journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 1, Article 22, 9 pp.

SOME PROPERTIES OF A NEW CLASS OF ANALYTIC FUNCTIONS DEFINED IN TERMS OF A HADAMARD PRODUCT

R. K. RAINA AND DEEPAK BANSAL

10/11, GANPATI VIHAR
OPPOSITE SECTOR 5, UDAIPUR 313002
RAJASTHAN, INDIA
rainark_7@hotmail.com

DEPARTMENT OF MATHEMATICS
SOBHASARIA ENGINEERING COLLEGE
GOKULPURA, NH-11, SIKAR-332001
RAJASTHAN, INDIA.
deepakbansal_79@yahoo.com

Received 15 April, 2007; accepted 12 January, 2008
Communicated by H.M. Srivastava

ABSTRACT. In this paper we introduce a new class $\mathcal{H}(\phi, \alpha, \beta)$ of analytic functions which is defined by means of a Hadamard product (or convolution) of two suitably normalized analytic functions. Several properties like, the coefficient bounds, growth and distortion theorems, radii of starlikeness, convexity and close-to-convexity are investigated. We further consider a subordination theorem, certain boundedness properties associated with partial sums, an integral transform of a certain class of functions, and some integral means inequalities. Several interesting consequnces of our main results are also pointed out.

Key words and phrases: Starlike function, Convex function, Close-to-convex function, Hadamard product, Ruscheweyh operator, Carlson and Schaffer operator, Subordination factor sequence, Schwarz function.

2000 *Mathematics Subject Classification.* Primary 30C45.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by $f(0) = f'(0) - 1 = 0$, and analytic in the open unit disk $\mathcal{U} = \{z;\ z \in \mathbb{C} : |z| < 1\}$, then $f(z)$ can be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Consider the subclass $\mathcal{T}$ of the class $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \tag{1.2}$$

The authors thank the referee for his comments.
120-07

then a function $f(z) \in \mathcal{A}$ is said to be in the class of uniformly β- starlike functions of order α (denoted by $USF(\alpha,\beta)$), if

$$\Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)}-1\right| \qquad (-1 \leqq \alpha < 1,\ \beta \geqq 0; z \in \mathcal{U}). \tag{1.3}$$

For $\alpha = 0$ in (1.3), we obtain the class of uniformly β-starlike functions which is denoted by $USF(\beta)$. Similarly, if $f(z) \in \mathcal{A}$ satisfies

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right| \qquad (-1 \leqq \alpha < 1,\ \beta \geqq 0; z \in \mathcal{U}), \tag{1.4}$$

then $f(z)$ is said to be in the class of uniformly β-convex functions of order α, and is denoted by $UCF(\alpha,\beta)$. When $\alpha = 0$ in (1.4), we obtain the class of uniformly β-convex functions which is denoted by $UCF(\beta)$.

The classes of uniformly convex and uniformly starlike functions have been extensively studied by Goodman ([2], [3]), Kanas and Srivastava [4], Kanas and Wisniowska [5], Ma and Minda [8] and Ronning [10].

If $f, g \in \mathcal{A}$ (where $f(z)$ is given by (1.1)), and $g(z)$ is defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.5}$$

then their Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.6}$$

We introduce here a class $\mathcal{H}(\phi,\alpha,\beta)$ which is defined as follows:
Suppose the function $\phi(z)$ is given by

$$\phi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k, \tag{1.7}$$

where $\mu_k \geqq 0$ $(\forall k \in \mathbb{N}\backslash\{1\})$. We say that $f(z) \in \mathcal{A}$ is in $\mathcal{H}(\phi,\alpha,\beta)$, provided that $(f * \phi)(z) \neq 0$, and

$$\Re\left\{\frac{z\left[(f*\phi)(z)\right]'}{(f*\phi)(z)}\right\} > \beta\left|\frac{z\left[(f*\phi)(z)\right]'}{(f*\phi)(z)}-1\right| + \alpha, \tag{1.8}$$
$$(-1 \leqq \alpha < 1,\ \beta \geqq 0; z \in \mathcal{U}).$$

Generally speaking, $\mathcal{H}(\phi,\alpha,\beta)$ consists of functions $F(z) = (f * \phi)(z)$ which are uniformly β-starlike functions of order α in $\mathcal{U}$.We also let

$$\mathcal{H}_{\mathcal{T}}(\phi,\alpha,\beta) = \mathcal{H}(\phi,\alpha,\beta) \cap \mathcal{T}. \tag{1.9}$$

Several known subclasses can be obtained from the class $\mathcal{H}(\phi,\alpha,\beta)$, by suitably choosing the values of the arbitrary function ϕ, and the parameters α and β. We mention below some of these subclasses of $\mathcal{H}(\phi,\alpha,\beta)$ consisting of functions $f(z) \in \mathcal{A}$. We observe that

$$\mathcal{H}\left\{\frac{z}{(1-z)^{\lambda+1}},\alpha,\beta\right\} = \mathcal{S}_p^{\lambda}(\alpha,\beta) \quad (-1 \leqq \alpha < 1,\ \beta \geqq 0,\ \lambda > -1; z \in \mathcal{U}), \tag{1.10}$$

in which case the function $\frac{z}{(1-z)^{\lambda+1}}$ is related to the Ruscheweyh derivative operator $D^{\lambda}f(z)$ ([12]) defined by

$$D^{\lambda}f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \qquad (\lambda > -1).$$

The class $\mathcal{S}_p^\lambda(\alpha,\beta)$ was studied by Rosy *et al.* [11] and Shams *et al.* [13] and this class also reduces to $\mathcal{S}(\alpha)$ and $\mathcal{K}(\alpha)$ which are, respectively, the familiar classes of starlike functions of order α $(0 \leqq \alpha < 1)$ and convex functions of order α $(0 \leqq \alpha < 1)$ (see [15]).

Also

$$\mathcal{H}\left\{\phi(a,c,z),\alpha,\beta\right\} = S(\alpha,\beta), \tag{1.11}$$

and in this case the function $\phi(a,c,z)$ is related to the Carlson and Shaffer operator $\mathcal{L}(a,c)f(z)$ ([1]) defined by

$$\mathcal{L}(a,c)f(z) = \phi(a,c,z) * f(z).$$

The class $S(\alpha,\beta)$ was studied by Murugusundaramoorthy and Magesh [9].

Further, we let

$$\mathcal{T}\mathcal{S}_p^\lambda(\alpha,\beta) = \mathcal{S}_p^\lambda(\alpha,\beta) \cap \mathcal{T}; \qquad \mathcal{T}S(\alpha,\beta) = S(\alpha,\beta) \cap \mathcal{T}. \tag{1.12}$$

The object of the present paper is to investigate the coefficient estimates, distortion properties and the radii of starlikeness, convexity and close-to-convexity for the class of functions $\mathcal{H}(\phi,\alpha,\beta)$. Further (for this class of functions), we obtain a subordination theorem, boundedness properties involving partial sums, properties relating to an integral transform and some integral mean inequalities. Several corollaries depicting interesting consequences of the main results are also mentioned.

2. COEFFICIENT ESTIMATES

We first mention a sufficient condition for the function $f(z)$ of the form (1.1) to belong to the class $\mathcal{H}(\phi,\alpha,\beta)$ given by the following result which can be established easily.

Theorem 2.1. *If $f(z) \in \mathcal{A}$ of the form (1.1) satisfies*

$$\sum_{k=2}^{\infty} \mathcal{B}_k\left(\mu_k;\alpha,\beta\right)|a_k| \leqq 1, \tag{2.1}$$

where

$$\mathcal{B}_k\left(\mu_k;\alpha,\beta\right) = \frac{\left\{k(\beta+1)-(\alpha+\beta)\right\}\mu_k}{1-\alpha}, \tag{2.2}$$

for some $\alpha(-1 \leqq \alpha < 1)$, $\beta(\beta \geqq 0)$ and $\mu_k \geqq 0$ $(\forall\, k \in \mathbb{N}\backslash\{1\})$, then $f(z) \in \mathcal{H}(\phi,\alpha,\beta)$.

Our next result shows that the condition (2.1) is necessary as well for functions of the form (1.2) to belong to the class of functions $\mathcal{H}_{\mathcal{T}}(\phi,\alpha,\beta)$.

Indeed, by using (1.2), (1.6) to (1.8), and in the process letting $z \to 1^-$ along the real axis, we arrive at the following:

Theorem 2.2. *A necessary and sufficient condition for $f(z)$ of the form (1.2) to be in $\mathcal{H}_{\mathcal{T}}(\phi,\alpha,\beta)$, $-1 \leqq \alpha < 1,\ \beta \geqq 0,\ \mu_k \geqq 0$ $(\forall k \in \mathbb{N}\backslash\{1\})$ is that*

$$\sum_{k=2}^{\infty} \mathcal{B}_k\left(\mu_k;\alpha,\beta\right)|a_k| \leqq 1, \tag{2.3}$$

where

$$\mathcal{B}_k\left(\mu_k;\alpha,\beta\right) = \frac{\left\{k(\beta+1)-(\alpha+\beta)\right\}\mu_k}{1-\alpha}. \tag{2.4}$$

Corollary 2.3. *Let $f(z)$ defined by (1.2) belong to the class $\mathcal{H}_{\mathcal{T}}(\phi,\alpha,\beta)$, then*

$$|a_k| \leqq \frac{1}{\mathcal{B}_k(\mu_k;\alpha,\beta)} \qquad (k \geqq 2). \tag{2.5}$$

The result is sharp (for each k), for functions of the form

$$f_k(z) = z - \frac{1}{\mathcal{B}_k(\mu_k;\alpha,\beta)} z^k \qquad (k = 2, 3, \dots), \tag{2.6}$$

where $\mathcal{B}_k(\mu_k;\alpha,\beta)$ is given by (2.4).

Remark 2.4. It is clear from (2.4) that if $\{\mu_k\}_{k=2}^{\infty}$ is a non-decreasing positive sequence, then $\{\mathcal{B}_k(\mu_k;\alpha,\beta)\}_{k=2}^{\infty}$ and $\left\{\frac{\mathcal{B}_k(\mu_k;\alpha,\beta)}{k}\right\}_{k=2}^{\infty}$ would also be non-decreasing positive sequences (being the product of two non-decreasing positive sequences).

Remark 2.5. By appealing to (1.10), we find that Theorems 2.1, 2.2 and Corollary 2.3 correspond, respectively, to the results due to Rosy *et al.* [11, Theorems 2.1, 2.2 and Corollary 2.3]. Similarly, making use of (1.11), then Theorems 2.1, 2.2 and Corollary 2.3, respectively, give the known theorems of Murugursundarmoorthy *et al.* [9, Theorems 2.1, 2.2 and Corollary 2.3].

3. GROWTH AND DISTORTION THEOREMS

In this section we state the following growth and distortion theorems for the class $\mathcal{H}_{\mathcal{T}}(\phi,\alpha,\beta)$. The results follow easily, therefore, we omit the proof details.

Theorem 3.1. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{H}_{\mathcal{T}}(\phi,\alpha,\beta)$. If $\{\mu_k\}_{k=2}^{\infty}$ is a positive non-decreasing sequence, then*

$$|z| - \frac{(1-\alpha)}{(2+\beta-\alpha)\mu_2}|z|^2 \leqq |f(z)| \leqq |z| + \frac{(1-\alpha)}{(2+\beta-\alpha)\mu_2}|z|^2. \tag{3.1}$$

The equality in (3.1) is attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{(2+\beta-\alpha)\mu_2}\, z^2. \tag{3.2}$$

Theorem 3.2. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{H}_{\mathcal{T}}(\phi,\alpha,\beta)$. If $\{\mu_k\}_{k=2}^{\infty}$ is a positive non-decreasing sequence, then*

$$1 - \frac{2\,(1-\alpha)}{(2+\beta-\alpha)\mu_2}|z| \leqq |f'(z)| \leqq 1 + \frac{2\,(1-\alpha)}{(2+\beta-\alpha)\mu_2}|z|. \tag{3.3}$$

The equality in (3.3) is attained for the function $f(z)$ given by (3.2).

In view of the relationships (1.10) and (1.11), Theorems 3.1 and 3.2 would yield the corresponding distortion properties for the classes $\mathcal{TS}_p^{\lambda}(\alpha,\beta)$ and $\mathcal{T}S(\alpha,\beta)$.

4. INTEGRAL TRANSFORM OF THE CLASS $\mathcal{H}_{\mathcal{T}}(\phi,\alpha,\beta)$

For $f(z) \in \mathcal{A}$, we define the integral transform

$$V_\mu(f(z)) = \int_0^1 \mu(t)\frac{f(tz)}{t}dt, \tag{4.1}$$

where μ is a real-valued non-negative weight function normalized, so that $\int_0^1 \mu(t)dt = 1$. In particular, when $\mu(t) = (1+\eta)t^\eta,\ \eta > -1$ then V_μ is a known Bernardi integral operator. On the other hand, if

$$\mu(t) = \frac{(1+\eta)^\delta}{\Gamma(\delta)} t^\eta \left(\log\frac{1}{t}\right)^{\delta-1} \qquad (\eta > -1,\ \delta > 0), \tag{4.2}$$

then V_μ becomes the Komatu integral operator (see [6]).

We first show that the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$ is closed under $V_\mu(f)$. By applying (1.2), (4.1) and (4.2), we straightforwardly arrive at the following result.

Theorem 4.1. *Let $f(z) \in \mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$, then $V_\mu(f(z)) \in \mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$.*

Following the usual methods of derivation, we can prove the following results:

Theorem 4.2. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$. Then $V_\mu(f(z))$ is starlike of order $\sigma(0 \leqq \sigma < 1)$ in $|z| < r_1$, where*

$$r_1 = \inf_k \left[\frac{\mathcal{B}_k(\mu_k; \alpha, \beta)(1-\sigma)}{(k-\sigma)\left(\frac{\eta+1}{\eta+k}\right)^\delta} \right]^{\frac{1}{k-1}} \qquad (k \geqq 2,\ \eta > -1,\ \delta > 0;\ z \in \mathcal{U}), \tag{4.3}$$

and $\mathcal{B}_k(\mu_k; \alpha, \beta)$ is given by (2.4). The result is sharp for the function $f(z)$ given by (3.2).

Theorem 4.3. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$. Then $V_\mu(f(z))$ is convex of order $\sigma(0 \leqq \sigma < 1)$ in $|z| < r_2$, where*

$$|z| < r_2 = \inf_k \left[\frac{\mathcal{B}_k(\mu_k; \alpha, \beta)(1-\sigma)}{k\,(k-\sigma)\left(\frac{\eta+1}{\eta+k}\right)^\delta} \right]^{\frac{1}{k-1}} \qquad (k \geqq 2,\ \eta > -1, \delta > 0;\ z \in \mathcal{U}), \tag{4.4}$$

and $\mathcal{B}_k(\mu_k; \alpha, \beta)$ is given by (2.4).

Theorem 4.4. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$. Then $V_\mu(f(z))$ is close-to-convex of order $\sigma(0 \leqq \sigma < 1)$ in $|z| < r_3$, where*

$$r_3 = \inf_k \left[\frac{\mathcal{B}_k(\mu_k; \alpha, \beta)(1-\sigma)}{k\left(\frac{\eta+1}{\eta+k}\right)^\delta} \right]^{\frac{1}{k-1}} \qquad (k \geqq 2,\ \eta > -1, \delta > 0;\ z \in \mathcal{U}), \tag{4.5}$$

and $\mathcal{B}_k(\mu_k; \alpha, \beta)$ is given by (2.4).

Remark 4.5. On choosing the arbitrary function $\phi(z)$, suitably in accordance with the subclass defined by (1.10), Theorems 4.1, 4.2 and 4.3, respectively, give the results due to Shams *et al.* [13, Theorems 1, 2 and 3]. Also, making use of (1.11), Theorems 4.1, 4.2, 4.3 and 4.4 yield the corresponding results for the class $\mathcal{T}S(\alpha, \beta)$.

5. SUBORDINATION THEOREM

Before stating and proving our subordination theorem for the class $\mathcal{H}(\phi, \alpha, \beta)$, we need the following definitions and a lemma due to Wilf [16].

Definition 5.1. For two functions f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$ (denoted by $f \prec g$), if there exists a Schwarz function $w(z)$, analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < |z| < 1$ $(z \in \mathcal{U})$, such that $f(z) = g(w(z))$.

Definition 5.2. A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is called a subordination factor sequence if whenever $f(z)$ is analytic, univalent and convex in $\mathcal{U}$, then

$$\sum_{k=1}^{\infty} b_k a_k z^k \prec f(z) \qquad (z \in \mathcal{U},\ a_1 = 1). \tag{5.1}$$

Lemma 5.1. *The sequence* $\{b_k\}_{k=1}^{\infty}$ *is a subordinating factor sequence if and if only*

$$\Re\left\{1+2\sum_{k=1}^{\infty} b_k z^k\right\} > 0, \qquad (z\in\mathcal{U}). \tag{5.2}$$

Theorem 5.2. *Let* $f(z)$ *of the form (1.1) satisfy the coefficient inequality (2.1), and* $\langle\mu_k\rangle_{k=2}^{\infty}$ *be a non-decreasing sequence, then*

$$\frac{(2+\beta-\alpha)\,\mu_2}{2\left\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\right\}}(f*g)(z)\prec g(z), \tag{5.3}$$
$$(-1\leqq\alpha<1,\ \beta\geqq 0,\ z\in\mathcal{U},\ \mu_k\geqq 0\ (\forall k\in\mathbb{N}\backslash\{1\}))$$

for every function $g(z)\in\mathcal{K}$ *(class of convex functions). In particular:*

$$\Re\left\{f(z)\right\} > \frac{-\left\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\right\}}{(2+\beta-\alpha)\,\mu_2}\ \ (z\in\mathcal{U})\,. \tag{5.4}$$

The constant factor

$$\frac{(2+\beta-\alpha)\,\mu_2}{2\left\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\right\}}, \tag{5.5}$$

in the subordination result (5.3) cannot be replaced by any larger one.

Proof. Let $f(z)$ defined by (1.1) satisfy the coefficient inequality (2.1). In view of (1.5) and Definition 5.2, the subordination (5.3) of our theorem will hold true if the sequence

$$\left\{\frac{(2+\beta-\alpha)\,\mu_2}{2\left\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\right\}}a_k\right\}_{k=1}^{\infty} \qquad (a_1=1), \tag{5.6}$$

is a subordinating factor sequence which by virtue Lemma 5.1 is equivalent to the inequality

$$\Re\left(1+\sum_{k=1}^{\infty}\frac{(2+\beta-\alpha)\,\mu_2}{\left\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\right\}}a_k z^k\right) > 0 \qquad (z\in\mathcal{U}). \tag{5.7}$$

In view of (2.1) and when $|z|=r$ $(0<r<1)$, we obtain

$$\begin{aligned}\Re&\left(1+\sum_{k=1}^{\infty}\frac{(2+\beta-\alpha)\,\mu_2}{\left\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\right\}}a_k z^k\right)\\ &\geqq 1-\frac{(2+\beta-\alpha)\,\mu_2}{\left\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\right\}}r\\ &\qquad -\sum_{k=2}^{\infty}\frac{(1-\alpha)}{\left\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\right\}}|a_k|r > 0.\end{aligned}$$

This evidently establishes the inequality (5.7), and consequently the subordination relation (5.3) of Theorem 5.2 is proved. The assertion (5.4) follows readily from (5.3) when the function $g(z)$ is selected as

$$g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k.$$

The sharpness of the multiplying factor in (5.3) can be established by considering a function $h(z)$ defined by

$$h(z)=z-\frac{1-\alpha}{(2+\beta-\alpha)\,\mu_2}z^2,$$

which belongs to the class $\mathcal{H}_{\mathcal{T}}(\phi,\alpha,\beta)$. Using (5.3), we infer that

$$\frac{(2+\beta-\alpha)\,\mu_2}{2\left\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\right\}}h(z)\prec\frac{z}{1-z}, \tag{5.8}$$

and it follows that

$$\inf_{|z|\leqq 1}\left\{\Re\left(\frac{(2+\beta-\alpha)\,\mu_2}{2\left\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\right\}}h(z)\right)\right\}=-\frac{1}{2}, \tag{5.9}$$

which completes the proof of Theorem 5.2. □

If we choose the sequence μ_k appropriately by comparing (1.7) with (1.10) and (1.11), we can deduce additional subordination results from Theorem 5.2.

6. PARTIAL SUMS

In this section we investigate the ratio of real parts of functions involving (1.2) and its sequence of partial sums defined by

$$f_1(z)=z;\quad \text{and}\quad f_N(z)=z-\sum_{k=2}^{N}|a_k|\,z^k \qquad (\forall k\in\mathbb{N}\backslash\{1\}), \tag{6.1}$$

and determine sharp lower bounds for $\Re\left\{f(z)/f_N(z)\right\}$, $\Re\left\{f_N(z)/f(z)\right\}$, $\Re\left\{f'(z)/f'_N(z)\right\}$ and $\Re\left\{f'_N(z)/f'(z)\right\}$.

Theorem 6.1. *Let $f(z)$ of the form (1.2) belong to $\mathcal{H}_{\mathcal{T}}(\phi,\alpha,\beta)$, and $\langle\mu_k\rangle_{k=2}^{\infty}$ be a non-decreasing sequence such that $\mu_2\geqq\frac{1-\alpha}{2+\beta-\alpha}$ $\left(0<\frac{1-\alpha}{2+\beta-\alpha}<1;-1\leqq\alpha<1,\beta\geqq 0\right)$, then*

$$\Re\left(\frac{f(z)}{f_N(z)}\right)\geqq 1-\frac{1}{\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)} \tag{6.2}$$

and

$$\Re\left(\frac{f_N(z)}{f(z)}\right)\geqq\frac{\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)}{\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)+1}, \tag{6.3}$$

where $\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)$ is given by (2.4). The results are sharp for every N, with the extremal functions given by

$$f(z)=z-\frac{1}{\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)}z^{N+1}\qquad (N\in\mathbb{N}\backslash\{1\}). \tag{6.4}$$

Proof. In order to prove (6.2), it is sufficient to show that

$$\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)\left[\frac{f(z)}{f_N(z)}-\left(1-\frac{1}{\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)}\right)\right]\prec\frac{1+z}{1-z}\qquad (z\in\mathcal{U}). \tag{6.5}$$

We can write

$$\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)\left[\frac{1-\sum\limits_{k=2}^{\infty}|a_k|\,z^{k-1}}{1-\sum\limits_{k=2}^{N}|a_k|\,z^{k-1}}-\left(1-\frac{1}{\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)}\right)\right]=\frac{1+w(z)}{1-w(z)}.$$

Obviously $w(0) = 0$, and

$$|w(z)| \leqq \frac{\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)\sum\limits_{k=N+1}^{\infty}|a_k|}{2-2\sum\limits_{k=2}^{N}|a_k| - \mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)\sum\limits_{k=N+1}^{\infty}|a_k|},$$

which is less than one if and only if

$$\sum_{k=2}^{N}|a_k| + \mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)\sum_{k=N+1}^{\infty}|a_k| \leqq 1. \tag{6.6}$$

In view of (2.3), this is equivalent to showing that

$$\sum_{k=2}^{N}\{\mathcal{B}_k(\mu_k;\alpha,\beta)-1\}|a_k| + \sum_{k=N+1}^{\infty}\{\mathcal{B}_k(\mu_k;\alpha,\beta) - \mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)\}|a_k| \geqq 0. \tag{6.7}$$

We observe that the first term of the first series in (6.7) is positive when $\mu_2 \geqq \frac{1-\alpha}{2+\beta-\alpha}$, which is true (in view of the hypothesis). Now, since $\{\mathcal{B}_k(\mu_k;\alpha,\beta)\}_{k=2}^{\infty}$ is a non-decreasing sequence (see Remark 2.4), therefore, all the other terms in the first series are positive. Also, the first term of the second series in (6.7) vanishes, and all other terms of this series also remain positive. Thus, the inequality (6.7) holds true. This completes the proof of (6.2). Finally, it can be verified that the equality in (6.2) is attained for the function given by (6.4) when, $z = re^{2\pi i/N}$ and $r \to 1^-$.

The proof of (6.3) is similar to (6.2), and is hence omitted. □

Similarly, we can establish the following theorem.

Theorem 6.2. *Let $f(z)$ of the form (1.2) belong to $\mathcal{H}_{\mathcal{T}}(\phi,\alpha,\beta)$, and $\langle\mu_k\rangle_{k=2}^{\infty}$ be a non-decreasing sequence such that $\mu_2 \geqq \frac{2(1-\alpha)}{2+\beta-\alpha}$ $\left(0 < \frac{1-\alpha}{2+\beta-\alpha} < 1; -1 \leqq \alpha < 1, \beta \geqq 0\right)$, then*

$$\Re\left(\frac{f'(z)}{f_N'(z)}\right) \geqq 1 - \frac{N+1}{\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)} \tag{6.8}$$

and

$$\Re\left(\frac{f_N'(z)}{f'(z)}\right) \geqq \frac{\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)}{N+1+\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)}, \tag{6.9}$$

where $\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)$ is given by (2.4). The results are sharp for every N, with the extremal functions given by (6.4).

Making use of (1.10) to (1.12), then Theorems 6.1 and 6.2 would yield the corresponding results for the classes $\mathcal{TS}_p^{\lambda}(\alpha,\beta)$ and $\mathcal{T}S(\alpha,\beta)$.

7. INTEGRAL MEANS INEQUALITIES

The following subordination result due to Littlewood [7] will be required in our investigation.

Lemma 7.1. *If $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$ with $f(z) \prec g(z)$, then*

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\mu}d\theta \leqq \int_0^{2\pi}\left|g(re^{i\theta})\right|^{\mu}d\theta, \tag{7.1}$$

where $\mu > 0,\ z = re^{i\theta}$ and $0 < r < 1$.

Application of Lemma 7.1 for functions $f(z)$ in the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$ gives the following result using known procedures.

Theorem 7.2. *Let $\mu > 0$. If $f(z) \in \mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$ is given by (1.2), and $\{\mu_k\}_{k=2}^{\infty}$, is a non-decreasing sequence, then, for $z = re^{i\theta}$ $(0 < r < 1)$:*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|f_1(re^{i\theta})\right|^{\mu} d\theta, \tag{7.2}$$

where

$$f_1(z) = z - \frac{(1-\alpha)}{(2+\beta-\alpha)\mu_2} z^2. \tag{7.3}$$

We conclude this paper by observing that several integral means inequalities can be deduced from Theorem 7.2 in view of the relationships (1.10) and (1.11).

REFERENCES

[1] B.C. CARLSON AND S.B. SHAFFER, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, **15** (2002), 737–745.

[2] A.W. GOODMAN, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), 87–92.

[3] A.W. GOODMAN, On uniformly starlike functions, *J. Math. Anal. Appl.*, **155** (1991), 364–370.

[4] S. KANAS AND H.M. SRIVASTAVA, Linear operators associated with k-uniformly convex functions, *Integral Transforms Spec. Funct.*, **9** (2000), 121–132.

[5] S. KANAS AND A. WISNIOWSKA, Comic domains and starlike functions, *Rev. Roumaine Math. Pures Appl.*, **45** (2000), 647–657.

[6] Y.C. KIM AND F. RONNING, Integral transforms of certain subclasses of analytic functions, *J. Math. Anal. Appl.*, **258** (2001), 466–489.

[7] J.E. LITTLEWOOD, On inequalities in the theory of functions, *Proc. London Math. Soc.*, **23** (1925), 481–519.

[8] W. MA AND D. MINDA, Uniformly convex functions, *Ann. Polon. Math*, **57** (1992), 165–175.

[9] G. MURUGUSUNDARAMOORTHY AND N. MAGESH, A new subclass of uniformly convex functions and a corresponding subclass of starlike functions with fixed second coefficient, *J. Inequal. Pure Appl. Math.*, **5**(4) (2004), Art. 85. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=438`].

[10] F. RONNING, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118** (1993), 189–196.

[11] T. ROSY, K.G. SUBRAMANIAN AND G. MURUGUSUNDARAMOORTHY, Neighbourhoods and partial sums of starlike functions based on Ruscheweyh derivatives, *J. Ineq. Pure Appl. Math.*, **4**(4) (2003), Art. 64. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=305`].

[12] S. RUSCHEWEYH, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, **49** (1975), 109–115.

[13] S. SHAMS, S.R. KULKARNI AND J.M. JAHANGIRI, On a class of univalent functions defined by Ruscheweyh derivatives, *Kyungpook Math. J.*, **43** (2003), 579–585.

[14] S. SHAMS, S.R. KULKARNI AND J.M. JAHANGIRI, Classes of Ruscheweyh type analytic univalent functions, *Southwest J. Pure. Appl. Math.*, **2** (2003), 105–110.

[15] H.M. SRIVASTAVA AND S. OWA (Eds.), Current Topics in Analytic FunctionTheory, World Scientific Publishing Company, Singapore, New Jersey, London and HongKong, 1992.

[16] H. S. WILF, Subordinating factor sequences for convex maps of the unit circle, *Proc. Amer. Math. Soc.*, **12** (1961), 689–693.