journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 9 (2008), Issue 4, Article 111, 5 pp.

λ-CENTRAL BMO ESTIMATES FOR COMMUTATORS OF N-DIMENSIONAL HARDY OPERATORS

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Received 12 April, 2008; accepted 13 October, 2008
Communicated by R.N. Mohapatra

ABSTRACT. This paper gives the λ-central BMO estimates for commutators of n-dimensional Hardy operators on central Morrey spaces.

Key words and phrases: Commutator, N-dimensional Hardy operator, λ-central BMO space, Central Morrey space.

2000 *Mathematics Subject Classification.* 26D15, 42B25, 42B99.

1. INTRODUCTION AND MAIN RESULTS

Let f be a locally integrable function on $\mathbb{R}^n$. The n-dimensional Hardy operators are defined by

$$\mathcal{H}f(x) := \frac{1}{|x|^n}\int_{|t|\leq|x|} f(t)dt, \quad \mathcal{H}^*f(x) := \int_{|t|>|x|}\frac{f(t)}{|t|^n}dt, \qquad x\in\mathbb{R}^n\setminus\{0\}.$$

In [4], Christ and Grafakos obtained results for the boundedness of $\mathcal{H}$ on $L^p(\mathbb{R}^n)$ spaces. They also found the exact operator norms of $\mathcal{H}$ on $L^p(\mathbb{R}^n)$ spaces, where $1<p<\infty$.

It is easy to see that $\mathcal{H}$ and $\mathcal{H}^*$ satisfy

$$\int_{\mathbb{R}^n} g(x)\mathcal{H}f(x)\,dx = \int_{\mathbb{R}^n} f(x)\mathcal{H}^*g(x)\,dx. \tag{1.1}$$

We have

$$|\mathcal{H}f(x)| \leq C_n Mf(x),$$

where M is the Hardy-Littlewood maximal operator which is defined by

$$Mf(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(t)|dt, \tag{1.2}$$

where the supremum is taken over all balls containing x.

The research is supported by the NNSF (Grant No. 10571014; 10871024) of People's Republic of China.

The author would like to express his thanks to Prof. Shanzhen Lu for his constant encourage. This paper is dedicated to him for his 70^{th} birthday. The author also would like to express his gratitude to the referee for his valuable comments.

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Recently, Fu et al. [2] gave the definition of commutators of n-dimensional Hardy operators.

Definition 1.1. Let b be a locally integrable function on $\mathbb{R}^n$. We define the commutators of n-dimensional Hardy operators as follows:

$$\mathcal{H}_b f := b\mathcal{H}f - \mathcal{H}(fb), \qquad \mathcal{H}_b^* f := b\mathcal{H}^* f - \mathcal{H}^*(fb).$$

In [2], Fu et al. gave the central BMO estimates for commutators of n-dimensional Hardy operators. In 2000, Alvarez, Guzmán-Partida and Lakey [1] studied the relationship between central BMO spaces and Morrey spaces. Furthermore, they introduced λ-central bounded mean oscillation spaces and central Morrey spaces, respectively.

Definition 1.2 (λ-central BMO space)**.** Let $1 < q < \infty$ and $-\frac{1}{q} < \lambda < \frac{1}{n}$. A function $f \in L^q_{loc}(\mathbb{R}^n)$ is said to belong to the λ-central bounded mean oscillation space $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{C\dot{M}O^{q,\lambda}(\mathbb{R}^n)} = \sup_{\mathrm{R}>0}\left(\frac{1}{|B(0,\mathrm{R})|^{1+\lambda q}}\int_{B(0,\mathrm{R})}|f(x) - f_{B(0,\mathrm{R})}|^q dx\right)^{\frac{1}{q}} < \infty. \tag{1.3}$$

Remark 1. *If two functions which differ by a constant are regarded as a function in the space $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$, then $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$ becomes a Banach space. Apparently, (1.3) is equivalent to the following condition (see* [1]*):*

$$\sup_{\mathrm{R}>0}\inf_{c\in\mathbb{C}}\left(\frac{1}{|B(0,\mathrm{R})|^{1+\lambda q}}\int_{B(0,\mathrm{R})}|f(x) - c|^q dx\right)^{\frac{1}{q}} < \infty.$$

Definition 1.3 (Central Morrey spaces, see [1])**.** Let $1 < q < \infty$ and $-\frac{1}{q} < \lambda < 0$. The central Morrey space $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ is defined by

$$\|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} = \sup_{\mathrm{R}>0}\left(\frac{1}{|B(0,\mathrm{R})|^{1+\lambda q}}\int_{B(0,\mathrm{R})}|f(x)|^q dx\right)^{\frac{1}{q}} < \infty. \tag{1.4}$$

Remark 2. *It follows from (1.3) and (1.4) that $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ is a Banach space continuously included in $C\dot{M}O^{q,\lambda}(\mathbb{R}^n)$.*

Inspired by [2], [3] and [5], we will establish the λ-central BMO estimates for commutators of n-dimensional Hardy operators on central Morrey spaces.

Theorem 1.1. *Let $\mathcal{H}_b$ be defined as above. Suppose $1 < p_1 < \infty$, $p_1' < p_2 < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, $-\frac{1}{q} < \lambda < 0, 0 \le \lambda_2 < \frac{1}{n}$ and $\lambda = \lambda_1 + \lambda_2$. If $b \in C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)$, then the commutator $\mathcal{H}_b$ is bounded from $\dot{B}^{p_1,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and satisfies the following inequality:*

$$\|\mathcal{H}_b f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} \le C\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}.$$

Let $\lambda_2 = 0$ in Theorem 1.1. We can obtain the central BMO estimates for commutators of n-dimensional Hardy operators, $\mathcal{H}_b$, on central Morrey spaces.

Corollary 1.2. *Let $\mathcal{H}_b$ be defined as above. Suppose $1 < p_1 < \infty$, $p_1' < p_2 < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$ and $-\frac{1}{q} < \lambda < 0$. If $b \in C\dot{M}O^{p_2}(\mathbb{R}^n)$, then the commutator $\mathcal{H}_b$ is bounded from $\dot{B}^{p_1,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and satisfies the following inequality:*

$$\|\mathcal{H}_b f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} \le C\|b\|_{C\dot{M}O^{p_2}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p_1,\lambda}(\mathbb{R}^n)}.$$

Similar to Theorem 1.1, we have:

Theorem 1.3. *Let $\mathcal{H}_b^*$ be defined as above. Suppose $1 < p_1 < \infty$, $p_1' < p_2 < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, $-\frac{1}{q} < \lambda < 0$, $0 \le \lambda_2 < \frac{1}{n}$ and $\lambda = \lambda_1 + \lambda_2$. If $b \in C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)$, then the commutator $\mathcal{H}_b^*$ is bounded from $\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)$ to $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and satisfies the following inequality:*

$$\|\mathcal{H}_b^* f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} \le C\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}.$$

Let $\lambda_2 = 0$ in Theorem 1.3. We can get the central BMO estimates for commutators of n-dimensional Hardy operators, $\mathcal{H}_b^*$, on central Morrey spaces.

Corollary 1.4. *Let $\mathcal{H}_b^*$ be defined as above. Suppose $1 < p_1 < \infty$, $p_1' < p_2 < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$ and $-\frac{1}{q} < \lambda < 0$. If $b \in C\dot{M}O^{p_2}(\mathbb{R}^n)$, then the commutator $\mathcal{H}_b^*$ is bounded from $\dot{B}^{p_1,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and satisfies the following inequality:*

$$\|\mathcal{H}_b^* f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} \le C\|b\|_{C\dot{M}O^{p_2}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p_1,\lambda}(\mathbb{R}^n)}.$$

2. PROOFS OF THEOREMS

Proof of Theorem 1.1. Let f be a function in $\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)$. For fixed $R > 0$, denote $B(0,R)$ by B. Write

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |\mathcal{H}_b f(x)|^q dx\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{|B|}\int_B \left|\frac{1}{|x|^n}\int_{B(0,|x|)} f(y)(b(x)-b(y))\,dy\right|^q dx\right)^{\frac{1}{q}} \\
&\le \left(\frac{1}{|B|}\int_B \left|\frac{1}{|x|^n}\int_{B(0,|x|)} f(y)(b(x)-b_B)\,dy\right|^q dx\right)^{\frac{1}{q}} \\
&\qquad + \left(\frac{1}{|B|}\int_B \left|\frac{1}{|x|^n}\int_{B(0,|x|)} f(y)(b(y)-b_B)\,dy\right|^q dx\right)^{\frac{1}{q}} \\
&:= I + J.
\end{aligned}$$

For $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, by Hölder's inequality and the boundedness of $\mathcal{H}$ from L^{p_1} to L^{p_1}, we have

$$\begin{aligned}
I &\le |B|^{-\frac{1}{q}}\left(\int_B |b(x)-b_B|^{p_2}dx\right)^{\frac{1}{p_2}}\left(\int_B |\mathcal{H}(f\chi_B)(x)|^{p_1}dx\right)^{\frac{1}{p_1}} \\
&\le C|B|^{-\frac{1}{q}}\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}|B|^{\frac{1}{p_2}+\lambda_2}\left(\int_B |f(x)|^{p_1}dx\right)^{\frac{1}{p_1}} \\
&= C|B|^{\lambda}\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\left(\frac{1}{|B|^{1+p_1\lambda_1}}\int_B |f(x)|^{p_1}dx\right)^{\frac{1}{p_1}} \\
&\le C|B|^{\lambda}\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}.
\end{aligned}$$

For J, we have

$$\begin{aligned}
J^q &= \frac{1}{|B|}\int_B \left|\frac{1}{|x|^n}\int_{B(0,|x|)} f(y)(b(y)-b_B)\,dy\right|^q dx \\
&= \frac{1}{|B|}\sum_{k=-\infty}^{0}\int_{2^kB\setminus 2^{k-1}B}\left|\frac{1}{|x|^n}\int_{B(0,|x|)} f(y)(b(y)-b_B)\,dy\right|^q dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C}{|B|}\sum_{k=-\infty}^{0}\frac{1}{|2^kB|^q}\int_{2^kB\backslash 2^{k-1}B}\left|\sum_{i=-\infty}^{k}\int_{2^iB\backslash 2^{i-1}B} f(y)(b(y)-b_B)\,dy\right|^q dx\\
&\leq \frac{C}{|B|}\sum_{k=-\infty}^{0}\frac{1}{|2^kB|^q}\int_{2^kB\backslash 2^{k-1}B}\left|\sum_{i=-\infty}^{k}\int_{2^iB\backslash 2^{i-1}B} f(y)(b(y)-b_{2^iB})\,dy\right|^q dx\\
&\qquad+\frac{C}{|B|}\sum_{k=-\infty}^{0}\frac{1}{|2^kB|^q}\int_{2^kB\backslash 2^{k-1}B}\left|\sum_{i=-\infty}^{k}\int_{2^iB\backslash 2^{i-1}B} f(y)(b_{2^iB}-b_B)\,dy\right|^q dx\\
&:= J_1+J_2
\end{aligned}
$$

By Hölder's inequality ($\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q}$), we have

$$
\begin{aligned}
J_1 &\leq \frac{C}{|B|}\sum_{k=-\infty}^{0}\frac{|2^kB|}{|2^kB|^q}\left\{\sum_{i=-\infty}^{k}|2^iB|^{\frac{1}{q'}}\left(\int_{2^iB}|f(y)|^{p_1}\,dy\right)^{\frac{1}{p_1}}\right.\\
&\qquad\left.\times\left(\int_{2^iB}|b(y)-b_{2^iB}|^{p_2}\,dy\right)^{\frac{1}{p_2}}\right\}^q\\
&\leq \frac{C}{|B|}\|b\|^q_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|^q_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\frac{|2^kB|}{|2^kB|^q}\left\{\sum_{i=-\infty}^{k}|2^iB|^{\lambda+1}\right\}^q\\
&\leq C|B|^{q\lambda}\|b\|^q_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|^q_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}.
\end{aligned}
$$

To estimate J_2, the following fact is applied.

For $\lambda_2\geq 0$,

$$
\begin{aligned}
|b_{2^iB}-b_B| &\leq \sum_{j=i}^{-1}|b_{2^{j+1}B}-b_{2^jB}|\\
&\leq \sum_{j=i}^{-1}\frac{1}{|2^jB|}\int_{2^jB}|b(y)-b_{2^{j+1}B}|dy\\
&\leq C\sum_{j=i}^{-1}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B}|^{p_2}dy\right)^{\frac{1}{p_2}}\\
&\leq C\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}|B|^{\lambda_2}\sum_{j=i}^{-1}2^{(j+1)n\lambda_2}\\
&\leq C\|b\|_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}|i||B|^{\lambda_2}.
\end{aligned}
$$

By Hölder's inequality ($\frac{1}{p_1}+\frac{1}{p_1'}=1$), we have

$$
\begin{aligned}
J_2 &= \frac{C}{|B|}\sum_{k=-\infty}^{0}\frac{1}{|2^kB|^q}\int_{2^kB\backslash 2^{k-1}B}\left|\sum_{i=-\infty}^{k}\int_{2^iB\backslash 2^{i-1}B} f(y)(b_{2^iB}-b_B)\,dy\right|^q dx\\
&\leq \frac{C}{|B|}\|b\|^q_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|^q_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\frac{|2^kB||B|^{q\lambda_2}}{|2^kB|^q}\left\{\sum_{i=-\infty}^{k}|i||2^iB|^{\lambda_1+1}\right\}^q
\end{aligned}
$$

$$\leq \frac{C}{|B|}\|b\|^q_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|^q_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\frac{|2^kB||B|^{q\lambda_2}|k|^q|2^kB|^{(\lambda_1+1)q}}{|2^kB|^q}$$
$$\leq C|B|^{q\lambda}\|b\|^q_{C\dot{M}O^{p_2,\lambda_2}(\mathbb{R}^n)}\|f\|^q_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)}.$$

Combining the estimates of I, J_1 and J_2, we get the required estimate for Theorem 1.1. □

Proof of Theorem 1.3. We omit the details here. □

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