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NOTE ON AN INTEGRAL INEQUALITY APPLICABLE IN PDEs

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ABSTRACT. The article presents and refines the results which were proven in [1]. We give a condition for obtaining the optimal constant of the integral inequality for the numerical analysis of a nonlinear system of PDEs.

Key words and phrases: Integral inequality, Nonlinear system of PDEs.

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1. INTRODUCTION

In [1] the following problem is considered and its application to nonlinear system of PDEs is described.

Theorem A. *Let* $a, b \in \mathbb{R}$, $a < 0$, $b > 0$ *and* $f \in C[a, b]$, *such that:* $0 < f \leq 1$ *on* $[a, b]$, f *is decreasing on* $[a, 0]$ *and*

$$\int_a^0 f dx = \int_0^b f dx$$

then

(a) *If* $p \geq 2$, *the inequality*

$$\int_a^b f^p dx \leq A_p \int_a^{\frac{a+b}{2}} f dx \tag{1.1}$$

holds for all $A_p \geq 2$.

(b) *If* $1 \leq p < 2$, *the inequality*

$$\int_a^b f^p dx \leq A_p \int_a^{\frac{a+b}{2}} f dx \tag{1.2}$$

holds for all $A_p \geq 4$.

In this note we improve the optimal A_p for the case $1 < p < 2$.

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2. Results

Theorem 2.1. *Let $a, b \in \mathbb{R}, a < 0, b > 0$ and $f \in C[a,b]$, such that $0 < f \leq 1$ on $[a,b]$, f is decreasing on $[a,0]$ and*

$$\int_a^0 f dx = \int_0^b f dx.$$

(i) *If $a+b \geq 0$, then for $1 \leq p$, this inequality holds*

$$\int_a^b f^p dx \leq 2 \int_a^{\frac{a+b}{2}} f dx. \tag{2.1}$$

(ii) *If $a+b < 0$ then*

(a) *If $p \geq 2$, the inequality*

$$\int_a^b f^p dx \leq A_p \int_a^{\frac{a+b}{2}} f dx \tag{2.2}$$

holds for all $A_p \geq 2$.

(b) *If $1 < p < 2$, the inequality*

$$\int_a^b f^p dx \leq A_p \int_a^{\frac{a+b}{2}} f dx \tag{2.3}$$

holds for all $A_p \geq 2\frac{1+x_{\max}^{p-1}}{1+x_{\max}}$, where $0 < x_{\max} \leq 1$ is the solution of

$$x^{p-1}(p-2) + x^{p-2}(p-1) - 1 = 0. \tag{2.4}$$

(c) *For $p = 1$ the inequality*

$$\int_a^b f dx \leq 4 \int_a^{\frac{a+b}{2}} f dx \tag{2.5}$$

holds.

Proof. As in the proof in [1], we consider two cases: (i) $a+b \geq 0$ and (ii) $a+b < 0$.
(i) First, we suppose that $a+b \geq 0$. Using the properties of the function f, we conclude, for $p \geq 1$, that:

$$\int_a^b f^p dx \leq \int_a^b f dx = 2\int_a^0 f dx \leq 2 \int_a^{\frac{a+b}{2}} f dx.$$

The constant $A_p = 2$ is the best possible. To prove sharpness, we choose $f = 1$.
(ii) Now we suppose that $a+b < 0$. Let $\varphi : [a,0] \to [0,b]$ be a function with the property

$$\int_x^0 f dt = \int_0^{\varphi(x)} f dt.$$

So, $\varphi(x)$ is differentiable and $\varphi(a) = b, \varphi(0) = 0$.

For arbitrary $x \in [a,0]$, such that $x + \varphi(x) \geq 0$, according to case (i) for $p \geq 1$, we obtain the inequality

$$\int_x^{\varphi(x)} f^p dt \leq 2 \int_x^{\frac{x+\varphi(x)}{2}} f dt.$$

In particular, for $x = a$,

$$\int_a^b f^p dt \leq 2 \int_a^{\frac{a+b}{2}} f dt.$$

If we suppose that $x + \varphi(x) < 0$ for arbitrary $x \in [a,0]$, then we define a new function

$\psi : [a, 0] \to \mathbb{R}$ by

$$\psi(x) = A_p \int_x^{\frac{x+\varphi(x)}{2}} f dt - \int_x^{\varphi(x)} f^p dt.$$

The function ψ is differentiable and

$$\psi'(x) = \frac{1}{2} A_p (1 + \varphi'(x)) f\left(\frac{x+\varphi(x)}{2}\right) - A_p f(x) - f^p(\varphi(x))\varphi'(x) + f^p(x)$$

and $\psi(0) = 0$.

If we prove that $\psi'(x) \le 0$ then the inequality

$$\int_x^{\varphi(x)} f^p dt \le A_p \int_x^{\frac{x+\varphi(x)}{2}} f dt$$

holds.

Using the properties of the functions f, φ and the fact that $f(\varphi(x))\varphi'(x) = -f(x)$, we consider $f(\varphi(x))\psi'(x)$ and try to conclude that $f(\varphi(x))\psi'(x) \le 0$ as follows:

$$\begin{aligned}
&f(\varphi(x))\psi'(x)\\
&= f(\varphi(x)) \left[\frac{1}{2} A_p (1 + \varphi'(x)) f\left(\frac{x+\varphi(x)}{2}\right) - A_p f(x) - f^p(\varphi(x))\varphi'(x) + f^p(x)\right]\\
&= \frac{1}{2} A_p f(\varphi(x)) f\left(\frac{x+\varphi(x)}{2}\right) + \frac{1}{2} A_p f(\varphi(x))\varphi'(x) f\left(\frac{x+\varphi(x)}{2}\right) - A_p f(x) f(\varphi(x))\\
&\qquad - f^p(\varphi(x))\varphi'(x) f(\varphi(x)) + f^p(x) f(\varphi(x))\\
&= \frac{1}{2} A_p f(\varphi(x)) f\left(\frac{x+\varphi(x)}{2}\right) - \frac{1}{2} A_p f(x) f\left(\frac{x+\varphi(x)}{2}\right) - A_p f(x) f(\varphi(x))\\
&\qquad + f^p(\varphi(x)) f(x) + f^p(x) f(\varphi(x))\\
&= \frac{1}{2} A_p [f(\varphi(x)) - f(x)] f\left(\frac{x+\varphi(x)}{2}\right) - A_p f(x) f(\varphi(x))\\
&\qquad + f^p(\varphi(x)) f(x) + f^p(x) f(\varphi(x)).
\end{aligned}$$

For $p \ge 1$, if $[f(\varphi(x)) - f(x)] \le 0$, then

$$\begin{aligned}
&f(\varphi(x))\psi'(x)\\
&\le \frac{1}{2} A_p [f(\varphi(x)) - f(x)] f\left(\frac{x+\varphi(x)}{2}\right) - A_p f(x) f(\varphi(x))\\
&\qquad + [f(\varphi(x)) f(x) + f(x) f(\varphi(x))]\\
&= \frac{1}{2} A_p [f(\varphi(x)) - f(x)] f\left(\frac{x+\varphi(x)}{2}\right) - (A_p - 2) f(x) f(\varphi(x)).
\end{aligned}$$

Then, obviously, $\psi'(x) \le 0$ for $A_p - 2 \ge 0$.

If we suppose that $[f(\varphi(x)) - f(x)] > 0$ then using the properties of φ, we can conclude that $f\left(\frac{x+\varphi(x)}{2}\right) \le f(x)$ and we estimate $f(\varphi(x))\psi'(x)$ as follows:

$$\begin{aligned}
&f(\varphi(x))\psi'(x)\\
&\le \frac{1}{2} A_p [f(\varphi(x)) - f(x)] f(x) - A_p f(x) f(\varphi(x)) + f^p(\varphi(x)) f(x) + f^p(x) f(\varphi(x))\\
&\le \frac{1}{2} A_p [f(\varphi(x)) - f(x)] f(x) - A_p f(x) f(\varphi(x)) + f(\varphi(x)) f(x) + f(x) f(\varphi(x))
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}A_p f^2(x) + \left(2 - \frac{1}{2}A_p\right) f(\varphi(x))f(x) \\
&\le -\frac{1}{2}A_p f^2(x) + \left(2 - \frac{1}{2}A_p\right) f^2(\varphi(x)) \\
&\le -\frac{1}{2}(A_p - 4) f^2(\varphi(x)).
\end{aligned}
$$

So, $\psi'(x) \le 0$ for $A_p - 4 \ge 0$.

Now, we will consider the sign of $f(\varphi(x))\psi'(x)$ for $p = 1$, $p \ge 2$, and $1 < p < 2$.

(a) For $p \ge 2$, we try to improve the constant $A_p \ge 4$ for the case $a + b < 0$ and $[f(\varphi(x)) - f(x)] > 0$. We can estimate $f(\varphi(x))\psi'(x)$ as follows:

$$
\begin{aligned}
&f(\varphi(x))\psi'(x) \\
&\le \frac{1}{2}A_p[f(\varphi(x)) - f(x)]f(x) - A_p f(x)f(\varphi(x)) + f^p(\varphi(x))f(x) + f^p(x)f(\varphi(x)) \\
&\le \frac{1}{2}A_p[f(\varphi(x)) - f(x)]f(x) - A_p f(x)f(\varphi(x)) + f^2(\varphi(x))f(x) + f^2(x)f(\varphi(x)) \\
&\le \frac{1}{2}f(x)[f(x) + f(\varphi(x))][2f(\varphi(x)) - A_p].
\end{aligned}
$$

Hence, $\psi'(x) \le 0$ for $A_p \ge 2$.

(b) For $1 < p < 2$, we can improve the constant $A_p \ge 4$ for the case $a + b < 0$ and $[f(\varphi(x)) - f(x)] > 0$. We can estimate $f(\varphi(x))\psi'(x)$ (for $0 < f(x) = y < f(\varphi(x)) = z \le 1$), as follows:

$$
\begin{aligned}
&f(\varphi(x))\psi'(x) \\
&\le \frac{1}{2}A_p[f(\varphi(x)) - f(x)]f(x) - A_p f(x)f(\varphi(x)) + f^p(\varphi(x))f(x) + f^p(x)f(\varphi(x)) \\
&\le y\left[-\frac{1}{2}A_p z - \frac{1}{2}A_p y + z^p + y^{p-1}z\right] \\
&= y\left[-\frac{1}{2}A_p z\left(1 + \frac{y}{z}\right) + z^p\left(1 + \left(\frac{y}{z}\right)^{p-1}\right)\right] \\
&\le yz\left[-\frac{1}{2}A_p\left(1 + \frac{y}{z}\right) + 1 + \left(\frac{y}{z}\right)^{p-1}\right].
\end{aligned}
$$

So, we conclude that $\psi'(x) \le 0$ if

$$
\left[-\frac{1}{2}A_p(1 + t) + 1 + t^{p-1})\right] < 0,
$$

for $0 < t = \frac{y}{z} \le 1$.

Therefore, for $1 < p < 2$ the constant $A_p \ge 2\max_{0<t\le 1} \frac{1+t^{p-1}}{1+t}$.

The function $\frac{1+t^{p-1}}{1+t}$ is concave on $(0, 1]$ and the point $t_{\max}$ where the maximum is achieved is a root of the equation

$$
t^{p-1}(p - 2) + t^{p-2}(p - 1) - 1 = 0.
$$

Numerically we get the following values of A_p :

for $p = 1.01$,	the constant $A_p \ge 3.8774$,
for $p = 1.99$,	the constant $A_p \ge 2.0056$,
for $p = 1.9999$,	the constant $A_p \ge 2.0001$.

If we consider the sequence $p_n = 2 - \frac{1}{n}$, then the $\lim_{n\to\infty} \frac{1+t^{p_n-1}}{1+t} = 1$, but we find that the point $t_{\max}$ where the function $\frac{1+t^{p_n-1}}{1+t}$ achieves the maximum is a fixed point of the function $g(x) = (1 - \frac{1+x}{n})^n$.

We use fixed point iteration to find the fixed point for the function $g(x) = (1 - \frac{1+x}{100})^{100}$, by starting with $t_0 = 0.2$ and iterating $t_k = g(t_{k-1})$, $k = 1, 2, ...7$:

$$
\begin{aligned}
t_0 &= 0.200000000000000,\\
t_1 &= 0.299016021496423,\\
t_2 &= 0.270488141422931,\\
t_3 &= 0.278419068898826,\\
t_4 &= 0.276191402436672,\\
t_5 &= 0.276815328895026,\\
t_6 &= 0.276640438571483,\\
t_7 &= 0.276689450339917.
\end{aligned}
$$

When $n \to \infty$, i.e. $p_n \to 2$, the point $t_{\max}$ where the function $\frac{1+t^{p_n-1}}{1+t}$ achieves the maximum is a fixed point of the function $g(x) = e^{-(1+x)}$.

We use fixed point iteration to find the fixed point for the function $g(x) = e^{-(1+x)}$, by starting with $t_0 = 0.2$ and iterating $t_k = g(t_{k-1}), k = 1, 2, ...7$:

$$
\begin{aligned}
t_0 &= 0.200000000000000,\\
t_1 &= 0.301194211912202,\\
t_2 &= 0.272206526577512,\\
t_3 &= 0.280212642489384,\\
t_4 &= 0.277978184195021,\\
t_5 &= 0.278600009316777,\\
t_6 &= 0.278426822683543,\\
t_7 &= 0.278475046663319
\end{aligned}
$$

If we consider the sequence $p_n = 1 + \frac{1}{n}$ then $\lim_{n\to\infty} \frac{1+t^{p_n-1}}{1+t} = \frac{2}{1+t}$, and $\sup_{t\in(0,1]} \frac{2}{1+t} = 2$ for $t \to 0+$.

(c) For $p = 1$,

- if $[f(\varphi(x)) - f(x)] \le 0$ then $\psi'(x) \le 0$ for $A_1 - 2 \ge 0$;
- if $[f(\varphi(x)) - f(x)] > 0$ then $\psi'(x) \le 0$ for $A_1 - 4 \ge 0$,

so, the best constant is $A_1 = 4$. □

REFERENCES

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