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INEQUALITIES IN $q-$FOURIER ANALYSIS

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ABSTRACT. In this paper we introduce the $q-$Bessel Fourier transform, the $q-$Bessel translation operator and the $q-$convolution product. We prove that the $q-$heat semigroup is contractive and we establish the $q-$analogue of Babenko inequalities associated to the $q-$Bessel Fourier transform. With applications and finally we enunciate a $q-$Bessel version of the central limit theorem.

Key words and phrases: Čebyšev functional, Grüss inequality, Bessel, Beta and Zeta function bounds.

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1. INTRODUCTION AND PRELIMINARIES

In introducing $q-$Bessel Fourier transforms, the $q-$Bessel translation operator and the $q-$convolution product we shall use the standard conventional notation as described in [4]. For further detailed information on $q-$derivatives, Jackson $q-$integrals and basic hypergeometric series we refer the interested reader to [4], [10], and [8].

The following two propositions will useful for the remainder of the paper.

Proposition 1.1. *Consider* $0 < q < 1$. *The series*

$$(w;q)_{\infty 1}\phi_1(0,w;q;z) = \sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n-1)}{2}} \frac{(wq^n;q)_\infty}{(q;q)_n} z^n,$$

defines an entire analytic function in z, w*, which is also symmetric in* z, w*:*

$$(w;q)_{\infty 1}\phi_1(0,w;q;z) = (z;q)_{\infty 1}\phi_1(0,z;q;w).$$

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Both sides can be majorized by

$$|(w;q)_{\infty 1}\phi_1(0,w;q;z)| \leq (-|w|;q)_\infty(-|z|;q)_\infty.$$

Finally, for all $n \in \mathbb{N}$ we have

$$(q^{1-n};q)_{\infty 1}\phi_1(0,q^{1-n};q;z) = (-z)^n q^{\frac{n(n-1)}{2}}(q^{1+n};q)_{\infty 1}\phi_1(0,q^{1+n};q;q^n z).$$

Proof. See [10]. □

Now we introduce the following functional spaces:

$$\mathbb{R}_q = \{\mp q^n, \quad n \in \mathbb{Z}\}, \quad \mathbb{R}_q^+ = \{q^n,\ n \in \mathbb{Z}\}.$$

Let $\mathcal{D}_q$, $\mathcal{C}_{q,0}$ and $\mathcal{C}_{q,b}$ denote the spaces of even smooth functions defined on $\mathbb{R}_q$ continuous at 0, which are respectively with compact support, vanishing at infinity and bounded. These spaces are equipped with the topology of uniform convergence, and by $\mathcal{L}_{q,p,v}$ the space of even functions f defined on $\mathbb{R}_q$ such that

$$\|f\|_{q,p,v} = \left[\int_0^\infty |f(x)|^p x^{2v+1} d_q x\right]^{\frac{1}{p}} < \infty.$$

We denote by $\mathcal{S}_q$ the $q-$analogue of the Schwartz space of even function f defined on $\mathbb{R}_q$ such that $D_q^k f$ is continuous at 0, and for all $n \in \mathbb{N}$ there is C_n such that

$$|D_q^k f(x)| \leq \frac{C_n}{(1+x^2)^n}, \quad \forall k \in \mathbb{N}, \forall x \in \mathbb{R}_q^+.$$

Ar the end of this section we introduce the $q-$Bessel operator as follows

$$\Delta_{q,v} f(x) = \frac{1}{x^2}\left[f(q^{-1}x) - (1+q^{2v})f(x) + q^{2v} f(qx)\right].$$

Proposition 1.2. *Given two functions f and g in $\mathcal{L}_{q,2,v}$ such that*

$$\Delta_{q,v} f, \Delta_{q,v} g \in \mathcal{L}_{q,2,v}$$

then

$$\int_0^\infty \Delta_{q,v} f(x) g(x) x^{2v+1} d_q x = \int_0^\infty f(x) \Delta_{q,v} g(x) x^{2v+1} d_q x.$$

2. THE NORMALIZED HAHN-EXTON $q-$BESSEL FUNCTION

The normalized Hahn-Exton $q-$Bessel function of order v is defined as

$$j_v(x,q) = \frac{(q,q)_\infty}{(q^{v+1},q)_\infty} x^{-v} J_v^{(3)}(x,q) = {}_1\phi_1(0,q^{v+1},q,qx^2), \quad \Re(v) > -1,$$

where $J_v^{(3)}(\cdot,q)$ is the Hahn-Exton $q-$bessel function, (see [12]).

Proposition 2.1. *The function*

$$x \mapsto j_v(\lambda x, q^2),$$

is a solution of the following $q-$difference equation

$$\Delta_{q,v} f(x) = -\lambda^2 f(x)$$

Proof. See [9]. □

In the following we put

$$c_{q,v} = \frac{1}{1-q} \cdot \frac{(q^{2v+2},q^2)_\infty}{(q^2,q^2)_\infty}.$$

Proposition 2.2. *Let $n, m \in \mathbb{Z}$ and $n \neq m$, then we have*

$$c_{q,v}^2 \int_0^\infty j_v(q^n x, q^2) j_v(q^m x, q^2) x^{2v+1} d_q x = \frac{q^{-2n(v+1)}}{1-q} \delta_{nm}.$$

Proof. See [10]. □

Proposition 2.3.

$$|j_v(q^n, q^2)| \leq \frac{(-q^2;q^2)_\infty (-q^{2v+2};q^2)_\infty}{(q^{2v+2};q^2)_\infty} \begin{cases} 1 & \text{if} \quad n \geq 0, \\ q^{n^2+(2v+1)n} & \text{if} \quad n < 0. \end{cases}$$

Proof. Use Proposition 1.1. □

3. $q-$BESSEL FOURIER TRANSFORM

The $q-$Bessel Fourier transform $\mathcal{F}_{q,v}$ is defined as follows

$$\mathcal{F}_{q,v}(f)(x) = c_{q,v} \int_0^\infty f(t) j_v(xt, q^2) t^{2v+1} d_q t.$$

Proposition 3.1. *The $q-$Bessel Fourier transform*

$$\mathcal{F}_{q,v} : \mathcal{L}_{q,1,v} \to \mathcal{C}_{q,0},$$

satisfying

$$\|\mathcal{F}_{q,v}(f)\|_{\mathcal{C}_{q,0}} \leq B_{q,v} \|f\|_{q,1,v},$$

where

$$B_{q,v} = \frac{1}{1-q} \frac{(-q^2;q^2)_\infty (-q^{2v+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

Proof. Use Proposition 2.3. □

Theorem 3.2. *Given $f \in \mathcal{L}_{q,1,v}$ then we have*

$$\mathcal{F}_{q,v}^2(f)(x) = f(x), \quad \forall x \in \mathbb{R}_q^+.$$

If $f \in \mathcal{L}_{q,1,v}$ and $\mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,1,v}$ then

$$\|\mathcal{F}_{q,v}(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

Proof. Let $t, y \in \mathbb{R}_q^+$, we put

$$\delta_{q,v}(t,y) = \begin{cases} \frac{1}{(1-q)t^{2v+2}} & \text{if} \quad t = y, \\ 0 & \text{if} \quad t \neq y. \end{cases}$$

It is not hard to see that

$$\int_0^\infty f(t) \delta_{q,v}(t,y) t^{2v+1} d_q t = f(y).$$

By Proposition 2.2, we can write

$$c_{q,v}^2 \int_0^\infty j_v(yx, q^2) j_v(tx, q^2) x^{2v+1} d_q x = \delta_{q,v}(t,y), \quad \forall t, y \in \mathbb{R}_q^+,$$

which leads to the result. □

Corollary 3.3. *The transformation*

$$\mathcal{F}_{q,v} : \mathcal{S}_q \to \mathcal{S}_q,$$

is an isomorphism, and

$$\mathcal{F}_{q,v}^{-1} = \mathcal{F}_{q,v}.$$

Proof. The result is deduced from properties of the space $\mathcal{S}_q$. □

4. $q-$BESSEL TRANSLATION OPERATOR

We introduce the $q-$Bessel translation operator as follows:

$$T_{q,x}^v f(y) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t, \quad \forall x, y \in \mathbb{R}_q^+, \forall f \in \mathcal{L}_{q,1,v}.$$

Proposition 4.1. *For any function $f \in \mathcal{L}_{q,1,v}$ we have*

$$T_{q,x}^v f(y) = T_{q,y}^v f(x),$$

and

$$T_{q,x}^v f(0) = f(x).$$

Proposition 4.2. *For all $x, y \in \mathbb{R}_q^+$, we have*

$$T_{q,x}^v j_v(\lambda y, q^2) = j_v(\lambda x, q^2) j_v(\lambda y, q^2).$$

Proof. Use Proposition 2.2. □

Proposition 4.3. *Let $f \in \mathcal{L}_{q,1,v}$ then*

$$T_{q,x}^v f(y) = \int_0^\infty f(z) D_v(x, y, z) z^{2v+1} d_q z,$$

where

$$D_v(x, y, z) = c_{q,v}^2 \int_0^\infty j_v(xt, q^2) j_v(yt, q^2) j_v(zt, q^2) t^{2v+1} d_q t.$$

Proof. Indeed,

$$\begin{aligned} T_{q,x}^v f(y) &= c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t \\ &= c_{q,v} \int_0^\infty \left[c_{q,v} \int_0^\infty f(z) j_v(zt, q^2) z^{2v+1} d_q t \right] j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t \\ &= \int_0^\infty f(z) \left[c_{q,v}^2 \int_0^\infty j_v(xt, q^2) j_v(yt, q^2) j_v(zt, q^2) t^{2v+1} d_q t \right] z^{2v+1} d_q z, \end{aligned}$$

which leads to the result. □

Proposition 4.4.

$$\lim_{z \to \infty} D_v(x, y, z) = 0$$

and

$$(1 - q) \sum_{s \in \mathbb{Z}} q^{(2v+2)s} D_v(x, y, q^s) = 1$$

Proof. To prove the first relation use Proposition 3.1. The second identity is deduced from Proposition 4.2: if $f = 1$ then $T_{q,x}^v f = 1$. □

Proposition 4.5. *Given $f \in \mathcal{S}_q$ then*

$$T_{q,x}^v f(y) = \sum_{n=0}^\infty \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2v+2}, q^2)_n} y^{2n} \Delta_{q,v}^n f(x).$$

Proof. By the use of Proposition 2.1 and the fact that

$$\Delta_{q,v}^n f(x) = (-1)^n c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) t^{2n} j_v(xt, q^2) t^{2v+1} d_q t.$$

□

Proposition 4.6. *If* $v = -\frac{1}{2}$ *then*

$$D_v(q^m, q^r, q^k) = \frac{q^{2(r-m)(k-m)-m}}{(1-q)(q;q)_\infty} (q^{2(r-m)+1}; q)_{\infty\, 1}\phi_1(0, q^{2(r-m)+1}, q; q^{2(k-m)+1}).$$

Proof. Indeed

$$\Delta_{q,v}^n = \frac{q^{-n(n+1)}}{x^{2n}} \sum_{k=-n}^{n} \begin{bmatrix} 2n \\ k+n \end{bmatrix}_q (-1)^{k+n} q^{\frac{(k+n)(k+n+1)}{2} - 2kn} \Lambda_q^k,$$

and use Proposition 4.5. □

5. $q-$CONVOLUTION PRODUCT

In harmonic analysis the positivity of the translation operator is crucial. It plays a central role in establishing some useful results, such as the property of the convolution product. Thus it is natural to investigate when this property holds for $T_{q,x}^v$. In the following we put

$$Q_v = \{q \in [0,1], \quad T_{q,x}^v \quad \text{is positive for all} \quad x \in \mathbb{R}_q^+\}.$$

Recall that $T_{q,x}^v$ is said to be positive if $T_{q,x}^v f \geq 0$ for $f \geq 0$.

Proposition 5.1. *If* $v = -\frac{1}{2}$ *then*

$$Q_v = [0, q_0],$$

where q_0 *is the first zero of the following function:*

$$q \mapsto {}_1\phi_1(0, q, q, q).$$

Proof. The operator $T_{q,x}^v$ is positive if and only if

$$D_v(x, y, q^s) \geq 0, \quad \forall x, y, q^s \in \mathbb{R}_q^+.$$

We replace $\frac{x}{y}$ by q^r, and we can choose $r \in \mathbb{N}$, because

$$T_{q,x}^v f(y) = T_{q,y}^v f(x),$$

thus we get

$$(q^{1+2s}, q)_{\infty\, 1}\phi_1(0, q^{1+2s}, q, q^{1+2r}) = \sum_{n=0}^{\infty} B_n(s, r), \quad \forall r, s \in \mathbb{N},$$

where

$$B_n(s,r) = \prod_{i=1}^{2n} \frac{q^{2r+i}}{1-q^i} \prod_{i=2n+2}^{\infty} (1 - q^{2s+i}) \left[(1 - q^{2s+2n+1}) - \frac{q^{2r+2n+1}}{1-q^{2n+1}} \right], \quad \forall n \in \mathbb{N}^*,$$

and

$$B_0(s,r) = \prod_{i=2}^{\infty} (1 - q^{2s+i}) \left[(1 - q^{2s+1}) - \frac{q^{2r+1}}{1-q} \right],$$

which leads to the result. □

In the rest of this work we choose $q \in Q_v$.

Proposition 5.2. *Given* $f \in \mathcal{L}_{q,1,v}$ *then*

$$\int_0^\infty T^v_{q,x} f(y) y^{2v+1} d_q y = \int_0^\infty f(y) y^{2v+1} d_q y.$$

The $q-$convolution product of both functions $f, g \in \mathcal{L}_{q,1,v}$ is defined by

$$f *_q g(x) = c_q \int_0^\infty T^v_{q,x} f(y) g(y) y^{2v+1} d_q y.$$

Proposition 5.3. *Given two functions* $f, g \in \mathcal{L}_{q,1,v}$ *then*

$$f *_q g \in \mathcal{L}_{q,1,v},$$

and

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(f)\mathcal{F}_{q,v}(g).$$

Proof. We have

$$\|f *_q g\|_{q,1,v} \leq \|f\|_{q,1,v}\|g\|_{q,1,v}.$$

On the other hand

$$\begin{aligned}\mathcal{F}_{q,v}(f *_q g)(\lambda) &= \int_0^\infty \left[\int_0^\infty f(x) T^v_{q,y} j_v(\lambda x, q^2) x^{2v+1} d_q x\right] g(y) y^{2v+1} d_q y \\ &= \mathcal{F}_{q,v}(f)(\lambda)\, \mathcal{F}_{q,v}(g)(\lambda).\end{aligned}$$

□

6. $q-$HEAT SEMIGROUP

The $q-$heat semigroup is defined by:

$$\begin{aligned}P^v_{q,t} f(x) &= G^v(\cdot, t, q^2) *_q f(x) \\ &= c_{q,v} \int_0^\infty T^v_{q,x} G^v(y, t, q^2) f(y) y^{2v+1} d_q y, \quad \forall f \in \mathcal{L}_{q,1,v}.\end{aligned}$$

$G^v(\cdot, t, q^2)$ is the $q-$Gauss kernel of $P^v_{q,t}$

$$G^v(x, t, q^2) = \frac{(-q^{2v+2}t, -q^{-2v}/t; q^2)_\infty}{(-t, -q^2/t; q^2)_\infty} e\left(-\frac{q^{-2v}}{t} x^2, q^2\right).$$

and $e(\cdot, q)$ the q-exponential function defined by

$$e(z, q) = \sum_{n=0}^\infty \frac{z^n}{(q, q)_n} = \frac{1}{(z; q)_\infty}, \quad |z| < 1.$$

Proposition 6.1. *The* $q-$*Gauss kernel* $G^v(\cdot, t, q^2)$ *satisfying*

$$\mathcal{F}_{q,v}\left\{G^v(\cdot, t, q^2)\right\}(x) = e(-tx^2, q^2),$$

and

$$\mathcal{F}_{q,v}\left\{e(-ty^2, q^2)\right\}(x) = G^v(x, t, q^2).$$

Proof. In [5], the Ramanujan identity was proved

$$\sum_{s\in\mathbb{Z}} \frac{z^s}{(bq^{2s}, q^2)_\infty} = \frac{\left(bz, \frac{q^2}{bz}, q^2, q^2\right)_\infty}{\left(b, z, \frac{q^2}{b}, q^2\right)_\infty},$$

which implies

$$\int_0^\infty e(-ty^2,q^2)y^{2n}y^{2v+1}d_qy = (1-q)\sum_s \frac{(q^{2n+2v+2})^s}{(-tq^{2s},q^2)_\infty}$$
$$= (1-q)\frac{\left(-tq^{2n+2v+2}, -\frac{q^{-2n-2v}}{t}, q^2, q^2\right)_\infty}{\left(-t, q^{2n+2v+2}, -\frac{q^2}{t}, q^2\right)_\infty}.$$

The following identity leads to the result

$$(a,q^2)_\infty = (a,q^2)_n(q^{2n}a,q^2)_\infty,$$

and

$$(aq^{-2n},q^2)_\infty = (-1)^n q^{-n^2+n}\left(\frac{a}{q^2}\right)^n \left(\frac{q^2}{a},q^2\right)_n (a,q^2)_\infty.$$

□

Proposition 6.2. *For any functions $f \in \mathcal{S}_q$, we have*

$$P^v_{q,t}f(x) = e(t\Delta_{q,v},q^2)f(x).$$

Proof. Indeed, if

$$c_{q,v}\int_0^\infty G^v(y,t,q^2)y^{2n}y^{2v+1}d_qy = (q^{2v+2},q^2)_n q^{-n(n+n)}t^n,$$

then

$$P^v_{q,t}f(x) = \sum_{n=0}^\infty \frac{q^{n(n+1)}}{(q^2,q^2)_n(q^{2v+2},q^2)_n}\left[c_{q,v}\int_0^\infty G^v(y,t,q^2)y^{2n}y^{2v+1}d_qy\right]\Delta^n_{q,v}f(x).$$

□

Theorem 6.3. *For $f \in \mathcal{L}_{q,p,v}$ and $1 \le p < \infty$, we have*

$$\|P^v_{q,t}f\|_{q,p,v} \le \|f\|_{q,p,v}.$$

Proof. If $p=1$ then

$$\|P^v_{q,t}f\|_{q,1,v} \le \|G^v(\cdot,t,q^2)\|_{q,1,v}\ \|f\|_{q,1,v} = \|f\|_{q,1,v}.$$

Now let $p>1$ and we consider the following function

$$g : y \mapsto T^v_{q,x}G^v(y,t;q^2).$$

In addition

$$\left\|P^v_{q,t}f\right\|^p_{q,p} \le c^p_{q,v}\int_0^\infty \left[\int_0^\infty |f(y)g(y)|\,y^{2v+1}d_qy\right]^p x^{2v+1}d_qx.$$

By the use of the Hölder inequality and the fact that $\|G^v(\cdot,t,q^2)\|_{q,1,v} = \frac{1}{c_{q,v}}$, the result follows immediately. □

7. $q-$WIENER ALGEBRA

For $u \in \mathcal{L}_{q,1,v}$ and $\lambda \in \mathbb{R}_q^+$, we introduce the following function

$$u_\lambda : x \mapsto \frac{1}{\lambda^{2v+2}} u\left(\frac{x}{\lambda}\right).$$

Proposition 7.1. *Given* $u \in \mathcal{L}_{q,1,v}$ *such that*

$$\int_0^\infty u(x) x^{2v+1} d_q x = 1,$$

then we have

$$\lim_{\lambda \to 0} \int_0^\infty f(x) u_\lambda(x) x^{2v+1} d_q x = f(0), \quad \forall f \in \mathcal{C}_{q,b}.$$

Corollary 7.2. *The following function*

$$G_\lambda^v : x \mapsto c_{q,v} G^v(x, \lambda^2, q^2),$$

checks the conditions of the preceding proposition.

Proof. Use Proposition 6.1. □

Theorem 7.3. *Given* $f \in \mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,p,v}$, $1 \leq p < \infty$ *and* f_λ *defined by*

$$f_\lambda(x) = c_q \int_0^\infty \mathcal{F}_{q,v}(f)(y) e(-\lambda^2 y^2, q^2) j_v(xy, q^2) y^{2v+1} d_q y.$$

then we have

$$\lim_{\lambda \to 0} \|f - f_\lambda\|_{q,p,v} = 0.$$

Proof. We have

$$f *_q G_\lambda^v(x) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) e(-\lambda^2 t^2, q^2) j_v(tx, q^2) t^{2v+1} d_q t.$$

In addition, for all $\varepsilon > 0$, there exists a function $h \in \mathcal{L}_{q,p,v}$ with compact support in $[q^k, q^{-k}]$ such that

$$\|f - h\|_{q,p,v} < \varepsilon,$$

however

$$\|G_\lambda^v *_q f - f\|_{q,p,v} \leq \|G_\lambda^v *_q (f - h)\|_{q,p,v} + \|G_\lambda^v *_q h - h\|_{q,p,v} + \|f - h\|_{q,p,v}.$$

By Theorem 6.3 we get

$$\|G_\lambda^v *_q (f - h)\|_{q,p,v} \leq \|f - h\|_{q,p,v}.$$

Now, we will prove that

$$\lim_{\lambda \to 0} \|G_\lambda^v *_q h - h\|_{q,p,v} = 0.$$

Indeed, by the use of Corollary 7.2 we get

$$\lim_{\lambda \to 0} \int_0^1 |G_\lambda^v *_q h(x) - h(x)|^p x^{2v+1} d_q x = 0.$$

On the other hand the following function is decreasing on the interval $[1, \infty[$:

$$u \mapsto u^{2v+2} G^v(u).$$

If $\lambda < 1$, then we deduce that

$$T_{q,q^i}^v G_\lambda^v(x) \leq T_{q,q^i}^v G(x).$$

We can use the dominated convergence theorem to prove that

$$\lim_{\lambda\to 0}\int_1^\infty |G_\lambda^v *_q h(x)-h(x)|^p x^{2v+1}d_qx=0.$$

□

Corollary 7.4. *Given* $f\in\mathcal{L}_{q,1,v}$ *then*

$$f(x)=c_{q,v}\int_0^\infty \mathcal{F}_{q,v}(f)(y)j_v(xy,q^2)y^{2v+1}d_qy,\quad \forall x\in\mathbb{R}_q^+.$$

Proof. The result is deduced by Theorem 7.3 and the following relation

$$(1-q)x^{2v+2}|f(x)-f_\lambda(x)|\le \|f-f_\lambda\|_{q,1,v}\quad \forall x\in\mathbb{R}_q^+.$$

□

Now we attempt to study the $q-$Wiener algebra denoted by

$$\mathcal{A}_{q,v}=\left\{f\in\mathcal{L}_{q,1,v},\quad \mathcal{F}_{q,v}(f)\in\mathcal{L}_{q,1,v}\right\}.$$

Proposition 7.5. *For* $1\le p\le\infty$*, we have*

(1) $\mathcal{A}_{q,v}\subset\mathcal{L}_{q,p,v}$ *and* $\overline{\mathcal{A}_{q,v}}=\mathcal{L}_{q,p,v}$.
(2) $\mathcal{A}_{q,v}\subset\mathcal{C}_{q,0}$ *and* $\overline{\mathcal{A}_{q,v}}=\mathcal{C}_{q,0}$.

Proof. 1. Given $h\in\mathcal{L}_{q,p,v}$ with compact support, and we put $h_n=h*_q G_{q^n}^v$. The function $h_n\in\mathcal{A}_{q,v}$ and by Theorem 7.3 we get

$$\lim_{n\to\infty}\|h-h_n\|_{q,p,v}=0.$$

2. If $f\in\mathcal{C}_{q,0}$, then there exist $h\in\mathcal{C}_{q,0}$ with compact support on $[q^k,q^{-k}]$, such that

$$\|f-h\|_{C_{q,0}}<\varepsilon,$$

and by Corollary 7.4 we prove that

$$\lim_{n\to\infty}\left[\sup_{x\in\mathbb{R}_q^+}|h(x)-h_n(x)|\right]=0.$$

□

Theorem 7.6. *For* $f\in\mathcal{L}_{q,2,v}\cap\mathcal{L}_{q,1,v}$*, we have*

$$\|\mathcal{F}_{q,v}(f)\|_{q,2,v}=\|f\|_{q,2,v}.$$

Proof. We put

$$f_n=f*_q G_{q^n}^v,$$

which implies

$$\mathcal{F}_{q,v}(f_n)(t)=e(-q^{2n}t^2,q^2)\mathcal{F}_{q,v}(f)(t),$$

by Corollary 7.4 we get

$$f_n(x)=c_q\int_0^\infty \mathcal{F}_{q,v}(f_n)(t)j_v(tx,q^2)t^{2v+1}d_qt.$$

On the other hand

$$\int_0^\infty f(x)f_n(x)x^{2v+1}d_qx=\int_0^\infty \mathcal{F}_{q,v}(f)(x)\mathcal{F}_{q,v}(f_n)(x)x^{2v+1}d_qx.$$

Theorem 7.3 implies

$$\lim_{n\to\infty}\int_0^\infty \mathcal{F}_{q,v}(f)(x)^2 e(-q^{2n}x^2,q^2)x^{2v+1}d_qx = \|f\|_{q,2,v}^2.$$

The sequence $e(-q^{2n}x^2,q^2)$ is increasing. By the use of the Fatou-Beppo-Levi theorem we deduce the result. □

Theorem 7.7.

(1) *The $q-$cosine Fourier transform $\mathcal{F}_{q,v}$ possesses an extension*

$$U : \mathcal{L}_{q,2,v} \to \mathcal{L}_{q,2,v}.$$

(2) *For $f \in \mathcal{L}_{q,2,v}$, we have*

$$\|U(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

(3) *The application U is bijective and*

$$U^{-1} = U.$$

Proof. Let the maps

$$u : \mathcal{A}_{q,v} \to \mathcal{A}_{q,v}, \quad f \mapsto \mathcal{F}_{q,v}(f).$$

Theorem 3.2 implies

$$\|u(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

The map u is uniformly continuous, with values in complete space $\mathcal{L}_{q,2,v}$. It has a prolongation U on $\overline{\mathcal{A}_{q,v}} = \mathcal{L}_{q,2,v}$. □

Proposition 7.8. *Given $1 < p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$, if $f \in \mathcal{L}_{q,p,v}$, then $\mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,p',v}$,*

$$\|\mathcal{F}_{q,v}(f)\|_{q,p',v} \le B_{p,q,v}\|f\|_{q,p,v},$$

where

$$B_{p,q,v} = B_{q,v}^{(\frac{2}{p}-1)}.$$

Proof. The result is a consequence of Proposition 3.1, Theorem 7.7 and the Riesz-Thorin theorem, see [13]. □

As an immediate consequence of Proposition 7.8, we have the following theorem:

Theorem 7.9. *Given $1 < p, p', r \le 2$ and*

$$\frac{1}{p} + \frac{1}{p'} - 1 = \frac{1}{r},$$

if $f \in \mathcal{L}_{q,p,v}$ and $g \in \mathcal{L}_{q,p',v}$, then

$$f *_q g \in \mathcal{L}_{q,r,v},$$

and

$$\|f *_q g\|_{q,r,v} \le B_{q,p,v}B_{q,p',v}B_{q,r',v}\|f\|_{q,p,v}\|g\|_{q,p',v},$$

where

$$\frac{1}{r} + \frac{1}{r'} = 1.$$

Proof. We can write

$$f *_q g = \mathcal{F}_{q,v}\left\{\mathcal{F}_{q,v}(f)\mathcal{F}_{q,v}(g)\right\},$$

the use of Proposition 7.8 and the Hölder inequality leads to the result. □

Now we are in a position to establish the hypercontractivity of the $q-$heat semigroup $P_{q,t}^v$. For more information about this notion, the reader can consult ([1, 2, 3]).

Proposition 7.10. *For $f \in \mathcal{L}_{q,p',v}$ and $t \in \mathbb{R}_q^+$, we have*

$$\|P_{q,t}^v f\|_{q,p,v} \leq B_{q,p',v} B_{q,p_1,v} c(r,q,v) t^{-\frac{v+1}{r}} \|f\|_{q,p',v},$$

where

$$1 < p' < p \leq 2, \quad \frac{1}{p} + \frac{1}{p_1} = 1, \quad \frac{1}{r} = \frac{1}{p'} - \frac{1}{p},$$

and

$$c(r,q,v) = \|e(-x^2, q^2)\|_{q,r,v}.$$

Proof. The result is deduced by the following relations

$$\mathcal{F}_{q,v}\left\{G^v(\cdot,t,q^2)\right\}(x) = e(-tx^2, q^2),$$

and

$$\|\mathcal{F}_{q,v}\left\{G^v(\cdot,t,q^2)\right\}\|_{q,r,v} = c(r,q,v) t^{-\frac{v+1}{r}}.$$

□

8. $q-$CENTRAL LIMIT THEOREM

In this section we study the analogoue of the well known central limit theorem with the aid of the $q-$Bessel Fourier transform.

For this, we consider the set $\mathcal{M}_q^+$ of positive and bounded measures on $\mathbb{R}_q^+$. The q-cosine Fourier transform of $\xi \in \mathcal{M}_q^+$ is defined by

$$\mathcal{F}_{q,v}(\xi)(x) = \int_0^\infty j_v(tx, q^2) t^{2v+1} d_q\xi(t).$$

The $q-$convolution product of two measures $\xi, \rho \in \mathcal{M}_q^+$ is given by

$$\xi *_q \rho(f) = \int_0^\infty T_{q,x}^v f(t) t^{2v+1} d_q\xi(x) d_q\rho(t),$$

and we have

$$\mathcal{F}_{q,v}(\xi *_q \rho) = \mathcal{F}_{q,v}(\xi)\mathcal{F}_{q,v}(\rho).$$

We begin by showing the following result

Proposition 8.1. *For $f \in \mathcal{A}_{q,v}$ and $\xi \in \mathcal{M}_q^+$, we have*

$$\int_0^\infty f(x) x^{2v+1} d_q\xi(x) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(x)\mathcal{F}_{q,v}(\xi)(x) x^{2v+1} d_q x.$$

As a direct consequence we may state

Corollary 8.2. *Given $\xi, \xi' \in \mathcal{M}_q^+$ such that*

$$\mathcal{F}_{q,v}(\xi) = \mathcal{F}_{q,v}(\xi'),$$

then $\xi = \xi'$.

Proof. By Proposition 8.1, we have

$$\int_0^\infty f(x) x^{2v+1} d_q\xi(x) = \int_0^\infty f(x) x^{2v+1} d_q\xi'(x), \quad \forall f \in \mathcal{A}_{q,v}.$$

from the assertion (2) of Proposition 7.5, we conclude that $\xi = \xi'$. □

Theorem 8.3. *Let $(\xi_n)_{n\geq 0}$ be a sequence of probability measures of $\mathcal{M}_q^+$ such that*

$$\lim_{n\to\infty} \mathcal{F}_{q,v}(\xi_n)(t) = \psi(t),$$

then there exists $\xi \in \mathcal{M}_q^+$ such that the sequence ξ_n converges strongly toward ξ, and

$$\mathcal{F}_{q,v}(\xi) = \psi.$$

Proof. We consider the map I_n defined by

$$I_n(u) = \int_0^\infty u(x)x^{2v+1}d_q\xi_n(x), \quad \forall f \in \mathcal{C}_{q,0}.$$

By the following inequality

$$|I_n(u)| \leq \|u\|_{C_{q,0}},$$

and by Proposition 8.1, we get

$$I_n(f) = c_{q,v}\int_0^\infty \mathcal{F}_{q,v}(f)(x)\mathcal{F}_{q,v}(\xi_n)(x)x^{2v+1}d_qx, \quad \forall f \in \mathcal{A}_{q,v},$$

which implies

$$\lim_{n\to\infty} I_n(f) = \int_0^\infty \mathcal{F}_{q,v}(f)(x)\psi(x)x^{2v+1}d_qx, \quad \forall f \in \mathcal{A}_{q,v}.$$

On the other hand, by assertion (2) of Proposition 7.5, and by the use of the Ascoli theorem (see [11]):

Consider a sequence of equicontinuous linear forms on $\mathcal{C}_{q,0}$ which converge on a dense part $\mathcal{A}_{q,v}$ then converge on the entire $\mathcal{C}_{q,0}$. We get

$$\lim_{n\to\infty} I_n(u) = \int_0^\infty \mathcal{F}_{q,v}(u)(x)\psi(x)x^{2v+1}d_qx, \quad \forall u \in \mathcal{C}_{q,0}.$$

Finally there exist $\xi \in \mathcal{M}_q^+$ such that

$$\lim_{n\to\infty}\int_0^\infty u(x)x^{2v+1}d_q\xi_n(x) = \int_0^\infty u(x)x^{2v+1}d_q\xi(x), \quad \forall u \in \mathcal{C}_{q,0}.$$

On the other hand

$$\mathcal{F}_{q,v}(\mathcal{A}_{q,v}) = \mathcal{A}_{q,v},$$

and

$$\int_0^\infty \mathcal{F}_{q,v}(f)(x)\mathcal{F}_{q,v}(\xi)(x)d_qx = \int_0^\infty \mathcal{F}_{q,v}(f)(x)\psi(x)x^{2v+1}d_qx, \quad \forall f \in \mathcal{A}_{q,v},$$

which implies

$$\mathcal{F}_{q,v}(\xi) = \psi.$$

□

Proposition 8.4. *Given $\xi \in \mathcal{M}_q^+$, and supposing that*

$$\sigma = \int_0^\infty t^2 t^{2v+1}d_q\xi(t) < \infty,$$

then

$$\mathcal{F}_{q,v}(\xi)(x) = 1 - \frac{q^2\sigma}{(q^2,q^2)_1(q^{2v+2},q^2)_1}x^2 + o(x^2).$$

Proof. We write

$$j_v(tx,q^2) = 1 - \frac{q^2t^2}{(q^2,q^2)_1(q^{2v+2},q^2)_1}x^2 + x^2\theta(tx)t^2,$$

where

$$\lim_{x\to 0}\theta(x) = 0,$$

then

$$\mathcal{F}_{q,v}(\xi)(x) = 1 - \frac{q^2\sigma}{(q^2,q^2)_1(q^{2v+2},q^2)_1}x^2 + \left[\int_0^\infty t^2\theta(tx)t^{2v+1}d_q\xi(t)\right]x^2.$$

□

Now we are in a position to present the $q-$central limit theorem.

Theorem 8.5. *Let $(\xi_n)_{n\geq 0}$ be a sequence of probability measures of $\mathcal{M}_q^+$ of total mass 1, satisfying*

$$\lim_{n\to\infty} n\sigma_n = \sigma, \quad \textit{where} \quad \sigma_n = \int_0^\infty t^2t^{2v+1}d_q\xi_n(t),$$

and

$$\lim_{n\to\infty} n\widetilde{\sigma}_n = 0, \quad \textit{where} \quad \widetilde{\sigma}_n = \int_0^\infty \frac{t^4}{1+t^2}t^{2v+1}d_q\xi_n(t),$$

then ξ_n^{*n} converge strongly toward a measure ξ defined by

$$d_q\xi(x) = c_{q,v}\mathcal{F}_{q,v}\left(e^{-\frac{q^2\sigma}{(q^2,q^2)_1(q^{2v+2},q^2)_1}t^2}\right)(x)d_qx.$$

Proof. We have

$$\mathcal{F}_{q,v}(\xi_n^{*n}) = (\mathcal{F}_{q,v}(\xi_n))^n,$$

and

$$\mathcal{F}_{q,v}(\xi_n)(x) = 1 - \frac{q^2\sigma_n}{(q^2,q^2)_1(q^{2v+2},q^2)_1}x^2 + \theta_n(x)x^2,$$

where

$$\theta_n(x) = \int_0^\infty t^2\theta(tx)t^{2v+1}d_q\xi_n(t).$$

Consequently

$$(\mathcal{F}_{q,v}(\xi_n))^n(x) = \exp\left[n\log\left[1 - \frac{q^2\sigma_n}{(q^2,q^2)_1(q^{2v+2},q^2)_1}x^2 + \theta_n(x)x^2\right]\right].$$

By the following inequality

$$|t^2\theta(tx)| \leq C_x\frac{t^4}{1+t^2}, \quad \forall t\in\mathbb{R}_q^+,$$

where C_x is some constant, the result follows immediately. □

REFERENCES

[1] K.I. BABENKO, An inequality in the theory of Fourier integrals, *Izv. Akad. Nauk SSSR*, **25** (1961), English transl., *Amer. Math. Soc.*

[2] W. BECKNER, Inequalities in Fourier analysis, *Ann.of Math.*,(2), **102** (1975), 159–182.

[3] A. FITOUHI, Inégalité de Babenko et inégalité logarithmique de Sobolev pour l'opérateur de Bessel, *C.R. Acad. Sci. Paris*, **305**(I) (1987), 877–880.

[4] G. GASPER AND M. RAHMAN, Basic hypergeometric series, *Encyclopedia of Mathematics and its Applications*, **35**, Cambridge University Press, 1990.

[5] M.E.H. ISMAIL, A simple proof of Ramanujan's ${}_1\psi_1$ sum, *Proc. Amer. Math. Soc.*, **63** (1977), 185–186.

[6] F.H. JACKSON, On q-Functions and a certain difference operator, *Transactions of the Royal Society of London*, **46** (1908), 253–281.

[7] F.H. JACKSON, On a q-definite integral, *Quarterly Journal of Pure and Application Mathematics*, **41** (1910), 193–203.

[8] T.H. KOORNWINDER, Special functions and q-commuting variables, in *Special Functions, q-Series and Related Topics*, M. E. H. Ismail, D. R. Masson and .Rahman (eds), Fields Institue Communications 14, American Mathematical Society, 1997, pp. 131–166.

[9] H.T. KOELINK AND R.F. SWARTTOUW, On the zeros of the Hahn-Exton q-Bessel function and associated q-Lommel polynomials, *Journal of Mathematical Analysis and Applications*, **186**(3) (1994), 690–710.

[10] T.H. KOORNWINDER AND R.F. SWARTTOUW, On q-Analogues of the Hankel and Fourier transform, *Trans. A.M.S.*, 1992, 333, 445–461.

[11] L. SCHWARTZ, *Analye Hilbertienne*, Hermann Paris-Collection Méthode, 1979.

[12] R.F. SWARTTOUW, The Hahn-Exton q-Bessel functions, PhD Thesis, The Technical University of Delft, 1992.

[13] G.O. THORIN, Kungl.Fysiogr.söllsk.i Lund Förh, **8** (1938), 166–170.