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CONVEXITY OF THE ZERO-BALANCED GAUSSIAN HYPERGEOMETRIC FUNCTIONS WITH RESPECT TO HÖLDER MEANS

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ABSTRACT. In this note we investigate the convexity of zero-balanced Gaussian hypergeometric functions and general power series with respect to Hölder means.

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1. INTRODUCTION AND PRELIMINARIES

For a given interval $I \subseteq [0, \infty)$, a function $f : I \to [0, \infty)$ is said to be multiplicatively convex if for all $r, s \in I$ and all $\lambda \in (0, 1)$ the inequality

$$f(r^{1-\lambda}s^{\lambda}) \leq f(r)^{1-\lambda}f(s)^{\lambda} \tag{1.1}$$

holds. The function f is said to be multiplicatively concave if

$$f(r^{1-\lambda}s^{\lambda}) \geq f(r)^{1-\lambda}f(s)^{\lambda} \tag{1.2}$$

for all $r, s \in I$ and all $\lambda \in (0, 1)$. If for $r \neq s$ the inequality (1.1) (respectively (1.2)) is strict, then f is said to be strictly multiplicatively convex (respectively multiplicatively concave). It can be proved (see the paper of C.P. Niculescu [15, Theorem 2.3]) that if f is continuous, then f is multiplicatively convex (respectively strictly multiplicatively convex) if and only if

$$f\left(\sqrt{rs}\right) \leq \sqrt{f(r)f(s)} \quad \left(\text{respectively } f(\sqrt{rs}) < \sqrt{f(r)f(s)}\right)$$

for all $r, s \in I$ with $r \neq s$. A similar characterization of the continuous (strictly) multiplicatively concave functions holds as well. In what follows, for simplicity of notation, the symbols H, G and A will stand, respectively, for the unweighted harmonic, geometric and arithmetic means of the positive numbers r and s, i.e.,

$$H \equiv H(r, s) = \frac{2rs}{r + s}, \quad G \equiv G(r, s) = \sqrt{rs}, \quad A \equiv A(r, s) = \frac{r + s}{2}.$$

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It is well-known that $H \leq G \leq A$.

For $a, b, c \in \mathbb{C}$ and $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric series is defined by

$$ {}_2F_1(a,b,c,r) := F(a,b,c,r) = \sum_{n\geq 0} d_n r^n = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{r^n}{n!}, \quad |r| < 1, \tag{1.3} $$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ is the well-known Pochhammer symbol. Recently we proved [6, Theorem 1.10] that the zero-balanced Gaussian hypergeometric function F, defined by $F(r) := {}_2F_1(a,b,a+b,r)$, for all $a, b > 0$ satisfies the following chain of inequalities

$$ F(G(r,s)) \leq G(F(r),F(s)) \leq F(1 - G(1-r,1-s)) \leq A(F(r),F(s)), \tag{1.4} $$

where $r, s \in (0,1)$. We note that in 1998 R. Balasubramanian, S. Ponnusamy and M. Vuorinen [3, Lemma 2.1] showed that the function $r \mapsto F'(r)/F(r)$ is strictly increasing on $(0,1)$ for all $a, b > 0$. Thus the function F is log-convex on $(0,1)$, i.e.

$$ F(A(r,s)) \leq G(F(r),F(s)) \tag{1.5} $$

holds, where $r, s \in (0,1)$. Because F is strictly increasing on $(0,1)$, combining (1.4) with (1.5), we easily obtain

$$ F(G(r,s)) \leq F(A(r,s)) \leq G(F(r),F(s)) \leq A(F(r),F(s)), \tag{1.6} $$

for all $a, b > 0$ and $r, s \in (0,1)$. In [6, Theorem 1.10] we deduced that for $a, b \in (0,1]$

$$ F(G(r,s)) \leq H(F(r),F(s)) \tag{1.7} $$

holds for all $r, s \in (0, x_0)$, where $x_0 = 0.7153318630\ldots$ is the unique positive root of the equation $2\log(1-x) + x/(1-x) = 0$. Moreover we conjectured [6, Remark 1.13] that (1.7) holds for all $r, s \in (0,1)$, which was proved recently by G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen [2, Theorem 3.7]. Using this result, (1.7) and the (HG) inequality imply

$$ F(H(r,s)) \leq F(G(r,s)) \leq H(F(r),F(s)), \tag{1.8} $$

where $a, b \in (0,1]$ and $r, s \in (0,1)$.

In fact, using (1.6) and (1.8) we have that for all $r, s \in (0,1)$ the inequality

$$ F(M(r,s)) \leq M(F(r),F(s)) \tag{1.9} $$

holds for certain conditions on a, b and for M being the unweighted harmonic, geometric and arithmetic mean. Let $I \subseteq \mathbb{R}$ be a nondegenerate interval and $M : I^2 \to I$ be a continuous function. We say that M is a mean on I if it satisfies the following condition $\min\{r,s\} \leq M(r,s) \leq \max\{r,s\}$ for all $r, s \in I$, $r \neq s$. Taking into account the inequalities (1.6) and (1.8) it is natural to ask whether the inequality (1.9) remains true for some other means as well?

Our aim in this paper is to partially answer this question for Hölder means.

2. Convexity of Hypergeometric Functions with Respect to Hölder Means

Let $I \subseteq \mathbb{R}$ be a nondegenerate interval and $\varphi : I \to \mathbb{R}$ be a strictly monotonic continuous function. The function $M_\varphi : I^2 \to I$, defined by

$$ M_\varphi(r,s) := \varphi^{-1}\left(A(\varphi(r),\varphi(s))\right) $$

is called the quasi-arithmetic mean associated to φ, while the function φ is called a generating function of the quasi-arithmetic mean M_φ (for more details see the works of J. Aczél [1], Z.

Daróczy [10] and J. Matkowski [11]). A function $f : I \to \mathbb{R}$ is said to be convex with respect to the mean M_φ (or $M_\varphi-$convex) if for all $r, s \in I$ and all $\lambda \in (0,1)$ the inequality

$$f(M_\varphi^{(\lambda)}(r,s)) \le M_\varphi^{(\lambda)}(f(r), f(s)) \tag{2.1}$$

holds, where

$$M_\varphi^{(\lambda)}(r,s) := \varphi^{-1}((1-\lambda)\varphi(r) + \lambda\varphi(s))$$

is the weighted version of M_φ. If for $r \ne s$ the inequality (2.1) is strict, then f is said to be strictly convex with respect to M_φ (for more details see D. Borwein, J. Borwein, G. Fee and R. Girgensohn [9], J. Matkowski and J. Rätz [12], [13]). It can be proved (see [9]) that f is (strictly) convex with respect to M_φ if and only if $\varphi \circ f \circ \varphi^{-1}$ is (strictly) convex in the usual sense on $\varphi(I)$. Among the quasi-arithmetic means the Hölder means are of special interest. They are associated to the function $\varphi_p : (0,\infty) \to \mathbb{R}$, defined by

$$\varphi_p(r) := \begin{cases} r^p, & \text{if } p \ne 0 \\ \log r, & \text{if } p = 0, \end{cases}$$

thus

$$M_{\varphi_p}(r,s) = H_p(r,s) = \begin{cases} [A(r^p, s^p)]^{1/p}, & \text{if } p \ne 0 \\ G(r,s), & \text{if } p = 0. \end{cases}$$

Our first mean result reads as follows.

Theorem 2.1. *For all $a, b > 0$ and $p \in [0,1]$ the hypergeometric function $r \mapsto F(r) := {}_2F_1(a,b,a+b,r)$ defined by (1.3) is convex on $(0,1)$ with respect to the Hölder means H_p.*

By Theorem 2.1, using the definition of convexity with respect to the Hölder means, we get that for all $\lambda, r, s \in (0,1)$, $a, b > 0$ and $p \in (0,1]$ the following inequality

$$F([(1-\lambda)r^p + \lambda s^p]^{1/p}) \le [(1-\lambda)[F(r)]^p + \lambda[F(s)]^p]^{1/p}$$

holds. Moreover, for all $\lambda, r, s \in (0,1)$ and $a, b > 0$

$$F(r^{1-\lambda}s^\lambda) \le [F(r)]^{1-\lambda}[F(s)]^\lambda, \tag{2.2}$$

i.e., the zero-balanced hypergeometric function is multiplicatively convex on $(0,1)$.

Proof of Theorem 2.1. First assume that $p = 0$. Then we need to prove that (2.2) holds. Using the first inequality in (1.4) and Theorem 2.3 due to C.P. Niculescu [15], the desired result follows. Note that in fact (2.2) can be proved using Hölder's inequality [14, Theorem 1, p. 50]. For this let us denote $P_n(r) = \sum_{k=0}^n d_k r^k$. Then by the Hölder inequality we have

$$\sum_{k=0}^n ({d_k}^{1-\lambda} r^{(1-\lambda)k})({d_k}^\lambda s^{\lambda k}) \le \left(\sum_{k=0}^n d_k r^k\right)^{1-\lambda} \left(\sum_{k=0}^n d_k s^k\right)^\lambda .$$

But this is equivalent to

$$P_n(r^{1-\lambda}s^\lambda) \le [P_n(r)]^{1-\lambda}[P_n(s)]^\lambda,$$

so using the fact that $\lim_{n\to\infty} P_n(r) = F(r)$, we obtain immediately (2.2).

Now assume that $p \ne 0$. In order to establish the convexity of F with respect to H_p we need to show that the function $\varphi_p \circ F \circ \varphi_p^{-1}$ is convex in the usual sense. Let us denote

$$f_G(r) := (\varphi_p \circ F \circ \varphi_p^{-1})(r) = [F(r^{1/p})]^p.$$

Setting $q := 1/p \ge 1$ we have $f_G(r) = [F(r^q)]^{1/q}$, thus a simple computation shows that

$$f_G'^{q-1} \frac{F'^{q)}}{F(r^q)}[F(r^q)]^{1/q} = \frac{1}{q} f_G(r) \frac{d(\log F(r^q))}{dr} \ge 0. \tag{2.3}$$

Recall that from [3, Lemma 2.1] due to R. Balasubramanian, S. Ponnusamy and M. Vuorinen, the function F is log-convex on $(0,1)$. On the other hand the function $r \mapsto r^q$ is convex on $(0,1)$. Thus by the monotonicity of F for all $\lambda, r, s \in (0,1)$ we obtain

$$F([(1-\lambda)r+\lambda s]^q) \leq F((1-\lambda)r^q+\lambda s^q) \leq [F(r^q)]^{1-\lambda}[F(s^q)]^{\lambda}.$$

This shows that $r \mapsto F(r^q)$ is log-convex and consequently $r \mapsto d(\log F(r^q))/dr$ is increasing. From (2.3), we obtain that f_G is increasing, therefore f_G' is increasing too as a product of two strictly positive and increasing functions. □

Taking into account the above proof we note that Theorem 2.1 may be generalized easily in the following way. The proof of the next theorem is similar, so we omit the details.

Theorem 2.2. *For all $a, b > 0$ and $p \in [0, m]$, where $m = 1, 2, \ldots,$ the function $r \mapsto F(r) := {}_2F_1(a, b, a+b, r^m)$ is convex on $(0,1)$ with respect to Hölder means H_p. In particular, the complete elliptic integral of the first kind, defined by*

$$\mathcal{K}(r) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-r^2\sin^2\theta}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1, r^2\right),$$

is convex on $(0,1)$ with respect to means H_p where $p \in [0,2]$. In other words, for all $\lambda, r, s \in (0,1)$ and $p \in (0,2]$ we have the following inequality

$$\mathcal{K}([(1-\lambda)r^p+\lambda s^p]^{1/p}) \leq [(1-\lambda)[\mathcal{K}(r)]^p+\lambda[\mathcal{K}(s)]^p]^{1/p}.$$

Moreover, for all $\lambda, r, s \in (0,1)$,

$$\mathcal{K}\left(r^{1-\lambda}s^{\lambda}\right) \leq [\mathcal{K}(r)]^{1-\lambda}[\mathcal{K}(s)]^{\lambda}$$

holds, i.e., the complete elliptic integral $\mathcal{K}$ is multiplicatively convex on $(0,1)$.

By the proof of Theorem 3.7 due to G.D. Anderson, M.K. Vamanmurthy and M. Vuorinen [2], we know that the function $x \mapsto 1/F(x)$ is concave on $(0,1)$ for all $a, b \in (0,1]$. This implies that we have

$$F\left(H_{-1}^{(\lambda)}(r,s)\right) \leq F((1-\lambda)r+\lambda s) \leq H_{-1}^{(\lambda)}(F(r),F(s)), \tag{2.4}$$

where $\lambda, r, s \in (0,1)$ and $a, b \in (0,1]$. Here we denoted with $H_{-1}^{(\lambda)}(r,s) := [(1-\lambda)/r+\lambda/s]^{-1}$ the weighted harmonic mean and we used the (HA) inequality between the weighted harmonic and arithmetic means of r and s. We note that in fact (2.4) shows that the function F is convex on $(0,1)$ for all $a, b \in (0,1]$ with respect to the Hölder mean H_{-1}.

The following result is similar to Theorem 2.2.

Theorem 2.3. *If $a, b, p > 0$ and $m = 1, 2, \ldots,$ then $r \mapsto f_m(r) := {}_2F_1(a, b, a+b, r^m) - 1$ is convex on $(0,1)$ with respect to the Hölder means H_p. In particular for $m = 1$ and $m = 2$ the functions $f_1(r) := {}_2F_1(a, b, a+b, r) - 1$ and $f_2(r) := 2\mathcal{K}(r)/\pi - 1$ are convex on $(0,1)$ with respect to means H_p, i.e. for all $\lambda, r, s \in (0,1)$ and $p > 0$ one has*

$$F([(1-\lambda)r^p+\lambda s^p]^{1/p}) \leq 1 + [(1-\lambda)[F(r)-1]^p+\lambda[F(s)-1]^p]^{1/p},$$

$$\frac{2}{\pi}\mathcal{K}([(1-\lambda)r^p+\lambda s^p]^{1/p}) \leq 1 + \left[(1-\lambda)\left(\frac{2}{\pi}\mathcal{K}(r)-1\right)^p+\lambda\left(\frac{2}{\pi}\mathcal{K}(s)-1\right)^p\right]^{1/p}.$$

In order to prove this result we need the following lemma due to M. Biernacki and J. Krzyż [8]. Note that this lemma is a special case of a more general lemma established by S. Ponnusamy and M. Vuorinen [16].

Lemma 2.4 ([8, 16]). *Let us suppose that the power series $f(x) = \sum_{n\geq 0} \alpha_n x^n$ and $g(x) = \sum_{n\geq 0} \beta_n x^n$ both converge for $|x| < 1$, where $\alpha_n \in \mathbb{R}$ and $\beta_n > 0$ for all $n \geq 0$. Then the ratio f/g is (strictly) increasing (decreasing) on $(0, 1)$ if the sequence $\{\alpha_n/\beta_n\}_{n\geq 0}$ is (strictly) increasing (decreasing).*

It is worth mentioning that this lemma was used, among other things, to prove many interesting inequalities for the zero-balanced Gaussian hypergeometric functions (see the papers of R. Balasubramanian, S. Ponnusamy and M. Vuorinen [3], [16]) and for the generalized (in particular, for the modified) Bessel functions of the first kind (see the papers of Á. Baricz and E. Neuman [4, 5, 6, 7] for more details).

Proof of Theorem 2.3. We just need to show that $r \mapsto [f_m(r^{1/p})]^p$ is convex on $(0, 1)$. Let us denote

$$\gamma(r) := [f_m(r^{1/p})]^p = [F(r^{m/p}) - 1]^p.$$

Setting $q := 1/p > 0$, we get $\gamma(r) = [F(r^{qm}) - 1]^{1/q}$. Thus a simple computation shows that

$$\gamma'(r) = m \left[\frac{r^{mq} F'^{mq})}{F(r^{mq}) - 1}\right] \cdot \left[\frac{F(r^{mq}) - 1}{r^q}\right]^{1/q}.$$

Now taking $r^{mq} := x \in (0, 1)$, we need only to prove that the function

$$x \mapsto m \left[\frac{xF'(x)}{F(x) - 1}\right] \cdot \left[\frac{F(x) - 1}{x^{1/m}}\right]^{1/q}$$

is strictly increasing. From Lemma 2.4 it follows that the fuction $x \mapsto xF'(x)/(F(x) - 1)$ is strictly increasing because

$$\frac{xF'(x)}{F(x) - 1} = \frac{\sum_{n\geq 1} n d_n x^n}{\sum_{n\geq 1} d_n x^n} = \frac{\sum_{n\geq 0} (n + 1) d_{n+1} x^n}{\sum_{n\geq 0} d_{n+1} x^n},$$

and clearly the sequence $(n + 1)d_{n+1}/d_{n+1} = n + 1$ is strictly increasing. Now since $1/q > 0$, it is enough to show that $x \mapsto (F(x) - 1)/\sqrt[m]{x}$ is increasing. We have that

$$x^{1+1/m} \frac{d}{dx} \left(\frac{F(x) - 1}{x^{1/m}}\right) = xF'(x) - \frac{F(x) - 1}{m} = \sum_{n\geq 1} n d_n x^n - \frac{1}{m} \sum_{n\geq 1} d_n x^n,$$

which is positive because by assumption $1/m \leq 1 \leq n$ and $d_n > 0$. □

3. CONVEXITY OF GENERAL POWER SERIES WITH RESPECT TO HÖLDER MEANS

Let us consider the power series

$$f(r) = \sum_{n\geq 0} A_n r^n \text{ (where } A_n > 0 \text{ for all } n \geq 0) \tag{3.1}$$

which is convergent for all $r \in (0, 1)$. In this section our aim is to generalize Theorems 2.1 and 2.3, i.e. to find conditions for the convexity of f with respect to Hölder means. From the proof of Theorem 2.1, it is clear that the fact that F is log-convex was sufficient for F to be convex with respect to H_p for $p \in (0, 1]$. Moreover, taking into account the proof of Theorem 2.3, we observe that the statement of this theorem holds for an arbitrary power series. Our main result in this section is the following theorem, which generalizes Theorems 2.2 and 2.3.

Theorem 3.1. *Let f be defined by (3.1), $m = 1, 2, \ldots,$ and for all $n \geq 0$ let us denote $B_n := (n + 1)A_{n+1}/A_n$. Then the following assertions are true:*

(a) *If the sequence B_n is (strictly) increasing then $r \mapsto f(r^m)$ is convex on $(0, 1)$ with respect to H_p for $p \in [0, m]$;*

(b) *If the sequence* $B_n - n$ *is (strictly) increasing then* $r \mapsto f(r^m)$ *is convex on* $(0,1)$ *with respect to* H_p *for* $p \in [0, \infty)$;
(c) *The function* $r \mapsto f(r^m) - 1$ *is convex on* $(0,1)$ *with respect to* H_p *for* $p \in (0, \infty)$.

Proof. (a) First assume that $p = 0$. Then a simple application of Hölder's inequality gives the multiplicative convexity of f. Now let $p \neq 0$. Then by Lemma 2.4, it is clear that $r \mapsto f'(r)/f(r)$ is (strictly) increasing on $(0,1)$. Let us denote

$$\phi(r) := (\varphi_p \circ f \circ \varphi_p^{-1})(r) = [f(r^{m/p})]^p.$$

Setting $q := m/p \geq 1$ we have $\phi(r) = [f(r^q)]^{m/q}$, thus a simple computation shows that

$$\phi'^{q-1} \frac{f'^{q})}{f(r^q)} [f(r^q)]^{m/q} = \frac{m}{q} \phi(r) \frac{d(\log f(r^q))}{dr} \geq 0. \tag{3.2}$$

On the other hand, the function $r \mapsto r^q$ is convex on $(0,1)$. Therefore because f is strictly increasing and log-convex, one has for all $\lambda, r, s \in (0,1)$, $r \neq s$

$$f([(1-\lambda)r + \lambda s]^q) \leq f((1-\lambda)r^q + \lambda s^q) \leq [f(r^q)]^{1-\lambda} [f(s^q)]^{\lambda}.$$

This shows that the function $r \mapsto f(r^q)$ is log-convex too and consequently the function $r \mapsto d(\log f(r^q))/dr$ is increasing. From (3.2) we obtain that ϕ is increasing, therefore ϕ' is increasing too as a product of two strictly positive and increasing functions.
(b) Let us denote $Q(r) := d(\log f(r))/dr = f'(r)/f(r)$. Using again Lemma 2.4, from the fact that the sequence $B_n - n$ is (strictly) increasing we get that

$$(1-r)Q(r) = (1-r)\frac{f'(r)}{f(r)} = \frac{\sum_{n \geq 0} [(n+1)A_{n+1} - nA_n] r^n}{\sum_{n \geq 0} A_n r^n}$$

is (strictly) increasing too. Thus the function $r \mapsto \log[(1-r)Q(r)]$ will be also (strictly) increasing, i.e. $d \log[(1-r)Q(r)]/dr \geq 0$ for all $r \in (0,1)$. This in turn implies that

$$\frac{Q'(r)}{Q(r)} \geq \frac{1}{1-r} \tag{3.3}$$

holds for all $r \in (0,1)$. Taking into account (3.2) for $q := m/p > 0$ we just need to show that

$$\phi'(r) = \frac{m}{q} \phi(r) \frac{d(\log f(r^q))}{dr} = \frac{m}{q} \phi(r) Q(r^q) \geq 0$$

is strictly increasing. Now using (3.3) we get that

$$\begin{aligned} \phi''(r) =& \frac{m}{q} \phi(r) Q(r^q) \left[\frac{m}{q} Q(r^q) + \frac{d \log(Q(r^q))}{dr} \right] \\ \geq& \frac{m}{q} \phi(r) Q(r^q) \left[\frac{m}{q} Q(r^q) + \frac{1}{1-r^q} \right] \geq 0, \end{aligned}$$

which completes the proof of this part.
(c) The proof of this part is similar to the proof of Theorem 2.3. We need to show that $r \mapsto [f(r^{m/p}) - 1]^p$ is convex on $(0,1)$. Let us denote $\sigma(r) := [f(r^{m/p}) - 1]^p$. Setting $q := 1/p > 0$ we get $\sigma(r) = [f(r^{qm}) - 1]^{1/q}$. Thus a simple computation shows that

$$\sigma'(r) = m \left[\frac{r^{mq} f'^{mq})}{f(r^{mq}) - 1} \right] \cdot \left[\frac{f(r^{mq}) - 1}{r^q} \right]^{1/q}.$$

Now taking $r^{mq} := x \in (0,1)$, we need only to prove that the function

$$x \mapsto m \left[\frac{x f'(x)}{f(x) - 1} \right] \cdot \left[\frac{f(x) - 1}{x^{1/m}} \right]^{1/q}$$

is strictly increasing. By Lemma 2.4 it is clear that $x \mapsto xf'(x)/(f(x)-1)$ is strictly increasing because

$$\frac{xf'(x)}{f(x)-1} = \frac{\sum_{n\geq 1} nA_n x^n}{\sum_{n\geq 1} A_n x^n} = \frac{\sum_{n\geq 0}(n+1)A_{n+1}x^n}{\sum_{n\geq 0} A_{n+1}x^n}$$

and clearly the sequence $(n+1)A_{n+1}/A_{n+1} = n+1$ is strictly increasing. Finally, since $1/q > 0$, it is enough to show that $x \mapsto (f(x)-1)/\sqrt[m]{x}$ is increasing. We have that

$$x^{1+1/m}\frac{d}{dx}\left(\frac{f(x)-1}{x^{1/m}}\right) = xf'(x) - \frac{f(x)-1}{m} = \sum_{n\geq 1} nA_n x^n - \frac{1}{m}\sum_{n\geq 1} A_n x^n$$

which is positive by the assumptions $1/m \leq 1 \leq n$ and $A_n > 0$. □

As we have seen in Theorem 3.1, the log-convexity of the power series was crucial in proving convexity properties with respect to Hölder means. The following theorem contains sufficient conditions for a differentiable log-convex function to be convex with respect to Hölder means.

Theorem 3.2. *Let $f : I \subseteq [0,\infty) \to [0,\infty)$ be a differentiable function.*

(a) *If the function f is (strictly) increasing and log-convex, then f is convex with respect to Hölder means H_p for $p \in [0,1]$.*
(b) *If the function f is (strictly) decreasing and log-convex, then f is convex with respect to Hölder means H_p for $p \in [1,\infty)$. Moreover, if f is decreasing then f is multiplicativelly convex if and only if it is convex with respect to Hölder means H_p for $p \in [0,\infty)$.*

Proof. (a) Suppose that $p = 0$. Then using the (AG) inequality, the monotonicity of f and the log-convexity property, one has

$$f(r^{1-\lambda}s^{\lambda}) \leq f((1-\lambda)r + \lambda s) \leq [f(r)]^{1-\lambda}[f(s)]^{\lambda}$$

for all $r, s \in I$ and $\lambda \in (0,1)$. Now assume that $p \neq 0$. Let us denote $g(r) := [f(r^{1/p})]^p$ and $q := 1/p \geq 1$. Then $g(r) = [f(r^q)]^{1/q}$ and

$$g'(r) = \frac{1}{q}g(r)\frac{d[\log f(r^q)]}{dr} > 0. \tag{3.4}$$

In this case $r \mapsto r^q$ is convex, thus

$$f([(1-\lambda)r + \lambda s]^q) \leq f((1-\lambda)r^q + \lambda s^q) \leq [f(r^q)]^{1-\lambda}[f(s^q)]^{\lambda} \tag{3.5}$$

holds for all $r, s \in I$ and $\lambda \in (0,1)$, which means that $r \mapsto f(r^q)$ is log-convex too. Thus, by (3.4), g' is increasing as a product of two increasing functions.
(b) Using the same notation as in part (a), $q := 1/p \in (0,1]$ and consequently $r \mapsto r^q$ is concave. But f is decreasing, thus (3.5) holds again. Now suppose that f is multiplicativelly convex and decreasing. For $p \in (0,1]$ we have $q := 1/p \geq 1$ and $r \mapsto r^q$ is log-concave. Thus

$$f([(1-\lambda)r + \lambda s]^q) \leq f((r^q)^{1-\lambda}(s^q)^{\lambda}) \leq [f(r^q)]^{1-\lambda}[f(s^q)]^{\lambda} \tag{3.6}$$

holds for all $r, s \in I$ and $\lambda \in (0,1)$. When $p \geq 1$, then $q := 1/p \in (0,1]$ and $r \mapsto r^q$ is concave. Thus using the fact that f is decreasing, one has

$$\begin{aligned} f([(1-\lambda)r + \lambda s]^q) &\leq f((1-\lambda)r^q + \lambda s^q) \\ &\leq f((r^q)^{1-\lambda}(s^q)^{\lambda}) \leq [f(r^q)]^{1-\lambda}[f(s^q)]^{\lambda} \end{aligned} \tag{3.7}$$

for all $r, s \in I$ and $\lambda \in (0,1)$. So (3.6) and (3.7) imply that $r \mapsto f(r^q)$ is log-convex and, consequently, g is convex. Finally it is clear that the convexity of f with respect to Hölder means $H_p, p \in [0,\infty)$ implies the convexity of f with respect to H_0 and this is the multiplicative convexity. □

The decreasing homeomorphism $m : (0,1) \to (0,\infty)$, defined by

$$m(r) := \frac{{}_2F_1(a,b,a+b,1-r^2)}{{}_2F_1(a,b,a+b,r^2)},$$

and other various forms of this function were studied by R. Balasubramanian, S. Ponnusamy and M. Vuorinen [3] and also by S.L. Qiu and M. Vuorinen [17] (see also the references therein). In [3, Theorem 1.8], the authors proved that for $a \in (0,2)$ and $b \in (0, 2-a]$ the inequality

$$m(G(r,s)) \geq H(m(r), m(s)) \tag{3.8}$$

holds for all $r, s \in (0,1)$. In [5, Corollary 4.4] we proved that in fact (3.8) holds for all $a, b > 0$ and $r, s \in (0,1)$. Our aim in what follows is to generalize (3.8). Recall that in [3], in order to prove (3.8), the authors proved that the function $L : (0,\infty) \to (0,\infty)$, defined by

$$L(t) := \frac{F(e^{-t})}{F(1-e^{-t})},$$

is convex. In order to generalize (3.8) we prove that in fact L is convex with respect to Hölder means H_p, $p \in [1,\infty)$.

Corollary 3.3. *If $a, b > 0$ and $p \geq 1$, then the function L is convex on $(0,\infty)$ with respect to Hölder means H_p, i.e. for all $\lambda, r, s \in (0,1)$ and $a, b > 0$, $p \geq 1$ we have*

$$\frac{1-\lambda}{[m(r)]^p} + \frac{\lambda}{[m(s)]^p} \geq \frac{1}{[m(\alpha(r,s))]^p} \iff H_p^{(\lambda)}\left(\frac{1}{m(r)}, \frac{1}{m(s)}\right) \geq \frac{1}{m(\alpha(r,s))},$$

where $\alpha(r,s) = \exp\left[-H_p^{(\lambda)}(\log(1/r), \log(1/s))\right]$ and

$$H_p^{(\lambda)}(r,s) = \begin{cases} [(1-\lambda)r^p + \lambda s^p)]^{1/p}, & \text{if } p \neq 0, \\ r^{1-\lambda}s^{\lambda}, & \text{if } p = 0 \end{cases}$$

is the weighted version of H_p.

Proof. By [5, Lemma 2.12] we know that L is strictly decreasing and log-convex. Thus by part (b) of Theorem 3.2 we get that L is convex on $(0,\infty)$ with respect to Hölder means H_p for $p \in [1,\infty)$. This means that

$$L\left(H_p^{(\lambda)}(t_1,t_2)\right) \leq H_p^{(\lambda)}(L(t_1), L(t_2))$$

holds for all $t_1, t_2 > 0$, $\lambda \in (0,1)$ and $a, b > 0$. Now let $e^{-t_1} := r^2 \in (0,1)$ and $e^{-t_2} := s^2 \in (0,1)$, then we obtain that $L(t_1) = 1/m(r)$, $L(t_2) = 1/m(s)$ and $L\left(H_p^{(\lambda)}(t_1,t_2)\right) = 1/m(\alpha(r,s))$. Clearly, when $\lambda = 1/2$ and $p = 1$, we get that $\alpha(r,s) = G(r,s)$, thus the inequality in Corollary 3.3 reduces to (3.8). □

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