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THE METHOD OF LOWER AND UPPER SOLUTIONS FOR SOME FOURTH-ORDER EQUATIONS

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Abstract

In this paper, by combining a new maximum principle of fourth-order equations with the theory of eigenline problems, we develop a monotone method in the presence of lower and upper solutions for some fourth-order ordinary differential equation boundary value problem. Our results indicate there is a relation between the existence of solutions of nonlinear fourth-order equation and the first eigenline of linear fourth-order equation.

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Contents

1. Introduction

This paper consider solutions of the fourth-order boundary value problem

$$u^{(4)}(x) = f(x, u(x), u''(x)), \quad 0 < x < 1, \tag{1.1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

where $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

Many authors [1] – [8], [10], [11], [13] – [17] have studied this problem. In [1, 4, 6, 8, 10, 16], Aftabizadeh *et al.* showed the existence of positive solution to (1.1) – (1.2) under some growth conditions of f and a non-resonance condition involving a two-parameter linear eigenvalue problem. These results are based upon the Leray–Schauder continuation method and topological degree. In [2, 5, 7, 11, 15], Agarwal *et al.* considered an equation of the form

$$u^{(4)}(x) = f(x, u(x)),$$

with diverse kind of boundary conditions by using the lower and upper solution method.

Recently, Bai [3] and Ma *et al.* [14] developed the monotone method for the problem (1.1) – (1.2) under some monotone conditions of f. More recently, with using Krasnosel'skii fixed point theorem, Li [13] showed the existence results of positive solutions for the following problem

$$u^{(4)} + \beta u'' - \alpha u = f(t, u), \quad 0 < t < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

where $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $\alpha, \beta \in \mathbb{R}$ and $\beta < 2\pi^2, \alpha \geq -\beta^2/4, \alpha/\pi^4 + \beta/\pi^2 < 1$.

In this paper, by the use of a new maximum principle of fourth-order equation and the theory of the eigenline problem, we intend to further relax the monotone condition of f and get the iteration solution. Our results indicate there exists some relation between the existence of positive solutions of nonlinear fourth-order equation and the first eigenline of linear fourth-order equation.

2. Maximum Principle

In this section, we prove a maximum principle for the operator

$$L : F \longrightarrow C[0,1]$$

defined by $Lu = u^{(4)} - au'' + bu$. Here $a, b \in \mathbb{R}$ satisfy

$$\frac{a}{\pi^2} + \frac{b}{\pi^4} + 1 > 0, \quad a^2 - 4b \geq 0, \quad a > -2\pi^2; \tag{2.1}$$

$u \in F$ and

$$F = \{u \in C^4[0,1] \mid u(0) = 0,\ u(1) = 0,\ u''(0) \leq 0,\ u''(1) \leq 0\}.$$

Lemma 2.1. *[12] Let $f(x)$ be continuous for $a \leq x \leq b$ and let $c < \lambda_1 = \pi^2/(b-a)^2$. Let u satisfies*

$$u''(x) + cu(x) = f(x), \quad \text{for } x \in (a,b),$$

$$u(a) = u(b) = 0.$$

Assume that $u(x_1) = u(x_2) = 0$ where $a \leq x_1 < x_2 \leq b$ and $u(x) \neq 0$ for $x_1 \leq x \leq x_2$. If either $f(x) \geq 0$ for all $x \in [x_1, x_2]$ or $f(x) \leq 0$ for all $x \in [x_1, x_2]$ and $f(x)$ is not identically zero on $[x_1, x_2]$, then $u(x)f(x) \leq 0$ for all $x \in [x_1, x_2]$.

Lemma 2.2. *If $u(x)$ satisfies*

$$u'' + cu(x) \geq 0, \quad \text{for } x \in (a,b)$$

$$u(a) \leq 0, \quad u(b) \leq 0,$$

where $c < \lambda_1 = \pi^2/(b-a)^2$. Then $u(x) \leq 0$, in $[a,b]$.

Proof. It follows by Lemma 2.1. □

Lemma 2.3. *If* $u \in F$ *satisfies* $Lu \geq 0$, *then* $u \geq 0$ *in* $[0,1]$.

Proof. Set $Ax = x''$. As $a, b \in \mathbb{R}$ satisfy (2.1), we have that

$$Lu = u^{(4)} - au'' + bu = (A - r_2)(A - r_1)u \geq 0,$$

where $r_{1,2} = (a \pm \sqrt{a^2 - 4b})/2 \geq -\pi^2$. In fact, $r_1 = (a + \sqrt{a^2 - 4b})/2 \geq r_2 = (a - \sqrt{a^2 - 4b})/2$. By $a/\pi^2 + b/\pi^4 + 1 > 0$, we have $a\pi^2 + b + \pi^4 > 0$, thus $a^2 + 4a\pi^2 + 4\pi^4 > a^2 - 4b$, because $a^2 - 4b \geq 0$, so

$$(a + 2\pi^2)^2 > (\sqrt{a^2 - 4b})^2.$$

Combining this together with $a > -2\pi^2$, we can conclude

$$a + 2\pi^2 > \sqrt{a^2 - 4b}.$$

Then, $r_1 \geq r_2 = (a - \sqrt{a^2 - 4b})/2 > -\pi^2$.

Let $y = (A - r_1)u = u'' - r_1 u$, then

$$(A - r_2)y \geq 0,$$

i.e.,

$$y'' - r_2 y \geq 0.$$

On the other hand, $u \in F$ yields that

$$y(0) = u''(0) - r_1 u(0) \leq 0, \quad y(1) = u''(1) - r_1 u(1) \leq 0. \tag{2.2}$$

Therefore, by the use of Lemma 2.2, there exists

$$y(x) \le 0, \quad x \in [0,1],$$

i.e.,

$$u''(x) - r_1 u(x) = y(x) \le 0.$$

This together with Lemma 2.2 and the fact that $u(0) = 0, \;\; u(1) = 0$ implies that $u(x) \ge 0$ in $[0,1]$. □

Remark 2.1. *Observe that $a, b \in \mathbb{R}$ satisfies (2.1) if and only if*

$$b \le 0, \quad \frac{a}{\pi^2} + \frac{b}{\pi^4} + 1 > 0, \quad a > -2\pi^2; \tag{2.3}$$

or

$$b > 0, \quad a > 0, \quad a^2 - 4b \ge 0; \tag{2.4}$$

or

$$b > 0, \quad 0 > a > -2\pi^2, \quad \frac{a}{\pi^2} + \frac{b}{\pi^4} + 1 > 0, \quad a^2 - 4b \ge 0. \tag{2.5}$$

From (2.3) and (2.4), we can easily conclude

$$r_1 = \frac{a + \sqrt{a^2 - 4b}}{2} \ge 0.$$

Therefore, (2.2) can be obtained under $u(0) \ge 0, u(1) \ge 0, u''(0) \le 0, u''(1) \le 0$, and F can be defined as

$$F = \{u \in C^4[0,1] \mid u(0) \ge 0, \;\; u(1) \ge 0, \;\; u''(0) \le 0, \;\; u''(1) \le 0\},$$

we refer the reader to [3, 13].

Lemma 2.4. *[7] Given* $(a, b) \in \mathbb{R}^2$*, the following problem*

$$u^{(4)} - au'' + bu = 0, \tag{2.6}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{2.7}$$

has a non-trivial solution if and only if

$$\frac{a}{(k\pi)^2} + \frac{b}{(k\pi)^4} + 1 = 0,$$

for some $k \in \mathbb{N}$.

3. The Monotone Method

In this section, we develop the monotone method for the fourth order two-point boundary value problem (1.1) – (1.2).

For given $a, b \in \mathbb{R}$ satisfying $a/\pi^2 + b/\pi^4 + 1 > 0$, $a^2 - 4b \geq 0$, $a > -2\pi^2$ and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, let

$$f_1(x,u,v) = f(x,u,v) + bu - av. \tag{3.1}$$

Then (1.1) is equal to

$$Lu = f_1(x,u,u''). \tag{3.2}$$

Definition 3.1. *Letting $\alpha \in C^4[0,1]$, we say that α is an upper solution for the problem (1.1) – (1.2) if α satisfies*

$$\begin{aligned} \alpha^{(4)}(x) &\geq f(x,\alpha(x),\alpha''(x)), \quad \text{for } x \in (0,1), \\ \alpha(0) &= 0, \quad \alpha(1) = 0, \\ \alpha''(0) &\leq 0, \quad \alpha''(1) \leq 0. \end{aligned}$$

Definition 3.2. *Letting $\beta \in C^4[0,1]$, we say β is a lower solution for the problem (1.1) – (1.2) if β satisfies*

$$\begin{aligned} \beta^{(4)}(x) &\leq f(x,\beta(x),\beta''(x)), \quad \text{for } x \in (0,1), \\ \beta(0) &= 0, \quad \beta(1) = 0, \\ \beta''(0) &\geq 0, \quad \beta''(1) \geq 0. \end{aligned}$$

Remark 3.1. *If a, b satisfy (2.3) or (2.4), the boundary values can be replaced by*

$$\alpha(0) \geq 0, \quad \alpha(1) \geq 0; \qquad \beta(0) \leq 0, \quad \beta(1) \leq 0.$$

It is clear that if α, β are upper and lower solutions of the problem (1.1) – (1.2) respectively, α, β are upper and lower solutions of the problem (3.2) – (1.2) respectively, too.

Theorem 3.1. *If there exist α and β, upper and lower solutions, respectively, for the problem (1.1) – (1.2) which satisfy*

$$\beta \leq \alpha \quad \text{and} \quad \beta'' + r(\alpha - \beta) \geq \alpha'', \tag{3.3}$$

and if $f : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies

$$f(x, u_2, v) - f(x, u_1, v) \geq -b(u_2 - u_1), \tag{3.4}$$

for $\beta(x) \leq u_1 \leq u_2 \leq \alpha(x),\ v \in \mathbb{R}$, and $x \in [0,1]$;

$$f(x, u, v_2) - f(x, u, v_1) \leq a(v_2 - v_1), \tag{3.5}$$

for $v_2 + r(\alpha - \beta) \geq v_1,\ \alpha'' - r(\alpha - \beta) \leq v_1,\ v_2 \leq \beta'' + r(\alpha - \beta),\ u \in \mathbb{R}$, and $x \in [0,1]$, where $a, b \in \mathbb{R}$ satisfy $a/\pi^2 + b/\pi^4 + 1 > 0,\ a^2 - 4b \geq 0,\ a > -2\pi^2$ and $r = (a - \sqrt{a^2 - 4b})/2$, then there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, non-increasing and non-decreasing, respectively, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the extremal solutions in $[\beta, \alpha]$ of the problem (1.1) – (1.2).

Proof. Consider the problem

$$u^{(4)}(x) - au''(x) + bu(x) = f_1(x, \eta(x), \eta''(x)), \qquad \text{for } x \in (0,1), \tag{3.6}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{3.7}$$

with $\eta \in C^2[0,1]$.

Since $a/\pi^2 + b/\pi^4 + 1 > 0$, with the use of Lemma 2.4 and Fredholm Alternative [9], the problem (3.6) – (3.7) has a unique solution u. Define $T : C^2[0,1] \longrightarrow C^4[0,1]$ by

$$T\eta = u. \tag{3.8}$$

Now, we divide the proof into three steps.

Step 1. We show

$$TC \subseteq C. \tag{3.9}$$

Here, $C = \{\eta \in C^2[0,1] \mid \beta \le \eta \le \alpha, \ \ \alpha'' - r(\alpha - \beta) \le \eta'' \le \beta'' + r(\alpha - \beta)\}$ is a nonempty bounded closed subset in $C^2[0,1]$.

In fact, for $\zeta \in C$, set $\omega = T\zeta$. By the definition of α, β and C, combining (3.1), (3.4), and (3.5), we have that

$$\begin{aligned}(\alpha - \omega)^{(4)}(x) &- a(\alpha - \omega)''(x) + b(\alpha - \omega)(x) \\ &\ge f_1(x, \alpha(x), \alpha''(x)) - f_1(x, \zeta(x), \zeta''(x)) \\ &= f(x, \alpha(x), \alpha''(x)) - f(x, \zeta(x), \zeta''(x)) \\ &\qquad - a(\alpha - \zeta)''(x) + b(\alpha - \zeta)(x) \\ &\ge 0, \end{aligned} \tag{3.10}$$

$$(3.11) \qquad (\alpha-\omega)(0)=0, \quad (\alpha-\omega)(1)=0,$$

$$(3.12) \qquad (\alpha-\omega)''(0)\le 0, \quad (\alpha-\omega)''(1)\le 0.$$

With the use of Lemma 2.3, we obtain that $\alpha \ge \omega$. Analogously, there holds $\omega \ge \beta$.

By the proof of Lemma 2.3, combining (3.10), (3.11), and (3.12), we have that

$$(\alpha-\omega)''(x) - r(\alpha-\omega)(x) \le 0, \quad x \in (0,1),$$

hence,

$$\omega''(x) + r(\alpha-\beta)(x) \ge \omega''(x) + r(\alpha-\omega)(x) \ge \alpha''(x), \quad \text{for } x \in (0,1),$$

i.e.,

$$\omega''(x) \ge \alpha''(x) - r(\alpha-\beta)(x), \quad \text{for } x \in (0,1).$$

Analogously,

$$\omega''(x) \le \beta''(x) + r(\alpha-\beta)(x), \quad \text{for } x \in (0,1).$$

Thus, (3.9) holds.

Step 2. Let $u_1 = T\eta_1$, $u_2 = T\eta_2$, where $\eta_1, \eta_2 \in C$ satisfy $\eta_1 \le \eta_2$ and $\eta_1'' + r(\alpha-\beta) \ge \eta_2''$. We show

$$(3.13) \qquad u_1 \le u_2, \quad u_1'' + r(\alpha-\beta) \ge u_2''.$$

In fact, by (3.4), (3.5), and the definition of $u_1,\ u_2$,

$$L(u_2-u_1)(x)=f_1(x,\eta_2(x),\eta_2''(x))-f_1(x,\eta_1(x),\eta_1''(x))\geq 0,$$

$$(u_2-u_1)(0)=(u_2-u_1)(1)=0,$$
$$(u_2-u_1)''(0)=(u_2-u_1)''(1)=0.$$

With the use of Lemma 2.3, we get that $u_1\leq u_2$. Similar to Step 1, we can easily prove $u_1''+r(\alpha-\beta)\geq u_2''$. Thus, (3.13) holds.

Step 3. The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are obtained by recurrence:

$$\alpha_0=\alpha,\quad \beta_0=\beta,\quad \alpha_n=T\alpha_{n-1},\quad \beta_n=T\beta_{n-1},\ n=1,2,\dots.$$

From the results of Step 1 and Step 2, we have that

$$\beta=\beta_0\leq\beta_1\leq\cdots\leq\beta_n\leq\cdots\leq\alpha_n\leq\cdots\leq\alpha_1\leq\alpha_0=\alpha, \tag{3.14}$$

$$\beta''=\beta_0'',\quad \alpha''=\alpha_0'',\quad \alpha''-r(\alpha-\beta)\leq\alpha_n'',\quad \beta_n''\leq\beta''+r(\alpha-\beta). \tag{3.15}$$

Moreover, from the definition of T (see (3.8)), we get

$$\alpha_n^{(4)}(x)-a\alpha_n''(x)+b\alpha_n(x)=f_1(x,\alpha_{n-1}(x),\alpha_{n-1}''(x)),$$

i.e.,

$$\begin{aligned}\alpha_n^{(4)}(x)&=f_1(x,\alpha_{n-1}(x),\alpha_{n-1}''(x))+a\alpha_n''(x)-b\alpha_n(x)\\ &\leq f_1(x,\alpha_{n-1}(x),\alpha_{n-1}''(x))+a[\beta''+r(\alpha-\beta)](x)-b\beta(x),\end{aligned} \tag{3.16}$$

$$\alpha_n(0) = \alpha_n(1) = \alpha_n''(0) = \alpha_n''(1) = 0. \tag{3.17}$$

Analogously,

$$\begin{aligned} \beta_n^{(4)}(x) &= f_1(x, \beta_{n-1}(x), \beta_{n-1}''(x)) + a\beta_n''(x) - b\beta_n(x) \\ &\leq f_1(x, \beta_{n-1}(x), \beta_{n-1}''(x)) + a[\beta'' + r(\alpha - \beta)](x) - b\beta(x), \end{aligned} \tag{3.18}$$

$$\beta_n(0) = \beta_n(1) = \beta_n''(0) = \beta_n''(1) = 0. \tag{3.19}$$

From (3.14), (3.15), (3.16), and the continuity of f_1, we have that there exists $M_{\alpha,\beta} > 0$ depending only on α and β (but not on n or x) such that

$$|\alpha_n^{(4)}(x)| \leq M_{\alpha,\beta}, \quad \text{for all } x \in [0,1]. \tag{3.20}$$

Using the boundary condition (3.17), we get that for each $n \in \mathbb{N}$, there exists $\xi_n \in (0,1)$ such that

$$\alpha_n'''(\xi_n) = 0. \tag{3.21}$$

This together with (3.20) yields

$$|\alpha_n'''(x)| = |\alpha_n'''(\xi_n) + \int_{\xi_n}^{x} \alpha_n^{(4)}(s)ds| \leq M_{\alpha,\beta}. \tag{3.22}$$

By combining (3.15) and (3.17), we can similarly get that there is $C_{\alpha,\beta} > 0$ depending only on α and β (but not on n or x) such that

$$|\alpha_n''(x)| \leq C_{\alpha,\beta}, \quad \text{for all } x \in [0,1], \tag{3.23}$$

$$|\alpha_n'(x)| \leq C_{\alpha,\beta}, \quad \text{for all } x \in [0,1]. \tag{3.24}$$

Thus, from (3.14), (3.22), (3.23), and (3.24), we know that $\{\alpha_n\}$ is bounded in $C^3[0,1]$. Similarly, $\{\beta_n\}$ is bounded in $C^3[0,1]$.

Now, by using the fact that $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded in $C^3[0,1]$, we can conclude that $\{\alpha_n\}, \{\beta_n\}$ converge uniformly to the extremal solutions in $[0,1]$ of the problem (3.2) – (1.2). Therefore, $\{\alpha_n\}, \{\beta_n\}$ converge uniformly to the extremal solutions in $[0,1]$ of the problem (1.1) – (1.2), too. □

Example 3.1. *Consider the boundary value problem*

$$u^{(4)}(x) = -5u''(x) - (u(x)+1)^2 + \sin^2 \pi x + 1, \tag{3.25}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{3.26}$$

It is clear that the results of [3, 7, 13, 14] can't apply to the example. On the other hand, it is easy to check that $\alpha = \sin \pi x$, $\beta = 0$ are upper and lower solutions of (3.25) – (3.26), respectively. Letting $a = -5$, $b = 4$, then all assumptions of Theorem 3.1 are fulfilled. Hence the problem (3.25) – (3.26) has at least one solution u, which satisfies $0 \leq u \leq \sin \pi x$.

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