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NEW INEQUALITIES OF HADAMARD'S TYPE FOR LIPSCHITZIAN MAPPINGS

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ABSTRACT. In this paper, we study several new inequalities of Hadamard's type for Lipschitzian mappings.

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1. INTRODUCTION

Let us begin by defining some mappings that we shall need. If f is a continuous function on a closed interval $[a, b]$ $(a < b)$, and for any $t \in (0, 1)$, let $u = ta + (1 - t)b$, then we can define

$$A(t) \triangleq \frac{1}{t(1-t)(b-a)^2} \int_a^u \left[\int_u^b f\left(tx + (1-t)y\right) dy \right] dx,$$

$$B(t) \triangleq \frac{1}{(1-t)(b-a)^2} \int_a^u \left[\int_u^b f\left(\frac{(b-y)x + (y-u)u}{t(b-a)} \right) dy \right] dx + \frac{1}{t(b-a)^2} \int_a^u \left[\int_u^b f\left(\frac{(u-x)u + (x-a)y}{(1-t)(b-a)} \right) dy \right] dx,$$

$$C(t) \triangleq \frac{t}{(1-t)(b-a)} \int_a^u f(x)dx + \frac{1-t}{t(b-a)} \int_u^b f(y)dy.$$

Let f be a continuous convex function on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

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The inequalities in (1.1) are called Hadamard inequalities [1] – [7]. In [2], the author of this paper gave extensions and refinements of the inequalities in (1.1). He obtained the following:

$$f(ta+(1-t)b) \le A(t) \le B(t) \le C(t) \le tf(a)+(1-t)f(b). \tag{1.2}$$

Recently, Dragomir *et al.* [3], Yang and Tseng [4] and Matic and Pečarić [5] proved some results for Lipschitzian mappings related to (1.1). In this paper, we will prove some new inequalities for Lipschitzian mappings related to the mappings A, B and C (or to (1.2)).

2. MAIN RESULTS

For the mapping $A(t)$, we have the following theorem:

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a M-Lipschitzian mapping. For any $t\in(0,1)$, then*

$$|A(t)-f(ta+(1-t)b)| \le \frac{M}{3}t(1-t)(b-a). \tag{2.1}$$

Proof. Obviously, we have

$$b-u=t(b-a), u-a=(1-t)(b-a). \tag{2.2}$$

From integral properties and (2.2), we have

$$f(ta+(1-t)b)=\frac{1}{t(1-t)(b-a)^2}\int_a^u\left[\int_u^b f(ta+(1-t)b)\,dy\right]dx. \tag{2.3}$$

For $x\in[a,u]$, when $y\in\left[u,-\frac{t}{1-t}x+\frac{u}{1-t}\right]$, then we have

$$ta+(1-t)b=u\ge tx+(1-t)y, \tag{2.4}$$

if $y\in\left[-\frac{t}{1-t}x+\frac{u}{1-t},b\right]$, then the inequality (2.4) reverses.

Using (2.2) – (2.4) and integral properties, we obtain

$$\begin{aligned}
&|A(t)-f(ta+(1-t)b)|\\
&\le \frac{1}{t(1-t)(b-a)^2}\int_a^u\left[\int_u^b |f(tx+(1-t)y)-f(u)|\,dy\right]dx\\
&\le \frac{M}{t(1-t)(b-a)^2}\int_a^u\left[\int_u^b |tx+(1-t)y-u|\,dy\right]dx\\
&= \frac{M}{t(1-t)(b-a)^2}\int_a^u\left[\int_u^{-\frac{t}{1-t}x+\frac{u}{1-t}} (u-(tx+(1-t)y))\,dy\right]dx\\
&\qquad + \frac{M}{t(1-t)(b-a)^2}\int_a^u\left[\int_{-\frac{t}{1-t}x+\frac{u}{1-t}}^b (tx+(1-t)y-u)\,dy\right]dx\\
&= \frac{M}{t(1-t)(b-a)^2}\int_a^u\left[\frac{t^2}{1-t}x^2+t\left(b-\frac{1+t}{1-t}u\right)x\right.\\
&\qquad \left.+\frac{1+t^2}{2(1-t)}u^2+\frac{1-t}{2}b^2-bu\right]dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{M}{t(b-a)}\left[\frac{t^2}{3(1-t)}(u^2+ua+a^2)+\frac{t}{2}\left(b-\frac{1+t}{1-t}u\right)(u+a)\right.\\
&\qquad\left.+\frac{1+t^2}{2(1-t)}u^2+\frac{1-t}{2}b^2-bu\right]\\
&= \frac{M}{t(b-a)}\left[\frac{2t^2-3t+3}{6(1-t)}\left(ta+(1-t)b\right)^2\right.\\
&\qquad+\frac{-t(t+3)a-3(t^2-3t+2)b}{6(1-t)}\left(ta+(1-t)b\right)\\
&\qquad\left.+\frac{t^2}{3(1-t)}a^2+\frac{t}{2}ab+\frac{1-t}{2}b^2\right]\\
&= \frac{M}{3}t(1-t)(b-a).
\end{aligned}$$

This completes the proof of Theorem 2.1. □

For the mapping $B(t)$, we have the following theorem:

Theorem 2.2. *Let f be defined as in Theorem 2.1. For any $t \in (0,1)$, then*

$$|B(t)-A(t)| \le \frac{M}{2}t(1-t)(b-a), \tag{2.5}$$

and

$$|B(t)-f\left(ta+(1-t)b\right)| \le \frac{M}{2}t(1-t)(b-a). \tag{2.6}$$

Proof. From $\frac{1}{t(1-t)} = \frac{1}{t}+\frac{1}{1-t}$, we have

$$A(t) = \left(\frac{1}{t}+\frac{1}{1-t}\right)\frac{1}{(b-a)^2}\int_a^u\left[\int_u^b f\left(tx+(1-t)y\right)dy\right]dx. \tag{2.7}$$

Let F be defined by

$$F(y) = tx+(1-t)y-\frac{(b-y)x+(y-u)u}{t(b-a)}.$$

Obviously, if $F(y)$ is the function of first degree for y, then $F(y)$ is monotone with y on $[u,b]$. And for any $x \in [a,u]$, $F(u) = tx+(1-t)u-x \ge 0$, $F(b) = tx+(1-t)b-u \ge 0$. Hence for $x \in [a,u]$ and $y \in [u,b]$, we get $F(y) \ge 0$, i.e.,

$$tx+(1-t)y \ge \frac{(b-y)x+(y-u)u}{t(b-a)}. \tag{2.8}$$

Let G be defined by

$$G(x) = \frac{(u-x)u+(x-a)y}{(1-t)(b-a)}-\left(tx+(1-t)y\right).$$

Using $G(x)$ and the same method as in the proof of (2.8), we obtain

$$\frac{(u-x)u+(x-a)y}{(1-t)(b-a)} \ge tx+(1-t)y, \tag{2.9}$$

where, $x \in [a,u]$ and $y \in [u,b]$.

Using (2.7) – (2.9) and (2.2), we obtain

$$\begin{aligned}
&|B(t)-A(t)| \\
&\quad\le \frac{1}{(1-t)(b-a)^2}\int_a^u\left[\int_u^b\left|f\left(\frac{(b-y)x+(y-u)u}{t(b-a)}\right)-f\left(tx+(1-t)y\right)\right|dy\right]dx \\
&\qquad+\frac{1}{t(b-a)^2}\int_a^u\left[\int_u^b\left|f\left(\frac{(u-x)u+(x-a)y}{(1-t)(b-a)}\right)-f\left(tx+(1-t)y\right)\right|dy\right]dx \\
&\quad\le \frac{M}{(1-t)(b-a)^2}\int_a^u\left[\int_u^b\left|\frac{(b-y)x+(y-u)u}{t(b-a)}-(tx+(1-t)y)\right|dy\right]dx \\
&\qquad+\frac{M}{t(b-a)^2}\int_a^u\left[\int_u^b\left|\frac{(u-x)u+(x-a)y}{(1-t)(b-a)}-(tx+(1-t)y)\right|dy\right]dx \\
&\quad= \frac{M}{(1-t)(b-a)^2}\int_a^u\left[\int_u^b\left(tx+(1-t)y-\frac{(b-y)x+(y-u)u}{t(b-a)}\right)dy\right]dx \\
&\qquad+\frac{M}{t(b-a)^2}\int_a^u\left[\int_u^b\left(\frac{(u-x)u+(x-a)y}{(1-t)(b-a)}-(tx+(1-t)y)\right)dy\right]dx \\
&\quad= \frac{M}{t(1-t)(b-a)^3}\int_a^u\left[\int_u^b\Big((b-a)(2t-1)\left(tx+(1-t)y\right)\right. \\
&\qquad\qquad\left.+2xy-(b+u)x-(a+u)y+2u^2\Big)dy\right]dx \\
&\quad= \frac{M}{(1-t)(b-a)^2}\int_a^u\left[\frac{1}{2}\left((b-a)(2t-1)(1-t)-(a+u)\right)(b+u)\right. \\
&\qquad\qquad\left.+(b-a)(2t-1)tx+2u^2\right]dx \\
&\quad= \frac{M}{2(b-a)}\Big[\left((b-a)(2t-1)(1-t)-(a+u)\right)(b+u) \\
&\qquad\qquad+(b-a)(2t-1)t(u+a)+4u^2\Big] \\
&\quad= \frac{M}{2}t(1-t)(b-a).
\end{aligned}$$

This completes the proof of inequality (2.5).

From (2.3) and $\frac{1}{t(1-t)}=\frac{1}{t}+\frac{1}{1-t}$, we have

$$f\left(ta+(1-t)b\right)=\left(\frac{1}{t}+\frac{1}{1-t}\right)\frac{1}{(b-a)^2}\int_a^u\left[\int_u^b f\left(ta+(1-t)b\right)dy\right]dx. \tag{2.10}$$

If $x\in[a,u]$ and $y\in[u,b]$, then we have

$$\begin{aligned}
ta+(1-t)b=u&\ge\frac{(b-y)x+(y-u)u}{t(b-a)}, \\
\frac{(u-x)u+(x-a)y}{(1-t)(b-a)}&\ge ta+(1-t)b=u.
\end{aligned} \tag{2.11}$$

Using (2.10)-(2.11) and (2.2), we obtain

$$\begin{aligned}
&|B(t) - f(ta + (1-t)b)| \\
&\leq \frac{1}{(1-t)(b-a)^2}\int_a^u \left[\int_u^b \left| f\left(\frac{(b-y)x + (y-u)u}{t(b-a)}\right) - f(u)\right| dy\right] dx \\
&\quad + \frac{1}{t(b-a)^2}\int_a^u \left[\int_u^b \left| f\left(\frac{(u-x)u + (x-a)y}{(1-t)(b-a)}\right) - f(u)\right| dy\right] dx \\
&\leq \frac{M}{(1-t)(b-a)^2}\int_a^u \left[\int_u^b \left| \frac{(b-y)x + (y-u)u}{t(b-a)} - u\right| dy\right] dx \\
&\quad + \frac{M}{t(b-a)^2}\int_a^u \left[\int_u^b \left| \frac{(u-x)u + (x-a)y}{(1-t)(b-a)} - u\right| dy\right] dx \\
&= \frac{M}{(1-t)(b-a)^2}\int_a^u \left[\int_u^b \left(u - \frac{(b-y)x + (y-u)u}{t(b-a)}\right) dy\right] dx \\
&\quad + \frac{M}{t(b-a)^2}\int_a^u \left[\int_u^b \left(\frac{(u-x)u + (x-a)y}{(1-t)(b-a)} - u\right) dy\right] dx \\
&= \frac{M}{t(1-t)(b-a)^3}\int_a^u \left[\int_u^b \left((b-a)(2t-1)u + (2x-u-a)y - (b+u)x + 2u^2\right) dy\right] dx \\
&= \frac{M}{(1-t)(b-a)^2}\int_a^u \left[-\frac{1}{2}(u+a)(b+u) + (b-a)(2t-1)u + 2u^2\right] dx \\
&= \frac{M}{b-a}\left[-\frac{1}{2}(u+a)(b+u) + (b-a)(2t-1)u + 2u^2\right] \\
&= \frac{M}{2}t(1-t)(b-a).
\end{aligned}$$

This completes the proof of inequality (2.6).

This completes the proof of Theorem (2.2). □

For the mapping $C(t)$, we have the following theorem:

Theorem 2.3. *Let f be defined as in Theorem 2.1. For any $t \in (0,1)$, then*

$$|C(t) - B(t)| \leq \frac{M}{2}t(1-t)(b-a), \tag{2.12}$$

and

$$|C(t) - (tf(a) + (1-t)f(b))| \leq Mt(1-t)(b-a). \tag{2.13}$$

Proof. From the integral property and (2.2), we have

$$C(t) = \frac{1}{(1-t)(b-a)^2}\int_a^u \left[\int_u^b f(x)dy\right] dx + \frac{1}{t(b-a)^2}\int_a^u \left[\int_u^b f(y)dy\right] dx. \tag{2.14}$$

If $x \in [a,u]$ and $y \in [u,b]$, then we have

$$x \leq \frac{(b-y)x + (y-u)u}{t(b-a)}, \qquad \frac{(u-x)u + (x-a)y}{(1-t)(b-a)} \leq y. \tag{2.15}$$

Using (2.14) – (2.15) and (2.2), we obtain

$$
\begin{aligned}
&|C(t)-B(t)| \\
&\le \frac{1}{(1-t)(b-a)^2}\int_a^u\left[\int_u^b\left|f\left(\frac{(b-y)x+(y-u)u}{t(b-a)}\right)-f(x)\right|dy\right]dx \\
&\quad+\frac{1}{t(b-a)^2}\int_a^u\left[\int_u^b\left|f\left(\frac{(u-x)u+(x-a)y}{(1-t)(b-a)}\right)-f(y)\right|dy\right]dx \\
&\le \frac{M}{(1-t)(b-a)^2}\int_a^u\left[\int_u^b\left|\frac{(b-y)x+(y-u)u}{t(b-a)}-x\right|dy\right]dx \\
&\quad+\frac{1}{t(b-a)^2}\int_a^u\left[\int_u^b\left|\frac{(u-x)u+(x-a)y}{(1-t)(b-a)}-y\right|dy\right]dx \\
&= \frac{M}{(1-t)(b-a)^2}\int_a^u\left[\int_u^b\left(\frac{(b-y)x+(y-u)u}{t(b-a)}-x\right)dy\right]dx \\
&\quad+\frac{1}{t(b-a)^2}\int_a^u\left[\int_u^b\left(y-\frac{(u-x)u+(x-a)y}{(1-t)(b-a)}\right)dy\right]dx \\
&= \frac{M}{t(1-t)(b-a)^3} \\
&\quad\times\int_a^u\left[\int_u^b\left((b-a)\left((1-t)y-tx\right)-2xy+(b+u)x+(a+u)y-2u^2\right)dy\right]dx \\
&= \frac{M}{(1-t)(b-a)^2}\int_a^u\left[\frac{1}{2}\left((b-a)(1-t)+(a+u)\right)(b+u)-(b-a)tx-2u^2\right]dx \\
&= \frac{M}{b-a}\left[\frac{1}{2}\left(\left((b-a)(1-t)+(a+u)\right)(b+u)-(b-a)t(u+a)\right)-2u^2\right] \\
&= \frac{M}{2}t(1-t)(b-a).
\end{aligned}
$$

This completes the proof of inequality (2.12).

From integral property and (2.2), we have

$$
tf(a)+(1-t)f(b)=\frac{t}{(1-t)(b-a)}\int_a^u f(a)dx+\frac{1-t}{t(b-a)}\int_u^b f(b)dy. \tag{2.16}
$$

Using (2.16) and (2.2), we obtain

$$
\begin{aligned}
&|C(t)-(tf(a)+(1-t)f(b))| \\
&\le \frac{1}{b-a}\left[\frac{t}{1-t}\int_a^u|f(x)-f(a)|\,dx+\frac{1-t}{t}\int_u^b|f(y)-f(b)|\,dy\right] \\
&\le \frac{M}{b-a}\left[\frac{t}{1-t}\int_a^u|x-a|dx+\frac{1-t}{t}\int_u^b|y-b|dy\right] \\
&= \frac{M}{b-a}\left[\frac{t}{1-t}\int_a^u(x-a)dx+\frac{1-t}{t}\int_u^b(b-y)dy\right] \\
&= \frac{M}{b-a}\left[\frac{t}{1-t}\left(\frac{1}{2}(u^2-a^2)-a(u-a)\right)+\frac{1-t}{t}\left(b(b-u)-\frac{1}{2}(b^2-u^2)\right)\right] \\
&= Mt(1-t)(b-a).
\end{aligned}
$$

This completes the proof of inequality (2.13).

This completes the proof of Theorem 2.3. ☐

Corollary 2.4. *Let f be defined as in Theorem 2.1. For any $t \in (0,1)$, then*

$$|C(t) - f(ta + (1-t)b)| \le Mt(1-t)(b-a), \tag{2.17}$$

$$|A(t) - C(t)| \le Mt(1-t)(b-a), \tag{2.18}$$

$$|A(t) - (tf(a) + (1-t)f(b))| \le 2Mt(1-t)(b-a), \tag{2.19}$$

$$|B(t) - (tf(a) + (1-t)f(b))| \le \frac{3M}{2}t(1-t)(b-a), \tag{2.20}$$

and

$$|f(ta + (1-t)b) - (tf(a) + (1-t)f(b))| \le 2Mt(1-t)(b-a). \tag{2.21}$$

Proof. Using (2.5) and (2.12), we get inequality (2.18):

$$|A(t) - C(t)| \le |A(t) - B(t)| + |B(t) - C(t)| \le Mt(1-t)(b-a).$$

Using the same method as the proof of (2.18), from (2.6) and (2.12), (2.13) and (2.18), (2.12) and (2.13), and (2.13) and (2.17), we get (2.17), (2.19), (2.20) and (2.21), respectively.

This completes the proof of Corollary 2.4. ☐

Remark 2.5. If we let $t = \frac{1}{2}$, then (2.13), (2.17) and (2.21) reduce to

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \le \frac{M}{4}(b-a), \tag{2.22}$$

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{M}{4}(b-a), \tag{2.23}$$

$$\left|f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2}\right| \le \frac{M}{2}(b-a). \tag{2.24}$$

(2.22) is better than (2.2) in [3]. (2.23) and (2.24) are (2.1) and (2.4) in [3], respectively.

Corollary 2.6. *Let f be a convex mapping on $[a,b]$, with $f'_+(a)$ and $f'_-(b)$ existing. For any $t \in (0,1)$, then we have*

$$0 \le A(t) - f(ta + (1-t)b) \le \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{3}t(1-t)(b-a), \tag{2.25}$$

$$0 \le B(t) - A(t) \le \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{2}t(1-t)(b-a), \tag{2.26}$$

$$0 \le B(t) - f(ta + (1-t)b) \le \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{2}t(1-t)(b-a), \tag{2.27}$$

$$0 \le C(t) - B(t) \le \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{2}t(1-t)(b-a), \tag{2.28}$$

$$0 \le tf(a) + (1-t)f(b) - C(t) \le \max\{|f'_+(a)|, |f'_-(b)|\}t(1-t)(b-a), \tag{2.29}$$

$$0 \le C(t) - f(ta + (1-t)b) \le \max\{|f'_+(a)|, |f'_-(b)|\}t(1-t)(b-a), \tag{2.30}$$

$$0 \le C(t) - A(t) \le \max\{|f'_+(a)|, |f'_-(b)|\}t(1-t)(b-a), \tag{2.31}$$

$$0 \le tf(a) + (1-t)f(b) - A(t) \le 2\max\{|f'_+(a)|, |f'_-(b)|\}t(1-t)(b-a), \tag{2.32}$$

$$0 \le tf(a) + (1-t)f(b) - B(t) \le \frac{3}{2}\max\{|f'_+(a)|, |f'_-(b)|\}t(1-t)(b-a), \tag{2.33}$$

and

$$\begin{aligned} 0 &\le tf(a) + (1-t)f(b) - f(ta + (1-t)b) \\ &\le 2\max\{|f'_+(a)|, |f'_-(b)|\}t(1-t)(b-a). \end{aligned} \tag{2.34}$$

Proof. For any $x, y \in [a, b]$, from the properties of convex functions, we have the following $\max\{|f'_+(a)|, |f'_-(b)|\}$-Lipschitzian inequality (see [7]):

$$|f(x) - f(y)| \le \max\{|f'_+(a)|, |f'_-(b)|\}|x - y|. \tag{2.35}$$

Using (1.2), (2.35), Theorem 2.1 – 2.2 and Corollary 2.4, we obtain (2.25) – (2.34).

This completes the proof of Corollary (2.6). □

Remark 2.7. The condition in Corollary 2.6 is better than that in Corollary 2.2, Corollary 4.2 and Theorem 3.3 of [3]. Actually, it is that f is a differentiable convex mapping on $[a, b]$ with $M = \sup_{t\in[a,b]} |f'(t)| < \infty$.

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