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A NUMERICAL METHOD IN TERMS OF THE THIRD DERIVATIVE FOR A DELAY INTEGRAL EQUATION FROM BIOMATHEMATICS

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Abstract

Let F be a Schatten p-operator and R, S positive operators. We show that the inequality $|F(R+S)^{\frac{1}{c}}|_p^c \le |FR^{\frac{1}{c}}|_p^c + |FS^{\frac{1}{c}}|_p^c$ for the Schatten p-norm $|\cdot|_p$ is true for $p \ge c = 1$ and for $p \ge c = 2$, conjecture it to be true for $p \ge c \in [1, 2]$, give counterexamples for the other cases, and present a numerical study for 2×2 matrices. Furthermore, we have a look at a generalisation of the inequality which involves an additional factor $\sigma(c, p)$.

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1. Introduction

Let H and K be complex Hilbert spaces and $0 < p \leq \infty$. Following [1], we denote by $c_p(H,K)$ the space of Schatten p-operators $T : H \longrightarrow K$, equipped with the Schatten p-norm or quasi-norm $|\cdot|_p$. Note that [1] deals only with the spaces $c_p(H) := c_p(H,H)$. The generalisations $c_p(H,K)$ can be found in text-books like [2] and [3] (there written as $B_p(H,K)$ and $S_p(H,K)$ respectively).

By $L(H)$ we denote the space of bounded linear operators on H, and by $L(H)_+$ the subset of positive operators. With $|T| := (T^*T)^{1/2} \in L(H)_+$ for $T \in L(H,K)$ we have for $p < \infty$

$$\begin{aligned} |T|_p^p &= \operatorname{tr} |T|^p \quad \text{for } T \in c_p(H,K) \text{, and consequently} \\ |T|_p^p &= \operatorname{tr} T^p \quad \text{for } T \in c_p(H)_+ := c_p(H) \cap L(H)_+. \end{aligned}$$

Applying $|T|_p = |T^*|_p$ for $T \in c_p(H,K)$, this shows in case of $p < \infty$

$$\left|FU^{\frac{1}{2}}\right|_p^2 = \left|U^{\frac{1}{2}}F^*\right|_p^2 = \left(\operatorname{tr}\,(FUF^*)^{\frac{p}{2}}\right)^{\frac{2}{p}} = |FUF^*|_{\frac{p}{2}}$$

for $F \in c_p(H,K)$ and $U \in L(H)_+$. Because $|\cdot|_\infty$ is the usual operator norm,

$$\left|FU^{\frac{1}{2}}\right|_p^2 = |FUF^*|_{\frac{p}{2}}$$

is also true for $p = \infty$, with the common convention $\frac{\infty}{2} := \infty$.

Our question, which arose while studying the integration of Schatten operator valued functions in [4], is: For what values of $p \in (0,\infty]$ and $c \in (0,\infty)$ is the Minkowski-like inequality

$$\text{(MS)} \qquad \left| F\,(R+S)^{\frac{1}{c}} \right|_p^c \leq \left| F R^{\frac{1}{c}} \right|_p^c + \left| F S^{\frac{1}{c}} \right|_p^c$$

true for all $F \in c_p(H, K)$ and $R, S \in L(H)_+$?

2. The Conjecture

Let H, K, p, c, F, R, S be as above.

Theorem 2.1. *Inequality (MS) is true for $p \geq c = 1$ and for $p \geq c = 2$.*

Proof. For $p \geq c = 1$, the triangle inequality for $|\cdot|_p$ shows

$$|F(R+S)|_p = |FR + FS|_p \leq |FR|_p + |FS|_p.$$

For $p \geq c = 2$, the triangle inequality for $|\cdot|_{\frac{p}{2}}$ shows

$$\left|F(R+S)^{\frac{1}{2}}\right|_p^2 = |F(R+S)F^*|_{\frac{p}{2}} \leq |FRF^*|_{\frac{p}{2}} + |FSF^*|_{\frac{p}{2}} = \left|FR^{\frac{1}{2}}\right|_p^2 + \left|FS^{\frac{1}{2}}\right|_p^2.$$

□

Theorem 2.1 suggests the following conjecture.

Conjecture 1. *Inequality (MS) is true for $p \geq c \in [1, 2]$.*

For $c \in (1, 2)$ we have at the present time no proof of this conjecture for other than trivial situations, not even for the special case of 2×2 matrices. However, some justification will be given in Section 4.

3. The Case $p < c$ and the Case $c \notin [1, 2]$

In this section we will demonstrate, by providing counterexamples, that inequality (MS) is not necessarily true for other values of (c, p) than those stated in Conjecture 1. We will offer one example for $0 < p < c < \infty$, and one for arbitrary p when $c < 1$ or $c > 2$, both examples using 2×2 matrices. The power U^t for $t > 0$ of a non-negative matrix U can be calculated easily with help of the spectral decomposition of U.

Example 3.1. *Inequality (MS) is violated for* $0 < p < c < \infty$ *by*

$$F := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. From $U^t = U$ for $U \in \{R, S, R+S\}$ and $t \in (0, \infty)$ we get

$$\left| F U^{\frac{1}{c}} \right|_p = |U|_p = (\operatorname{tr} U^p)^{\frac{1}{p}} = (\operatorname{tr} U)^{\frac{1}{p}},$$

yielding

$$\left| F R^{\frac{1}{c}} \right|_p = 1, \quad \left| F S^{\frac{1}{c}} \right|_p = 1, \quad \left| F (R+S)^{\frac{1}{c}} \right|_p = 2^{\frac{1}{p}},$$

and using $p < c$,

$$\left| F R^{\frac{1}{c}} \right|_p^c + \left| F S^{\frac{1}{c}} \right|_p^c = 2 < 2^{\frac{c}{p}} = \left| F (R+S)^{\frac{1}{c}} \right|_p^c.$$

□

The second example makes use of an inequality which is interesting in its own right. Seeming simple, it is surprisingly fiddly to prove:

Lemma 3.1. *For* $x \in (0,1) \cup (2,\infty)$ *we have*

$$\left(1+\frac{1}{\sqrt{5}}\right)\left(\frac{3+\sqrt{5}}{2}\right)^x + \left(1-\frac{1}{\sqrt{5}}\right)\left(\frac{3-\sqrt{5}}{2}\right)^x < 1+3^x.$$

Proof. Setting $r := \sqrt{5}$, $\alpha_1 := 1+\frac{1}{r}$, $\alpha_2 := 1-\frac{1}{r}$, and $\omega := \frac{3+r}{2}$, we have to show

$$\alpha_1\,\omega^x + \alpha_2\,\omega^{-x} < 1+3^x.$$

The case $x \in (2,\infty)$: Set $f(x) := \alpha_1\,\omega^x$, $g(x) := \alpha_2\,\omega^{-x}$, $h(x) := 1+3^x$ for $x \in (0,\infty)$. Because $\alpha_2 > 0$ and $\omega > 1$, g is strictly decreasing, thus $f(x)+g(x) < f(x)+g(2)$ for $x > 2$. We will show $f(x)+g(2) < h(x)$ for $x > 2$. Because $f(2)+g(2) = h(2)$, this is done if we prove $f'(x) < h'(x)$ for $x > 2$, which is equivalent to $\alpha_1\,(\frac{\omega}{3})^x\,\ln\omega < \ln 3$. This inequality is true for $x = 2$. All factors of its left side are positive, and $\omega < 3$, so the left side is strictly decreasing for $x \geq 2$. Hence the inequality is true for $x > 2$ as well.

The case $x \in (0,1)$: After substituting $s := \omega^x$ and setting $\delta := \frac{\ln 3}{\ln\omega}$, we have to prove the equivalent inequality

$$s+\frac{1}{s}+\frac{1}{r}\left(s-\frac{1}{s}\right) < 1+s^\delta$$

for $s \in (1,\omega)$, which can be done by building a sandwich with a suitable polynomial function inside: Set

$$\varphi(s) := s + \frac{1}{s} + \frac{1}{r}\left(s - \frac{1}{s}\right), \quad p(s) := 2\left(1 + \frac{s-1}{\omega - 1}\right),$$
$$q(s) := \frac{(s-1)(s-\omega)}{(3-1)(3-\omega)}\left(\varphi(3) - p(3)\right)$$

for $s > 0$. The claim is

$$\varphi(s) < p(s) + q(s) < 1 + s^{\delta}$$

for $s \in (1, \omega)$. The left inequality is verified by the fact that $s \cdot (p(s) + q(s) - \varphi(s))$ defines a polynomial of degree 3 with three zeros $\{1, \omega, 3\}$, where $1 < \omega < 3$, and with positive leading coefficient $\lambda := \frac{1}{2}(\varphi(3) - p(3))/(3 - \omega)$. To prove the second inequality, we inspect

$$\psi(s) := 1 + s^{\delta} - p(s) - q(s)$$

for $s > 0$ and get $\psi''(s) = \delta(\delta - 1)s^{\delta-2} - 2\lambda$. Because $1 < \delta < 2$, ψ'' has a unique zero

$$s_0 := \left(\frac{\delta(\delta-1)}{2\lambda}\right)^{\frac{1}{2-\delta}}, \quad 1 < s_0 < \omega,$$

with $\psi''(s) > 0$ for $s \in (0, s_0)$ and $\psi''(s) < 0$ for $s \in (s_0, \infty)$. Now $\psi(1) = 0$, $\psi'(1) > 0$, and $\psi''(s) > 0$ for $s \in (1, s_0)$ show $\psi(s) > 0$ for $s \in (1, s_0]$, while $\psi(s_0) > 0$, $\psi(\omega) = 0$, $\psi'(\omega) < 0$, and $\psi''(s) < 0$ for $s \in (s_0, \omega)$ show $\psi(s) > 0$ for $s \in [s_0, \omega)$. □

Example 3.2. *Inequality (MS) is violated for $0 < p \le \infty$ and $c < 1$ as well as $c > 2$ by*

$$F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad R := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Proof. Evaluation of the matrix powers for $t \in (0, \infty)$ gives

$$R^t = R, \; S^t = \begin{pmatrix} \frac{1}{2}(\alpha_1\,\omega^t + \alpha_2\,\omega^{-t}) & \frac{1}{r}(\omega^{-t} - \omega^t) \\ \frac{1}{r}(\omega^{-t} - \omega^t) & \frac{1}{2}(\alpha_2\,\omega^t + \alpha_1\,\omega^{-t}) \end{pmatrix},$$

$$(R+S)^t = \frac{1}{2}\begin{pmatrix} 1+3^t & 1-3^t \\ 1-3^t & 1+3^t \end{pmatrix}$$

with $r := \sqrt{5}$, $\alpha_1 := 1 + \frac{1}{r}$, $\alpha_2 := 1 - \frac{1}{r}$, $\omega := \frac{3+r}{2}$. For $U \in \{R, S, R+S\}$ we get in case of $p < \infty$

$$|FU^t|_p = \left(\operatorname{tr}\,(FU^{2t}F^*)^{\frac{p}{2}}\right)^{\frac{1}{p}} = \sqrt{u_t}$$

with u_t being the top left entry of U^{2t}. Using $|FU^t|_\infty^2 = |FU^{2t}F^*|_\infty$, the case $p = \infty$ yields the same result, thus for all p:

$$\left|FR^{\frac{1}{c}}\right|_p = 0, \quad \left|FS^{\frac{1}{c}}\right|_p = \sqrt{\frac{1}{2}\,(\alpha_1\,\omega^{2/c} + \alpha_2\,\omega^{-2/c})},$$

$$\left|F(R+S)^{\frac{1}{c}}\right|_p = \sqrt{\frac{1}{2}\,(1 + 3^{2/c})}.$$

Substituting $\frac{2}{c}$ by x, we have to prove $\alpha_1\,\omega^x + \alpha_2\,\omega^{-x} < 1 + 3^x$ for $x \in (2, \infty)$ and for $x \in (0, 1)$, which is the statement of Lemma 3.1. □

4. Some Numerical Evidence

To justify Conjecture 1, we present the results of a numerical study performed with 2×2 matrices.

From functional calculus it is known: For an operator $T \geq 0$ on a complex Hilbert space the powers T^α, T^β for $\alpha, \beta \in (0, \infty)$ obey the rule $T^\alpha T^\beta = T^{\alpha+\beta}$. If T is invertible, then T^α can be defined for $\alpha \leq 0$ as well, and $T^\alpha T^\beta = T^{\alpha+\beta}$ is true for all $\alpha, \beta \in \mathbb{R}$.

Before turning to the matrix case, we note the following general lemma.

Lemma 4.1. *Let H, K, F be as above and $\alpha \in (0, \infty)$.*

(a) Let $T \in L(H)_+$. Then $FT^\alpha = 0$ if and only if $FT = 0$.

(b) Let $R, S \in L(H)_+$. Then $F(R+S)^\alpha = 0$ if and only if $FR^\alpha = 0$ and $FS^\alpha = 0$.

Proof. (a) Suppose $FT^\alpha = 0$. Then $|FT^{\alpha/2}|^2 = |FT^\alpha F^*| = 0$, hence $FT^{\alpha/2} = 0$. Repeated application yields $\beta \in (0,1)$ with $FT^\beta = 0$, thus $FT = FT^\beta T^{1-\beta} = 0$.

Now suppose $FT = 0$. There is nothing to prove in the case of $\alpha = 1$, so assume $\alpha \neq 1$. If T is invertible, then $FT^\alpha = FTT^{\alpha-1} = 0$. If T is not invertible, then we have $0 \in \sigma(T)$, the spectrum of T. Choose polynomials $f_n \in \mathbb{R}[t]$ for $n \in \mathbb{N}$ such that $f_n(x) \to x^\alpha$ for $n \to \infty$ uniformly for $x \in \sigma(T)$. Then $f_n(T) \to T^\alpha$ and $Ff_n(T) \to FT^\alpha$ for $n \to \infty$, hence $Ff_n(T) = f_n(0)F \to 0$ for $n \to \infty$, thus $FT^\alpha = 0$.

(b) Part (a) shows:

$$\begin{aligned} FR^\alpha = 0 \ \wedge \ FS^\alpha = 0 &\iff FR = 0 \ \wedge \ FS = 0 \\ &\implies F(R+S) = 0 \\ &\iff F(R+S)^\alpha = 0. \end{aligned}$$

To prove the missing implication, suppose $F(R+S) = 0$. Then $FRF^* + FSF^* = 0$. Because $FRF^* \geq 0$ and $FSF^* \geq 0$, we get $FRF^* = 0$, thus $|FR^{1/2}|^2 = |FRF^*| = 0$ and $FR^{1/2} = 0$. Applying (a) again gives $FR = 0$. Symmetry shows $FS = 0$. □

We will also use the following well-known property of 2×2 matrices:

Lemma 4.2. *A complex 2×2 matrix M is positive semidefinite if and only if there exist $a, b \in [0, \infty)$ and $\gamma \in \mathbb{C}$ with $|\gamma|^2 \leq ab$ such that*

$$M = \begin{pmatrix} a & \gamma \\ \overline{\gamma} & b \end{pmatrix}.$$

Lemma 4.1(b) shows that, when checking Conjecture 1, one may assume the denominator to be non-zero, or setting $\frac{0}{0} := 0$, in

$$q_{c,p}(F, R, S) := \frac{\left| F\,(R+S)^{\frac{1}{c}} \right|_p^c}{\left| FR^{\frac{1}{c}} \right|_p^c + \left| FS^{\frac{1}{c}} \right|_p^c}.$$

We are searching for the supremum of $q_{c,p}(F, R, S)$ over all complex 2×2 matrices F, R, S with $R, S \geq 0$. For $r \in [0, \infty)$ and $x \in \mathbb{C}$ define $r \wedge x := x$

if $|x| \le r$ and $r \wedge x := (r/|x|)\,x$ otherwise. Lemma 4.2 shows that R has the structure

$$R = \begin{pmatrix} \alpha^2 & |\alpha\,\beta| \wedge \gamma \\ |\alpha\,\beta| \wedge \overline{\gamma} & \beta^2 \end{pmatrix} =: P(\alpha, \beta, \gamma)$$

with $\alpha, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$, and a corresponding representation is valid for the matrix S. This means that we have to deal with six complex and four real variables, resulting in a 16-dimensional real optimisation problem: For $\lambda = (\lambda_1, \ldots, \lambda_{16}) \in \mathbb{R}^{16}$ we set

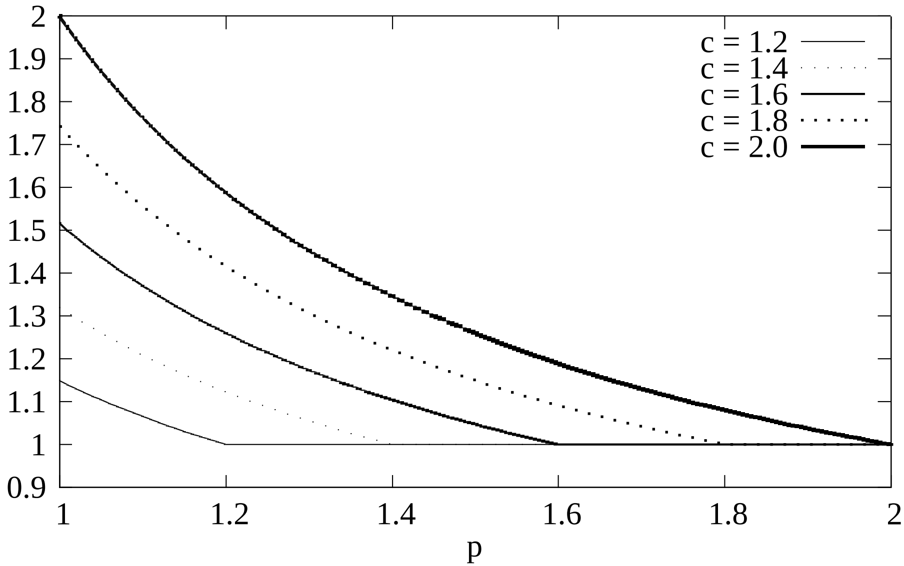
$$\begin{aligned} F_\lambda &:= \begin{pmatrix} \lambda_1 + \lambda_2\, i & \lambda_3 + \lambda_4\, i \\ \lambda_5 + \lambda_6\, i & \lambda_7 + \lambda_8\, i \end{pmatrix}, \\ R_\lambda &:= P(\lambda_9, \lambda_{10}, \lambda_{11} + \lambda_{12}\, i), \\ S_\lambda &:= P(\lambda_{13}, \lambda_{14}, \lambda_{15} + \lambda_{16}\, i) \end{aligned}$$

and are asking for

$$\sigma(c, p) := \sup_{\lambda \in \mathbb{R}^{16}} q_{c,p}(F_\lambda, R_\lambda, S_\lambda).$$

To attack this problem, GNU Octave [5], version 2.1.57, was utilised. It offers a function for determining the singular values of a matrix, which can be employed for calculating the Schatten norms. For the optimisation task the implementation [6], version 2002/05/09, with standard parameters of the Downhill Simplex Method of Nelder and Mead ([7], 10.4) was used. The results are in perfect agreement with Conjecture 1. For visualisation, approximations for $\sigma(c, p)$ for $c \in \{1.2, 1.4, 1.6, 1.8, 2.0\}$ have been calculated and plotted with a step size of 0.01 for p, see Figure 1.

Figure 1: Experimental approximations of $\sigma(c,p)$.



The apparently smooth shape of $p \mapsto \sigma(c,p)$ for $p \le c$, together with the fact that for each p a new random starting point λ was used for the Nelder-Mead algorihm, gives some confidence in the validity of the data.

A closer inspection of some of the calculated numerical values suggests

$$\sigma(2,1) = 2, \quad \sigma\left(\tfrac{3}{2},1\right) = \sigma\left(\tfrac{9}{5},\tfrac{6}{5}\right) = 2^{\frac{1}{2}}, \quad \sigma\left(\tfrac{8}{5},\tfrac{6}{5}\right) = \sigma\left(2,\tfrac{3}{2}\right) = 2^{\frac{1}{3}},$$
$$\sigma\left(\tfrac{5}{4},1\right) = \sigma\left(\tfrac{3}{2},\tfrac{5}{4}\right) = \sigma\left(\tfrac{7}{4},\tfrac{7}{5}\right) = \sigma\left(2,\tfrac{8}{5}\right) = 2^{\frac{1}{4}}, \quad \sigma\left(\tfrac{6}{5},1\right) = \sigma\left(\tfrac{9}{5},\tfrac{3}{2}\right) = 2^{\frac{1}{5}},$$

which leads to the idea to look at $\log_2 \sigma(c,p)$. It seems there is a linear dependency of $\log_2 \sigma(c,p)$ from c if $c \geq p$. This observation will be made precise in the next section.

A Minkowski-Type Inequality for the Schatten Norm

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5. Generalisation of (MS)

It is natural to generalise (MS) and to ask for the smallest $\sigma(c,p) \in [0,\infty]$ for $c \in (0,\infty)$ and $p \in (0,\infty]$ such that

$$\left| F\,(R+S)^{\frac{1}{c}} \right|_p^c \leq \sigma(c,p) \left(\left| F R^{\frac{1}{c}} \right|_p^c + \left| F S^{\frac{1}{c}} \right|_p^c \right)$$

for all $F \in c_p(H,K)$ and $R, S \in L(H)_+$ (and for all complex Hilbert spaces H and K). It is tempting to call $\sigma(c,p)$ the *Schatten-Minkowski constant* for (c,p). By choosing $F \neq 0$ and setting R to be the identity and $S := 0$ it can be seen that $\sigma(c,p) \geq 1$. Now Conjecture 1 can be re-phrased using $\sigma(c,p)$, and, motivated by the numerical results, we add another conjecture:

Conjecture 2. *(a) For* $1 \leq c \leq 2$ *and* $p \geq c$ *we have* $\sigma(c,p) = 1$.
(b) For $0 \leq c \leq 2$ *and* $p \leq c$ *we have* $\sigma(c,p) = 2^{\frac{c}{p}-1}$.

Again, the cases $c = 1$ and $c = 2$ are not too difficult to prove:

Theorem 5.1.

(a) $\sigma(1,p) = \begin{cases} 1 & \text{for } p \geq 1 \\ 2^{\frac{1}{p}-1} & \text{for } p \leq 1 \end{cases}$

(b) $\sigma(2,p) = \begin{cases} 1 & \text{for } p \geq 2 \\ 2^{\frac{2}{p}-1} & \text{for } p \leq 2 \end{cases}.$

Proof. $\sigma(1,p) \leq 1$ for $p \geq 1$ and $\sigma(2,p) \leq 1$ for $p \geq 2$ is the subject of Theorem 2.1, while $\sigma(c,p) \geq 1$ is noted above. Example 3.1 tells us that

$\sigma(c,p) \geq 2^{c/p-1}$ for $0 < p \leq c < \infty$, yielding

$$\sigma(1,p) \geq 2^{\frac{1}{p}-1} \text{ for } p \leq 1 \qquad \text{and} \qquad \sigma(2,p) \geq 2^{\frac{2}{p}-1} \text{ for } p \leq 2.$$

Now for the missing '$\leq$' inequalities. For the case $c = 1$, recall the inequality between the power means of degrees $p \leq 1$ and 1, see e.g. [8], 8.12, which reads

$$\left(\frac{\alpha^p + \beta^p}{2}\right)^{\frac{1}{p}} \leq \frac{\alpha + \beta}{2} \quad \text{or equivalently} \quad \alpha^p + \beta^p \leq 2^{1-p}(\alpha+\beta)^p$$

for $\alpha, \beta \in [0,\infty)$. Together with the quasi-norm inequality of $|\cdot|_p$ this gives

$$|F(R+S)|_p^p \leq |FR|_p^p + |FS|_p^p \leq 2^{1-p}\left(|FR|_p + |FS|_p\right)^p$$

and thus $|F(R+S)|_p \leq 2^{\frac{1}{p}-1}\left(|FR|_p + |FS|_p\right)$.

For the case $c = 2$, start with the power means inequality for the degrees $p \leq 2$ and 2,

$$\left(\frac{\alpha^p + \beta^p}{2}\right)^{\frac{1}{p}} \leq \left(\frac{\alpha^2 + \beta^2}{2}\right)^{\frac{1}{2}} \quad \text{or equivalently} \quad \alpha^p + \beta^p \leq 2^{1-\frac{p}{2}}\left(\alpha^2+\beta^2\right)^{\frac{p}{2}}$$

for $\alpha, \beta \in [0,\infty)$. Together with the quasi-norm inequality of $|\cdot|_{\frac{p}{2}}$ this gives

$$\begin{aligned}\left|F(R+S)^{\frac{1}{2}}\right|_p^p &= |F(R+S)F^*|_{\frac{p}{2}}^{\frac{p}{2}} \\ &\leq |FRF^*|_{\frac{p}{2}}^{\frac{p}{2}} + |FSF^*|_{\frac{p}{2}}^{\frac{p}{2}} \\ &= \left|FR^{\frac{1}{2}}\right|_p^p + \left|FS^{\frac{1}{2}}\right|_p^p \leq 2^{1-\frac{p}{2}}\left(\left|FR^{\frac{1}{2}}\right|_p^2 + \left|FS^{\frac{1}{2}}\right|_p^2\right)^{\frac{p}{2}}\end{aligned}$$

and consequently

$$\left|F(R+S)^{\frac{1}{2}}\right|_p^2 \le 2^{\frac{2}{p}-1}\left(\left|FR^{\frac{1}{2}}\right|_p^2 + \left|FS^{\frac{1}{2}}\right|_p^2\right).$$

□

6. Conclusion

Starting with Conjecture 1, which we proved for the cases $c = 1$ and $c = 2$ in Theorem 2.1, a numerical study of 2×2 matrices led to the generalised Conjecture 2, which we also proved for $c = 1$ and $c = 2$ in Theorem 5.1.

The given proofs make use of the (quasi-) triangle inequality of the Schatten (quasi-) norm. Another ingredient to Theorem 5.1 is the power means inequality. Presumably, a combination of these inequalities shall also be central when dealing with the case $c \neq 1, 2$. However, it is unclear how to apply the triangle inequality in this situation, because there is no obvious way to get from $F(R+S)^{1/c}$ to an expression where R and S can be separated.

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