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## DIFFERENCE OF GENERAL INTEGRAL MEANS

GEORGE A. ANASTASSIOU

Department of Mathematical Sciences  
The University of Memphis  
Memphis, TN 38152, U.S.A.

*E*Mail: [ganastss@memphis.edu](mailto:ganastss@memphis.edu)

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Abstract

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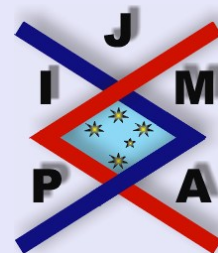


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## Abstract

In this paper we present sharp estimates for the difference of general integral means with respect to even different finite measures. This is achieved by the use of the Ostrowski and Fink inequalities and the Geometric Moment Theory Method. The produced inequalities are with respect to the supnorm of a derivative of the involved function.

*2000 Mathematics Subject Classification:* 26D10, 26D15, 28A25, 60A10, 60E15.

*Key words:* Inequalities, Averages of functions, General averages or means, Moments.

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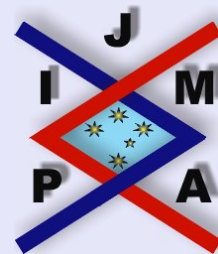
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# 1. Introduction

Here our work is motivated by the works of J. Duoandikoetxea [5] and P. Cerone [4]. We use Ostrowski's ([8]) and Fink's ([6]) inequalities along with the Geometric Moment Theory Method, see [7], [1], [3], to prove our results.

We compare general averages of functions with respect to various finite measures over different subintervals of a domain, even disjoint. Our estimates are sharp and the inequalities are attained. They are with respect to the supnorm of a derivative of the involved function  $f$ .

To the best of our knowledge this type of work is totally new.



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## 2. Results

### Part A

As motivation we give the following proposition.

**Proposition 2.1.** *Let  $\mu_1, \mu_2$  be finite Borel measures on  $[a, b] \subseteq \mathbb{R}$ ,  $[c, d]$ ,  $[\tilde{e}, g] \subseteq [a, b]$ ,  $f \in C^1([a, b])$ . Denote  $\mu_1([c, d]) = m_1 > 0$ ,  $\mu_2([\tilde{e}, g]) = m_2 > 0$ . Then*

$$(2.1) \quad \left| \frac{1}{m_1} \int_c^d f(x) d\mu_1 - \frac{1}{m_2} \int_{\tilde{e}}^g f(x) d\mu_2 \right| \leq \|f'\|_\infty (b - a).$$

*Proof.* From the mean value theorem we have

$$|f(x) - f(y)| \leq \|f'\|_\infty (b - a) =: \gamma, \quad \forall x, y \in [a, b],$$

that is,

$$-\gamma \leq f(x) - f(y) \leq \gamma, \quad \forall x, y \in [a, b],$$

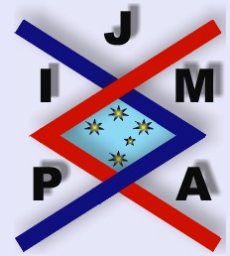
and by fixing  $y$  we get

$$-\gamma \leq \frac{1}{m_1} \int_c^d f(x) d\mu_1 - f(y) \leq \gamma.$$

The last statement holds  $\forall y \in [\tilde{e}, g]$ . Hence

$$-\gamma \leq \frac{1}{m_1} \int_c^d f(x) d\mu_1 - \frac{1}{m_2} \int_{\tilde{e}}^g f(x) d\mu_2 \leq \gamma,$$

proving the claim. □



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As a related result we have

**Corollary 2.2.** Let  $f \in C^1([a, b])$ ,  $[c, d], [\tilde{c}, g] \subseteq [a, b] \subseteq \mathbb{R}$ . Then we have

$$(2.2) \quad \left| \frac{1}{d-c} \int_c^d f(x) dx - \frac{1}{g-\tilde{c}} \int_{\tilde{c}}^g f(x) dx \right| \leq \|f'\|_\infty \cdot (b-a).$$

We use the following famous Ostrowski inequality, see [8], [2].

**Theorem 2.3.** Let  $f \in C^1([a, b])$ ,  $x \in [a, b]$ . Then

$$(2.3) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{\|f'\|_\infty}{2(b-a)} ((x-a)^2 + (x-b)^2),$$

and inequality (2.3) is sharp, see [2].

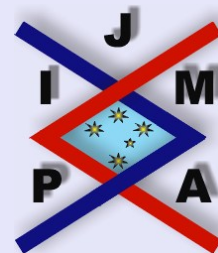
We also have

**Corollary 2.4.** Let  $f \in C^1([a, b])$ ,  $x \in [c, d] \subseteq [a, b] \subseteq \mathbb{R}$ . Then

$$(2.4) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{\|f'\|_\infty}{2(b-a)} \max \{((c-a)^2 + (c-b)^2), ((d-a)^2 + (d-b)^2)\}.$$

*Proof.* Obvious. □

We denote by  $\mathcal{P}([a, b])$  the power set of  $[a, b]$ . We give the following.



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**Theorem 2.5.** Let  $f \in C^1([a, b])$ ,  $\mu$  be a finite measure on  $([c, d], \mathcal{P}([c, d]))$ , where  $[c, d] \subseteq [a, b] \subseteq \mathbb{R}$  and  $m := \mu([c, d]) > 0$ . Then

1.

$$(2.5) \quad \left| \frac{1}{m} \int_{[c,d]} f(x) d\mu - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{\|f'\|_\infty}{2(b-a)} \max\{((c-a)^2 + (c-b)^2), ((d-a)^2 + (d-b)^2)\}.$$

2. Inequality (2.5) is attained when  $d = b$ .

*Proof.* 1) By (2.4) integrating against  $\mu/m$ .

2) Here (2.5) collapses to

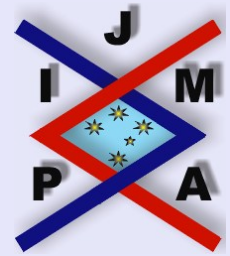
$$(2.6) \quad \left| \frac{1}{m} \int_{[c,b]} f(x) d\mu - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{\|f'\|_\infty}{2} (b-a).$$

We prove that (2.6) is attained. Take

$$f^*(x) = \frac{2x - (a+b)}{b-a}, \quad a \leq x \leq b.$$

Then  $f^{*'}(x) = \frac{2}{b-a}$  and  $\|f^{*'}\|_\infty = \frac{2}{b-a}$ , along with

$$\int_a^b f^*(x) dx = 0.$$



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Therefore (2.6) becomes

$$(2.7) \quad \left| \frac{1}{m} \int_{[c,b]} f^*(x) d\mu \right| \leq 1.$$

Finally pick  $\frac{\mu}{m} = \delta_{\{b\}}$  the Dirac measure supported at  $\{b\}$ , then (2.7) turns to equality.  $\square$

We further have

**Corollary 2.6.** *Let  $f \in C^1([a, b])$  and  $[c, d] \subseteq [a, b] \subseteq \mathbb{R}$ . Let  $M(c, d) := \{\mu : \mu \text{ a measure on } ([c, d], \mathcal{P}([c, d])) \text{ of finite positive mass}\}$ , denoted  $m := \mu([c, d])$ . Then*

1. *The following result holds*

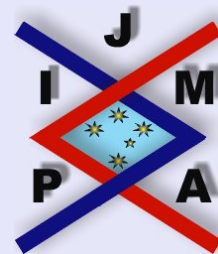
$$(2.8) \quad \sup_{\mu \in M(c,d)} \left| \frac{1}{m} \int_{[c,d]} f(x) d\mu - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{\|f'\|_\infty}{2(b-a)} \max\{((c-a)^2 + (c-b)^2), ((d-a)^2 + (d-b)^2)\}$$

$$= \frac{\|f'\|_\infty}{2(b-a)} \times \begin{cases} (d-a)^2 + (d-b)^2, & \text{if } d+c \geq a+b \\ (c-a)^2 + (c-b)^2, & \text{if } d+c \leq a+b \end{cases}$$

$$(2.9) \quad \leq \frac{\|f'\|_\infty}{2} (b-a).$$

*Inequality (2.9) becomes equality if  $d = b$  or  $c = a$  or both.*



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2. The following result holds

$$(2.10) \quad \sup_{\substack{\text{all } c,d \\ a \leq c < d \leq b}} \left( \sup_{\mu \in M(c,d)} \left| \frac{1}{m} \int_{[c,d]} f(x) d\mu - \frac{1}{b-a} \int_a^b f(x) dx \right| \right) \leq \frac{\|f'\|_\infty}{2} (b-a).$$

Next we restrict ourselves to a subclass of  $M(c, d)$  of finite measures  $\mu$  with given first moment and by the use of the Geometric Moment Theory Method, see [7], [1], [3], we produce an inequality sharper than (2.8). For that we need

**Lemma 2.7.** *Let  $\nu$  be a probability measure on  $([a, b], \mathcal{P}([a, b]))$  such that*

$$(2.11) \quad \int_{[a,b]} x d\nu = d_1 \in [a, b]$$

is given. Then

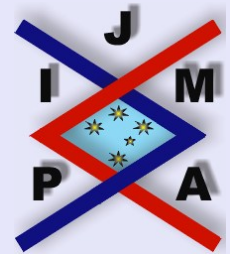
i)

$$(2.12) \quad U_1 := \sup_{\nu \text{ as in (2.11)}} \int_{[a,b]} (x-a)^2 d\nu = (b-a)(d_1-a),$$

and

ii)

$$(2.13) \quad U_2 := \sup_{\nu \text{ as in (2.11)}} \int_{[a,b]} (x-b)^2 d\nu = (b-a)(b-d_1).$$



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*Proof.* i) We observe the graph

$$G_1 = \{(x, (x - a)^2) : a \leq x \leq b\},$$

which is a convex arc above the  $x$ -axis. We form the closed convex hull of  $G_1$  and we call it  $\widehat{G}_1$  which has as an upper concave envelope the line segment  $\ell_1$  from  $(a, 0)$  to  $(b, (b - a)^2)$ . We consider the vertical line  $x = d_1$  which cuts  $\ell_1$  at the point  $Q_1$ . Then  $U_1$  is the distance from  $(d_1, 0)$  to  $Q_1$ . By using the equal ratios property of similar triangles related here we get

$$\frac{d_1 - a}{b - a} = \frac{U_1}{(b - a)^2},$$

which proves the claim.

ii) We observe the graph

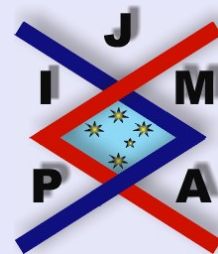
$$G_2 = \{(x, (x - b)^2) : a \leq x \leq b\},$$

which is a convex arc above the  $x$ -axis. We form the closed convex hull of  $G_2$  and we call it  $\widehat{G}_2$  which has as an upper concave envelope the line segment  $\ell_2$  from  $(b, 0)$  to  $(a, (b - a)^2)$ . We consider the vertical line  $x = d_1$  which intersects  $\ell_2$  at the point  $Q_2$ .

Then  $U_2$  is the distance from  $(d_1, 0)$  to  $Q_2$ . By using the equal ratios property of the related similar triangles we obtain

$$\frac{U_2}{(b - a)^2} = \frac{b - d_1}{b - a},$$

which proves the claim. □




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Furthermore we need

**Lemma 2.8.** *Let  $[c, d] \subseteq [a, b] \subseteq \mathbb{R}$  and let  $\nu$  be a probability measure on  $([c, d], \mathcal{P}([c, d]))$  such that*

$$(2.14) \quad \int_{[c,d]} x \, d\nu = d_1 \in [c, d]$$

is given. Then

(i)

$$(2.15) \quad U_1 := \sup_{\nu \text{ as in (2.14)}} \int_{[c,d]} (x - a)^2 \, d\nu = d_1(c + d - 2a) - cd + a^2,$$

and

(ii)

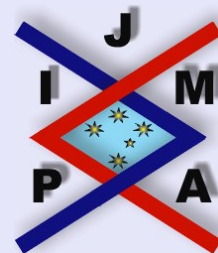
$$(2.16) \quad U_2 := \sup_{\nu \text{ as in (2.14)}} \int_{[c,d]} (x - b)^2 \, d\nu = d_1(c + d - 2b) - cd + b^2.$$

(iii) *The following also holds:*

$$(2.17) \quad \sup_{\nu \text{ as in (2.14)}} \int_{[c,d]} [(x - a)^2 + (x - b)^2] \, d\nu = U_1 + U_2.$$

*Proof.* (i) We see that

$$\int_c^d (x - a)^2 \, d\nu = (c - a)^2 + 2(c - a)(d_1 - c) + \int_c^d (x - c)^2 \, d\nu.$$




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Using (2.12) which is applied on  $[c, d]$ , we find

$$\begin{aligned} \sup_{\nu \text{ as in (2.14)}} \int_c^d (x-a)^2 d\nu &= (c-a)^2 + 2(c-a)(d_1-c) \\ &\quad + \sup_{\nu \text{ as in (2.14)}} \int_c^d (x-c)^2 d\nu \\ &= (c-a)^2 + 2(c-a)(d_1-c) + (d-c)(d_1-c) \\ &= d_1(c+d-2a) - cd + a^2, \end{aligned}$$

proving the claim.

(ii) We see that

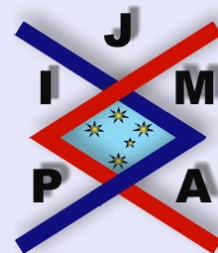
$$\int_c^d (x-b)^2 d\nu = (b-d)^2 + 2(b-d)(d-d_1) + \int_c^d (x-d)^2 d\nu.$$

Using (2.13) which is applied on  $[c, d]$ , we obtain

$$\begin{aligned} \sup_{\nu \text{ as in (2.14)}} \int_c^d (x-b)^2 d\nu &= (b-d)^2 + 2(b-d)(d-d_1) \\ &\quad + \sup_{\nu \text{ as in (2.14)}} \int_c^d (x-d)^2 d\nu \\ &= (b-d)^2 + 2(b-d)(d-d_1) + (d-c)(d-d_1) \\ &= d_1(c+d-2b) - cd + b^2, \end{aligned}$$

proving the claim.

(iii) Similar to Lemma 2.7 and above and obvious on noting that  $(x-a)^2 + (x-b)^2$  is convex, etc.  $\square$



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Now we are ready to present

**Theorem 2.9.** Let  $[c, d] \subseteq [a, b] \subseteq \mathbb{R}$ ,  $f \in C^1([a, b])$ ,  $\mu$  a finite measure on  $([c, d], \mathcal{P}([c, d]))$  of mass  $m := \mu([c, d]) > 0$ . Assume that

$$(2.18) \quad \frac{1}{m} \int_c^d x d\mu = d_1, \quad c \leq d_1 \leq d,$$

is given.

Then

$$(2.19) \quad \sup_{\mu \text{ as above}} \left| \frac{1}{m} \int_c^d f(x) d\mu - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{\|f'\|_\infty}{(b-a)} \left[ d_1((c+d) - (a+b)) - cd + \frac{a^2 + b^2}{2} \right].$$

*Proof.* Denote

$$\beta(x) := \frac{\|f'\|_\infty}{2(b-a)} ((x-a)^2 + (x-b)^2),$$

then by Theorem 2.3 we have

$$-\beta(x) \leq f(x) - \frac{1}{b-a} \int_a^b f(x) dx \leq \beta(x), \quad \forall x \in [c, d].$$

Thus

$$-\frac{1}{m} \int_c^d \beta(x) d\mu \leq \frac{1}{m} \int_c^d f(x) d\mu - \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{1}{m} \int_c^d \beta(x) d\mu,$$




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and

$$\left| \frac{1}{m} \int_c^d f(x) d\mu - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{m} \int_c^d \beta(x) d\mu =: \theta.$$

Here  $\nu := \frac{\mu}{m}$  is a probability measure subject to (2.18) on  $([c, d], \mathcal{P}([c, d]))$  and

$$\begin{aligned} \theta &= \frac{\|f'\|_\infty}{2(b-a)} \left( \int_c^d (x-a)^2 \frac{d\mu}{m} + \int_c^d (x-b)^2 \frac{d\mu}{m} \right) \\ &= \frac{\|f'\|_\infty}{2(b-a)} \left( \int_c^d (x-a)^2 d\nu + \int_c^d (x-b)^2 d\nu \right). \end{aligned}$$

Using (2.14), (2.15), (2.16) and (2.17) we get

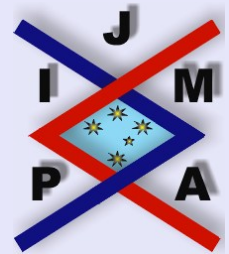
$$\begin{aligned} \theta &\leq \frac{\|f'\|_\infty}{2(b-a)} \left\{ (d_1(c+d-2a) - cd + a^2) + (d_1(c+d-2b) - cd + b^2) \right\} \\ &= \frac{\|f'\|_\infty}{(b-a)} \left[ d_1((c+d) - (a+b)) - cd + \frac{a^2 + b^2}{2} \right], \end{aligned}$$

proving the claim. □

We make the following remark.

**Remark 1 (Remark on Theorem 2.9).** 1. Case of  $c+d \geq a+b$ , using  $d_1 \leq d$  we obtain

$$(2.20) \quad d_1((c+d) - (a+b)) - cd + \frac{a^2 + b^2}{2} \leq \frac{(d-a)^2 + (d-b)^2}{2}.$$




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2. Case of  $c + d \leq a + b$ , using  $d_1 \geq c$  we find that

$$(2.21) \quad d_1((c + d) - (a + b)) - cd + \frac{a^2 + b^2}{2} \leq \frac{(c - a)^2 + (c - b)^2}{2}.$$

Hence under (2.18) inequality (2.19) is sharper than (2.8).

We also give

**Corollary 2.10.** *Let all the assumptions in Theorem 2.9 hold. Then*

$$(2.22) \quad \left| \frac{1}{m} \int_c^d f(x) d\mu - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{\|f'\|_\infty}{(b - a)} \left[ d_1((c + d) - (a + b)) - cd + \frac{a^2 + b^2}{2} \right].$$

By Remark 1, inequality (2.22) is sharper than (2.5).

### Part B

Here we follow Fink's work [6]. We require the following theorem.

**Theorem 2.11 ([6]).** *Let  $f: [a, b] \rightarrow \mathbb{R}$ ,  $f^{(n-1)}$  is absolutely continuous on  $[a, b]$ ,  $n \geq 1$ . Then*

$$(2.23) \quad f(x) = \frac{n}{b - a} \int_a^b f(t) dt$$



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$$+ \sum_{k=1}^{n-1} \left( \frac{n-k}{k!} \right) \left( \frac{f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k}{b-a} \right) \\ + \frac{1}{(n-1)!(b-a)} \int_a^b (x-t)^{n-1} k(t,x) f^{(n)}(t) dt,$$

where

$$(2.24) \quad k(t,x) := \begin{cases} t-a, & a \leq t \leq x \leq b, \\ t-b, & a \leq x < t \leq b. \end{cases}$$

For  $n = 1$  the sum in (2.23) is taken as zero.

We also need Fink's inequality

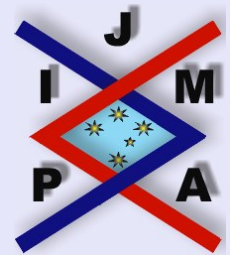
**Theorem 2.12 ([6]).** Let  $f^{(n-1)}$  be absolutely continuous on  $[a, b]$  and  $f^{(n)} \in L_\infty(a, b)$ ,  $n \geq 1$ . Then

$$(2.25) \quad \left| \frac{1}{n} \left( f(x) + \sum_{k=1}^{n-1} F_k(x) \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!(b-a)} [(b-x)^{n+1} + (x-a)^{n+1}], \quad \forall x \in [a, b],$$

where

$$(2.26) \quad F_k(x) := \left( \frac{n-k}{k!} \right) \left( \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a} \right).$$

Inequality (2.25) is sharp, in the sense that it is attained by an optimal  $f$  for any  $x \in [a, b]$ .




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We give

**Corollary 2.13.** *Let  $f^{(n-1)}$  be absolutely continuous on  $[a, b]$  and  $f^{(n)} \in L_\infty(a, b)$ ,  $n \geq 1$ . Then  $\forall x \in [c, d] \subseteq [a, b]$  we have*

$$\begin{aligned}
 & \left| \frac{1}{n} \left( f(x) + \sum_{k=1}^{n-1} F_k(x) \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!(b-a)} [(b-x)^{n+1} + (x-a)^{n+1}] \\
 (2.27) \quad & \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!} (b-a)^n.
 \end{aligned}$$

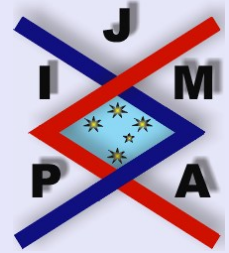
Also we have

**Proposition 2.14.** *Let  $f^{(n-1)}$  be absolutely continuous on  $[a, b]$  and  $f^{(n)} \in L_\infty(a, b)$ ,  $n \geq 1$ . Let  $\mu$  be a finite measure of mass  $m > 0$  on*

$$([c, d], \mathcal{P}([c, d])), \quad [c, d] \subseteq [a, b] \subseteq \mathbb{R}.$$

Then

$$\begin{aligned}
 (2.28) \quad K := & \left| \frac{1}{n} \left( \frac{1}{m} \int_{[c,d]} f(x) d\mu + \sum_{k=1}^{n-1} \frac{1}{m} \int_{[c,d]} F_k(x) d\mu \right) \right. \\
 & \left. - \frac{1}{b-a} \int_a^b f(x) dx \right|
 \end{aligned}$$




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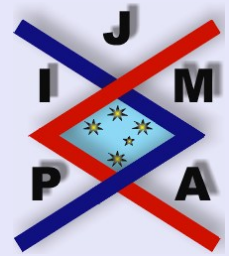
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$$\begin{aligned}
 &\leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!(b-a)} \left[ \frac{1}{m} \int_{[c,d]} (b-x)^{n+1} d\mu \right. \\
 &\quad \left. + \frac{1}{m} \int_{[c,d]} (x-a)^{n+1} d\mu \right] \\
 (2.29) \quad &\leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!} (b-a)^n.
 \end{aligned}$$

*Proof.* By (2.27). □

Similarly, based on Theorem A of [6] we also conclude

**Proposition 2.15.** *Let  $f^{(n-1)}$  be absolutely continuous on  $[a, b]$  and  $f^{(n)} \in L_p(a, b)$ , where  $1 < p < \infty$ ,  $n \geq 1$ . Let  $\mu$  be a finite measure of mass  $m > 0$  on  $[c, d]$ ,  $\mathcal{P}([c, d])$ ,  $[c, d] \subseteq [a, b] \subseteq \mathbb{R}$ .*

Here  $p' > 1$  such that  $\frac{1}{p} + \frac{1}{p'} = 1$ . Then

$$\begin{aligned}
 &\left| \frac{1}{n} \left( \frac{1}{m} \int_{[c,d]} f(x) d\mu + \sum_{k=1}^{n-1} \frac{1}{m} \int_{[c,d]} F_k(x) d\mu \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 &\leq \left( \frac{B((n-1)p' + 1, p' + 1)^{1/p'} \|f^{(n)}\|_p}{n!(b-a)} \right) \\
 &\quad \cdot \left( \frac{1}{m} \int_{[c,d]} ((x-a)^{np'+1} + (b-x)^{np'+1})^{1/p'} d\mu \right) \\
 (2.30) \quad &\leq \left( \frac{B((n-1)p' + 1, p' + 1)^{1/p'} (b-a)^{n-1+\frac{1}{p'}}}{n!} \right) \|f^{(n)}\|_p.
 \end{aligned}$$

We make the following remark.

**Remark 2.** Clearly we have the following for

$$(2.31) \quad g(x) := (b-x)^{n+1} + (x-a)^{n+1} \leq (b-a)^{n+1}, \quad a \leq x \leq b,$$

where  $n \geq 1$ . Here  $x = \frac{a+b}{2}$  is the only critical number of  $g$  and

$$g''\left(\frac{a+b}{2}\right) = n(n+1)\frac{(b-a)^{n-1}}{2^{n-2}} > 0,$$

giving that  $g\left(\frac{a+b}{2}\right) = \frac{(b-a)^{n+1}}{2^n} > 0$  is the global minimum of  $g$  over  $[a, b]$ . Also  $g$  is convex over  $[a, b]$ . Therefore for  $[c, d] \subseteq [a, b]$  we have

$$(2.32) \quad \begin{aligned} M &:= \max_{c \leq x \leq d} \{(x-a)^{n+1} + (b-x)^{n+1}\} \\ &= \max\{(c-a)^{n+1} + (b-c)^{n+1}, (d-a)^{n+1} + (b-d)^{n+1}\}. \end{aligned}$$

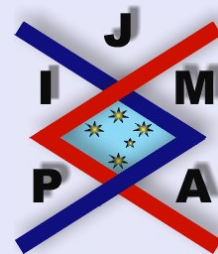
We get further that

$$(2.33) \quad M = \begin{cases} (d-a)^{n+1} + (b-d)^{n+1}, & \text{if } c+d \geq a+b \\ (c-a)^{n+1} + (b-c)^{n+1}, & \text{if } c+d \leq a+b. \end{cases}$$

If  $d = b$  or  $c = a$  or both then

$$(2.34) \quad M = (b-a)^{n+1}.$$

Based on Remark 2 we give




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**Theorem 2.16.** *Let all assumptions, terms and notations be as in Proposition 2.14. Then*

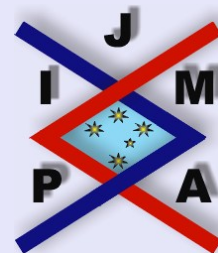
1.

$$\begin{aligned}
 (2.35) \quad K &\leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!(b-a)} \max\{(c-a)^{n+1} + (b-c)^{n+1}, \\
 &\quad (d-a)^{n+1} + (b-d)^{n+1}\} \\
 &= \frac{\|f^{(n)}\|_\infty}{n(n+1)!(b-a)} \\
 &\quad \times \left\{ \begin{array}{l} (d-a)^{n+1} + (b-d)^{n+1}, \quad \text{if } c+d \geq a+b, \\ (c-a)^{n+1} + (b-c)^{n+1}, \quad \text{if } c+d \leq a+b \end{array} \right\} \\
 (2.36) \quad &\leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!} (b-a)^n,
 \end{aligned}$$

where  $K$  is as in (2.28). If  $d = b$  or  $c = a$  or both, then (2.36) becomes equality. When  $d = b$ ,  $\frac{\mu}{m} = \delta_{\{b\}}$  and  $f(x) = \frac{(x-a)^n}{n!}$ ,  $a \leq x \leq b$ , then inequality (2.35) is attained, i.e. it becomes equality, proving that (2.35) is a sharp inequality.

2. We also have

$$(2.37) \quad \sup_{\mu \in M(c,d)} K \leq R.H.S \text{ (2.35)}$$



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and

$$(2.38) \quad \sup_{\substack{\text{all } c,d \\ a \leq c \leq d \leq b}} \left( \sup_{\mu \in M(c,d)} K \right) \leq R.H.S (2.36)$$

*Proof.* It remains to prove only the sharpness, via attainability of (2.35) when  $d = b$ . In that case (2.35) collapses to

$$(2.39) \quad \left| \frac{1}{n} \left( \frac{1}{m} \int_{[c,d]} f(x) d\mu + \sum_{k=1}^{n-1} \frac{1}{m} \int_{[c,b]} F_k(x) d\mu \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!} (b-a)^n.$$

The optimal measure here will be  $\frac{\mu}{m} = \delta_{\{b\}}$  and then (2.39) becomes

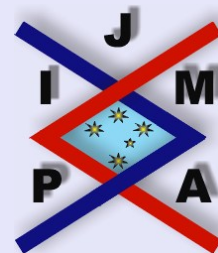
$$(2.40) \quad \left| \frac{1}{n} \left( f(b) + \sum_{k=1}^{n-1} F_k(b) \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!} (b-a)^n.$$

The optimal function here will be

$$f^*(x) = \frac{(x-a)^n}{n!}, \quad a \leq x \leq b.$$

Then we see that

$$f^{*(k-1)}(x) = \frac{(x-a)^{n-k+1}}{(n-k+1)!}, \quad k-1 = 0, 1, \dots, n-2,$$



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and  $f^{*(k-1)}(a) = 0$  for  $k - 1 = 0, 1, \dots, n - 2$ . Clearly here  $F_k(b) = 0$ ,  $k = 1, \dots, n - 1$ . Also we have

$$\frac{1}{b-a} \int_a^b f^*(x) dx = \frac{(b-a)^n}{(n+1)!} \quad \text{and} \quad \|f^{*(n)}\|_\infty = 1.$$

Putting all these elements in (2.40) we have

$$\left| \frac{(b-a)^n}{nn!} - \frac{(b-a)^n}{(n+1)!} \right| = \frac{(b-a)^n}{n(n+1)!},$$

proving the claim.  $\square$

Next, we again restrict ourselves to the subclass of  $M(c, d)$  of finite measures  $\mu$  with given first moment and by the use of the Geometric Moment Theory Method, see [7], [1], [3], we produce an inequality sharper than (2.37). For that we need the following result.

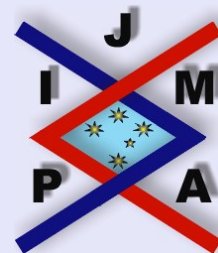
**Lemma 2.17.** *Let  $[c, d] \subseteq [a, b] \subseteq \mathbb{R}$  and  $\nu$  be a probability measure on  $([c, d], \mathcal{P}([c, d]))$  such that*

$$(2.41) \quad \int_{[c,d]} x d\nu = d_1 \in [c, d]$$

is given,  $n \geq 1$ . Then

$$(2.42) \quad W_1 := \sup_{\nu \text{ as in (2.41)}} \int_{[c,d]} (x-a)^{n+1} d\nu$$

$$(2.43) \quad = \left( \sum_{k=0}^n (d-a)^{n-k} (c-a)^k \right) (d_1 - d) + (d-a)^{n+1}.$$



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*Proof.* We observe the graph

$$G_1 = \{(x, (x - a)^{n+1}) : c \leq x \leq d\},$$

which is a convex arc above the  $x$ -axis. We form the closed convex hull of  $G_1$  and we call it  $\widehat{G}_1$ , which has as an upper concave envelope the line segment  $\bar{\ell}_1$  from  $(c, (c - a)^{n+1})$  to  $(d, (d - a)^{n+1})$ . Call  $\ell_1$  the line through  $\bar{\ell}_1$ . The line  $\ell_1$  intersects the  $x$ -axis at  $(t, 0)$ , where  $a \leq t \leq c$ . We need to determine  $t$ : the slope of  $\ell_1$  is

$$\tilde{m} = \frac{(d - a)^{n+1} - (c - a)^{n+1}}{d - c} = \sum_{k=0}^n (d - a)^{n-k} (c - a)^k.$$

The equation of line  $\ell_1$  is

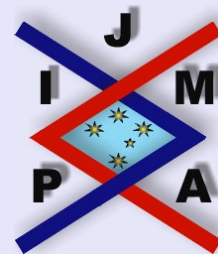
$$y = \tilde{m} \cdot x + (d - a)^{n+1} - \tilde{m}d.$$

Hence  $\tilde{m}t + (d - a)^{n+1} - \tilde{m}d = 0$  and

$$t = d - \frac{(d - a)^{n+1}}{\tilde{m}}.$$

Next we consider the moment right triangle with vertices  $(t, 0)$ ,  $(d, 0)$  and  $(d, (d - a)^{n+1})$ . Clearly  $(d_1, 0)$  is between  $(t, 0)$  and  $(d, 0)$ . Consider the vertical line  $x = d_1$ , it intersects  $\bar{\ell}_1$  at  $Q$ . Clearly then  $W_1 = \text{length}((d_1, 0), Q)$ , the line segment of which length we find by the formed two similar right triangles with vertices  $\{(t, 0), (d_1, 0), Q\}$  and  $\{(t, 0), (d, 0), (d, (d - a)^{n+1})\}$ . We have the equal ratios

$$\frac{d_1 - t}{d - t} = \frac{W_1}{(d - a)^{n+1}},$$



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i.e.

$$W_1 = (d - a)^{n+1} \left( \frac{d_1 - t}{d - t} \right).$$

□

We also need

**Lemma 2.18.** *Let  $[c, d] \subseteq [a, b] \subseteq \mathbb{R}$  and  $\nu$  be a probability measure on  $([c, d], \mathcal{P}([c, d]))$  such that*

$$(2.44) \quad \int_{[c,d]} x d\nu = d_1 \in [c, d]$$

is given,  $n \geq 1$ . Then

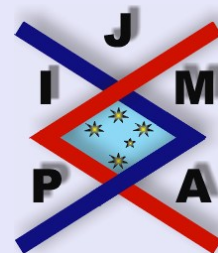
1.

$$(2.45) \quad \begin{aligned} W_2 &:= \sup_{\nu \text{ as in (2.44)}} \int_{[c,d]} (b - x)^{n+1} d\nu \\ &= \left( \sum_{k=0}^n (b - c)^{n-k} (b - d)^k \right) (c - d_1) + (b - c)^{n+1}. \end{aligned}$$

2. The following result holds

$$(2.46) \quad \sup_{\nu \text{ as in (2.44)}} \int_{[c,d]} [(x - a)^{n+1} + (b - x)^{n+1}] d\nu = W_1 + W_2,$$

where  $W_1$  is as in (2.42).



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*Proof.* 1. We observe the graph

$$G_2 = \{(x, (b-x)^{n+1}) : c \leq x \leq d\},$$

which is a convex arc above the  $x$ -axis. We form the closed convex hull of  $G_2$  and we call it  $\widehat{G}_2$ , which has as an upper concave envelope the line segment  $\bar{\ell}_2$  from  $(c, (b-c)^{n+1})$  to  $(d, (b-d)^{n+1})$ . Call  $\ell_2$  the line through  $\bar{\ell}_2$ . The line  $\ell_2$  intersects the  $x$ -axis at  $(t^*, 0)$ , where  $d \leq t^* \leq b$ . We need to determine  $t^*$ : The slope of  $\ell_2$  is

$$\tilde{m}^* = \frac{(b-c)^{n+1} - (b-d)^{n+1}}{c-d} = - \left( \sum_{k=0}^n (b-c)^{n-k} (b-d)^k \right).$$

The equation of line  $\ell_2$  is

$$y = \tilde{m}^* x + (b-c)^{n+1} - \tilde{m}^* c.$$

Hence

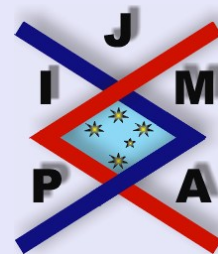
$$\tilde{m}^* t^* + (b-c)^{n+1} - \tilde{m}^* c = 0$$

and

$$t^* = c - \frac{(b-c)^{n+1}}{\tilde{m}^*}.$$

Next we consider the moment right triangle with vertices  $(c, (b-c)^{n+1})$ ,  $(c, 0)$ ,  $(t^*, 0)$ . Clearly  $(d_1, 0)$  is between  $(c, 0)$  and  $(t^*, 0)$ . Consider the vertical line  $x = d_1$ , it intersects  $\bar{\ell}_2$  at  $Q^*$ . Clearly then

$$W_2 = \text{length}(\overline{(d_1, 0), Q^*}),$$



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the line segment of which length we find by the formed two similar right triangles with vertices  $\{Q^*, (d_1, 0), (t^*, 0)\}$  and  $\{(c, (b - c)^{n+1}), (c, 0), (t^*, 0)\}$ . We have the equal ratios

$$\frac{t^* - d_1}{t^* - c} = \frac{W_2}{(b - c)^{n+1}},$$

i.e.

$$W_2 = (b - c)^{n+1} \left( \frac{t^* - d_1}{t^* - c} \right).$$

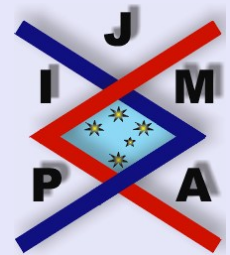
2. Similar to that above and obvious. □

We make the following useful remark.

**Remark 3.** By Lemmas 2.17, 2.18 we obtain

$$\begin{aligned} (2.47) \quad \lambda &:= W_1 + W_2 \\ &= \left( \sum_{k=0}^n (d - a)^{n-k} (c - a)^k \right) (d_1 - d) \\ &\quad + \left( \sum_{k=0}^n (b - c)^{n-k} (b - d)^k \right) (c - d_1) + (d - a)^{n+1} + (b - c)^{n+1} \\ &> 0, \end{aligned}$$

$$n \geq 1.$$




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We present the following important result.

**Theorem 2.19.** *Let  $f^{(n-1)}$  be absolutely continuous on  $[a, b]$  and  $f^{(n)} \in L_\infty(a, b)$ ,  $n \geq 1$ . Let  $\mu$  be a finite measure of mass  $m > 0$  on  $([c, d], \mathcal{P}([c, d]))$ ,  $[c, d] \subseteq [a, b] \subseteq \mathbb{R}$ . Furthermore we assume that*

$$(2.48) \quad \frac{1}{m} \int_{[c,d]} x d\mu = d_1 \in [c, d]$$

is given. Then

$$(2.49) \quad \sup_{\mu \text{ as above}} K \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!(b-a)} \lambda,$$

and

$$(2.50) \quad K \leq R.H.S (2.49),$$

where  $K$  is as in (2.28) and  $\lambda$  is as in (2.47).

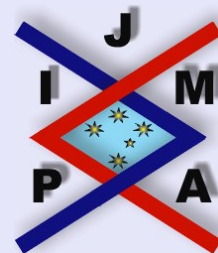
*Proof.* By Proposition 2.14 and Lemmas 2.17 and 2.18. □

We make the following remark.

**Remark 4.** *We compare  $M$  as in (2.32) and (2.33) and  $\lambda$  as in (2.47). We easily obtain that*

$$(2.51) \quad \lambda \leq M.$$

*As a result we have that (2.50) is sharper than (2.35) and (2.49) is sharper than (2.37). That is reasonable since we restricted ourselves to a subclass of  $M(c, d)$  of measures  $\mu$  by assuming the moment condition (2.48).*




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We finish with the following comment.

**Remark 5.**

I) When  $c = a$  and  $d = b$  then  $d_1$  plays no role in the best upper bounds we found with the Geometric Moment Theory Method. That is, the restriction on measures  $\mu$  via the first moment  $d_1$  has no effect in producing sharper estimates as it happens when  $a < c < d < b$ . More precisely we notice that:

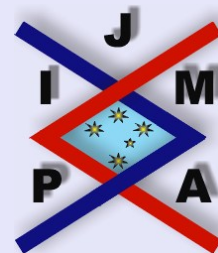
(a)

$$(2.52) \quad R.H.S.(2.19) = \frac{\|f'\|_\infty}{2}(b - a) = R.H.S.(2.9),$$

(b) by (2.47) here  $\lambda = (b - a)^{n+1}$  and

$$(2.53) \quad R.H.S.(2.49) = \frac{\|f^{(n)}\|_\infty}{n(n + 1)!}(b - a)^n = R.H.S.(2.36).$$

II) Further differences of general means over any  $[c_1, d_1]$  and  $[c_2, d_2]$  subsets of  $[a, b]$  (even disjoint) with respect to  $\mu_1$  and  $\mu_2$ , respectively, can be found by straightforward application of the above results and the triangle inequality.




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