.5), (2.6) gives

$$B_n < e^{1+\gamma-\frac{6(6n+1)\gamma-9}{(6n+1)(12n+11)}}\left(n-\frac{n\ln n}{2n+\ln n+11/6}\right). \tag{2.7}$$

Lemma 2.3 is proved. □

3. Proof of Theorem 1.1

For $n \in \mathbb{N}$ and $1 \le k \le n$, let

$$c_k = \frac{[(k+1)S_{k+1}]^{kS_k}}{(kS_k)^{kS_{k-1}}} \tag{3.1}$$

with assumption $S_0 = 0$, then

$$\left(\prod_{k=1}^{n} c_k^{1/k}\right)^{-\frac{1}{S_n}} = \frac{1}{(n+1)S_{n+1}}. \tag{3.2}$$

By using the discrete weighted arithmetic-geometric mean inequality and interchanging the order of summations,

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{\frac{1}{S_n}} &= \sum_{n=1}^{\infty}\left[\prod_{k=1}^{n}(c_k a_k)^{1/k}\right]^{\frac{1}{S_n}}\left(\prod_{k=1}^{n} c_k^{1/k}\right)^{-\frac{1}{S_n}} \\ &\le \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} c_k^{1/k}\right)^{-\frac{1}{S_n}}\frac{1}{S_n}\sum_{k=1}^{n}\frac{c_k a_k}{k} \\ &\le \sum_{n=1}^{\infty}\frac{1}{(n+1)S_{n+1}S_n}\sum_{k=1}^{n}\frac{c_k a_k}{k} \\ &= \sum_{k=1}^{\infty}\frac{c_k a_k}{k}\sum_{n=k}^{\infty}\frac{1}{(n+1)S_n S_{n+1}} \\ &= \sum_{k=1}^{\infty}\frac{c_k a_k}{k}\sum_{n=k}^{\infty}\left(\frac{1}{S_n}-\frac{1}{S_{n+1}}\right) \\ &= \sum_{k=1}^{\infty}\frac{c_k a_k}{kS_k} = \sum_{k=1}^{\infty}\left[\frac{(k+1)S_{k+1}}{kS_k}\right]^{kS_k} a_k = \sum_{n=1}^{\infty} B_n a_n.\end{aligned} \tag{3.3}$$

Substituting (2.7) into (3.3) leads to (1.7). The proof of Theorem 1.1 is complete.

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