

# Journal of Inequalities in Pure and Applied Mathematics

## ON A NEW MULTIPLE EXTENSION OF HILBERT'S INTEGRAL INEQUALITY

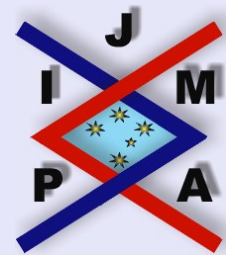
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Abstract

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## Abstract

This paper gives a new multiple extension of Hilbert's integral inequality with a best constant factor, by introducing a parameter  $\lambda$  and the  $\Gamma$  function. Some particular results are obtained.

*2000 Mathematics Subject Classification:* 26D15.

*Key words:* Hilbert's integral inequality; Weight coefficient,  $\Gamma$  function.

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# 1. Introduction

If  $f, g \geq 0$  satisfy

$$0 < \int_0^\infty f^2(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^2(x)dx < \infty,$$

then

$$(1.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}},$$

where the constant factor  $\pi$  is the best possible (cf. Hardy et al. [2]). Inequality (1.1) is well known as Hilbert's integral inequality, which had been extended by Hardy [1] as:

If  $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0$  satisfy

$$0 < \int_0^\infty f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(x)dx < \infty,$$

then

$$(1.2) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}},$$



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where the constant factor  $\frac{\pi}{\sin(\pi/p)}$  is the best possible. Inequality (1.2) is called Hardy- Hilbert's integral inequality, and is important in analysis and its applications (cf. Mitrinović et al.[6]).

Recently, by introducing a parameter  $\lambda$ , Yang [9] gave an extension of (1.2) as:

If  $\lambda > 2 - \min\{p, q\}$ ,  $f, g \geq 0$  satisfy

$$0 < \int_0^\infty x^{1-\lambda} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_0^\infty x^{1-\lambda} g^q(x) dx < \infty,$$

then

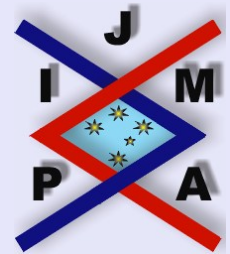
$$(1.3) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < k_\lambda(p) \left\{ \int_0^\infty x^{1-\lambda} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{1-\lambda} g^q(x) dx \right\}^{\frac{1}{q}},$$

where the constant factor  $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$  is the best possible ( $B(u, v)$  is the  $\beta$  function). For  $\lambda = 1$ , inequality (1.3) reduces to (1.2).

On the problem for multiple extension of (1.1), [3, 4] gave some new results and Yang [8] gave an improvement of their works as:

If  $n \in \mathbb{N} \setminus \{1\}$ ,  $p_i > 1$ ,  $\sum_{i=1}^n \frac{1}{p_i} = 1$ ,  $\lambda > n - \min_{1 \leq i \leq n} \{p_i\}$ ,  $f_i \geq 0$ , satisfy

$$0 < \int_0^\infty x^{n-1-\lambda} f_i^{p_i}(x) dx < \infty \quad (i = 1, 2, \dots, n),$$



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then

$$(1.4) \quad \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n$$

$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{p_i + \lambda - n}{p_i}\right) \left\{ \int_0^\infty x^{n-1-\lambda} f_i^{p_i}(x) dx \right\}^{\frac{1}{p_i}},$$

where the constant factor  $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{p_i + \lambda - n}{p_i}\right)$  is the best possible. For  $n = 2$ , inequality (1.4) reduces to (1.3). It follows that (1.4) is a multiple extension of (1.3), (1.2) and (1.1).

In 2003, Yang et. al [11] provided an extensive account of the above results.

The main objective of this paper is to build a new extension of (1.1) with a best constant factor other than (1.4), and give some new particular results. That is

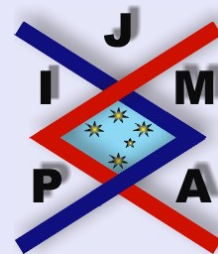
**Theorem 1.1.** *If  $n \in \mathbb{N} \setminus \{1\}$ ,  $p_i > 1$ ,  $\sum_{i=1}^n \frac{1}{p_i} = 1$ ,  $\lambda > 0$ ,  $f_i \geq 0$ , satisfy*

$$0 < \int_0^\infty x^{p_i-1-\lambda} f_i^{p_i}(x) dx < \infty \quad (i = 1, 2, \dots, n),$$

then

$$(1.5) \quad \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n$$

$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right) \left\{ \int_0^\infty x^{p_i-1-\lambda} f_i^{p_i}(x) dx \right\}^{\frac{1}{p_i}},$$



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where the constant factor  $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$  is the best possible. In particular,

(a) for  $\lambda = 1$ , we have

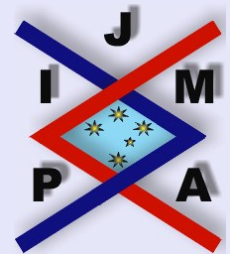
$$(1.6) \quad \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^n f_i(x_i)}{\sum_{j=1}^n x_j} dx_1 \cdots dx_n < \prod_{i=1}^n \Gamma\left(\frac{1}{p_i}\right) \left\{ \int_0^\infty x^{p_i-2} f_i^{p_i}(x) dx \right\}^{\frac{1}{p_i}};$$

(b) for  $n = 2$ , using the symbol of (1.3) and setting  $\tilde{k}_\lambda(p) = B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ , we have

$$(1.7) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < \tilde{k}_\lambda(p) \left\{ \int_0^\infty x^{p-1-\lambda} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q-1-\lambda} g^q(x) dx \right\}^{\frac{1}{q}},$$

where the constant factors in (1.6) and (1.7) are still the best possible.

In order to prove the theorem, we introduce some lemmas.



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## 2. Some Lemmas

**Lemma 2.1.** *If  $k \in \mathbb{N}$ ,  $r_i > 1$  ( $i = 1, 2, \dots, k + 1$ ), and  $\sum_{i=1}^{k+1} r_i = \lambda(k)$ , then*

$$(2.1) \quad \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(1 + \sum_{j=1}^k u_j\right)^{\lambda(k)}} \prod_{j=1}^k u_j^{r_j-1} du_1 \cdots du_k = \frac{\prod_{i=1}^{k+1} \Gamma(r_i)}{\Gamma(\lambda(k))}.$$

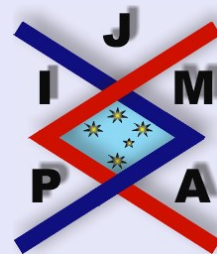
*Proof.* We establish (2.1) by mathematical induction. For  $k = 1$ , since  $r_1 + r_2 = \lambda(1)$ , and (see [7])

$$(2.2) \quad B(p, q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} du = B(q, p) \quad (p, q > 0),$$

we have (2.1). Suppose for  $k \in \mathbb{N}$ , that (2.1) is valid. Then for  $k + 1$ , since  $r_1 + \sum_{i=2}^{k+1} r_i = \lambda(k + 1)$ , by setting  $v = u_1 / \left(1 + \sum_{j=2}^{k+1} u_j\right)$ , we obtain

$$(2.3) \quad \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(1 + \sum_{j=1}^{k+1} u_j\right)^{\lambda(k+1)}} \prod_{j=1}^{k+1} u_j^{r_j-1} du_1 \cdots du_{k+1}$$

$$= \int_0^\infty \cdots \int_0^\infty \frac{v^{r_1-1} \left(1 + \sum_{j=2}^{k+1} u_j\right)^{r_1} \prod_{j=2}^{k+1} u_j^{r_j-1}}{\left(1 + \sum_{j=2}^{k+1} u_j\right)^{\lambda(k+1)} (1+v)^{\lambda(k+1)}} dv du_2 \cdots du_{k+1}$$



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$$= \int_0^\infty \cdots \int_0^\infty \frac{\prod_{j=2}^{k+1} u_j^{r_j-1}}{\left(1 + \sum_{j=2}^{k+1} u_j\right)^{\lambda(k+1)-r_1}} du_2 \cdots du_{k+1} \\ \times \int_0^\infty \frac{v^{r_1-1}}{(1+v)^{\lambda(k+1)}} dv.$$

In view of (2.2) and the assumption of  $k$ , we have

$$(2.4) \quad \int_0^\infty \frac{v^{r_1-1}}{(1+v)^{\lambda(k+1)}} dv = \frac{1}{\Gamma(\lambda(k+1))} \Gamma\left(\sum_{i=2}^{k+1} r_i\right) \Gamma(r_1);$$

$$(2.5) \quad \int_0^\infty \cdots \int_0^\infty \frac{\prod_{j=2}^{k+1} u_j^{r_j-1}}{\left(1 + \sum_{j=2}^{k+1} u_j\right)^{\lambda(k+1)-r_1}} du_2 \cdots du_{k+1} = \frac{\prod_{i=2}^{k+1} \Gamma(r_i)}{\Gamma\left(\sum_{i=2}^{k+1} r_i\right)}.$$

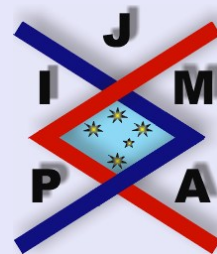
Then, by (2.5), (2.4) and (2.3), we find

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{j=1}^{k+1} u_j^{r_j-1}}{\left(1 + \sum_{j=1}^{k+1} u_j\right)^{\lambda(k+1)}} du_1 \cdots du_{k+1} = \frac{\prod_{i=1}^{k+2} \Gamma(r_i)}{\Gamma(\lambda(k+1))}.$$

Hence (2.1) is valid for  $k \in \mathbb{N}$  by induction. The lemma is proved.  $\square$

**Lemma 2.2.** If  $n \in \mathbb{N} \setminus \{1\}$ ,  $p_i > 1$  ( $i = 1, 2, \dots, n$ ),  $\sum_{i=1}^n \frac{1}{p_i} = 1$  and  $\lambda > 0$ , set the weight coefficient  $\omega(x_i)$  as

$$(2.6) \quad \omega(x_i) := x_i^{\frac{\lambda}{p_i}} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{j=1, (j \neq i)}^n x_j^{(\lambda-p_j)/p_j}}{\left(\sum_{j=1}^n x_j\right)^\lambda} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n.$$



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Then, each  $\omega(x_i)$  is constant, that is

$$(2.7) \quad \omega(x_i) = \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma\left(\frac{\lambda}{p_j}\right), \quad (i = 1, 2, \dots, n).$$

*Proof.* Fix  $i$ . Setting  $\tilde{p}_n = p_i$ , and  $\tilde{p}_j = p_j$ ,  $u_j = x_j/x_i$ , for  $j = 1, 2, \dots, i-1$ ;  $\tilde{p}_j = p_{j+1}$ ,  $u_j = x_{j+1}/x_i$ , for  $j = i, i+1, \dots, n-1$  in (2.6), by simplification, we have

$$(2.8) \quad \omega(x_i) = \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(1 + \sum_{j=1}^{n-1} u_j\right)^\lambda} \prod_{j=1}^{n-1} u_j^{-1 + \frac{\lambda}{p_j}} du_1 \cdots du_{n-1}.$$

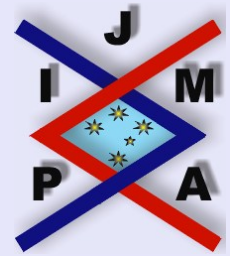
Substitution of  $n-1$  for  $k$ ,  $\lambda$  for  $\lambda(k)$  and  $\lambda/\tilde{p}_j$  for  $r_j$  ( $j = 1, 2, \dots, n$ ) into (2.1), in view of (2.8), we have

$$\omega(x_i) = \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma\left(\frac{\lambda}{\tilde{p}_j}\right) = \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma\left(\frac{\lambda}{p_j}\right).$$

Hence, (2.7) is valid. The lemma is proved.  $\square$

**Lemma 2.3.** As in the assumption of Lemma 2.2, for  $0 < \varepsilon < \lambda$ , we have

$$(2.9) \quad \begin{aligned} I &:= \varepsilon \int_1^\infty \cdots \int_1^\infty \frac{\prod_{i=1}^n x_i^{(\lambda - p_i - \varepsilon)/p_i}}{\left(\sum_{j=1}^n x_j\right)^\lambda} dx_1 \cdots dx_n \\ &\geq \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right) \quad (\varepsilon \rightarrow 0^+). \end{aligned}$$



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*Proof.* Setting  $u_i = x_i/x_n$  ( $i = 1, 2, \dots, n - 1$ ) in the following, we find

$$\begin{aligned}
 I &= \varepsilon \int_1^\infty x_n^{-1-\varepsilon} \left[ \int_{x_n^{-1}}^\infty \dots \int_{x_n^{-1}}^\infty \frac{\prod_{i=1}^{n-1} u_i^{(\lambda-p_i-\varepsilon)/p_i}}{\left(1 + \sum_{j=1}^{n-1} u_j\right)^\lambda} du_1 \dots du_{n-1} \right] dx_n \\
 &\geq \varepsilon \int_1^\infty x_n^{-1-\varepsilon} \left[ \int_0^\infty \dots \int_0^\infty \frac{\prod_{i=1}^{n-1} u_i^{(\lambda-p_i-\varepsilon)/p_i}}{\left(1 + \sum_{j=1}^{n-1} u_j\right)^\lambda} du_1 \dots du_{n-1} \right] dx_n \\
 (2.10) \quad &- \varepsilon \int_1^\infty x_n^{-1} \sum_{j=1}^{n-1} A_j(x_n) dx_n,
 \end{aligned}$$

where, for  $j = 1, 2, \dots, n - 1$ ,  $A_j(x_n)$  is defined by

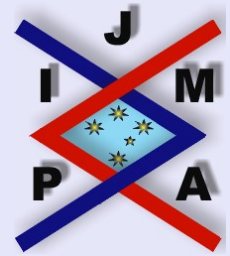
$$(2.11) \quad A_j(x_n) := \int \dots \int_{D_j} \frac{\prod_{i=1}^{n-1} u_i^{(\lambda-p_i-\varepsilon)/p_i}}{\left(1 + \sum_{j=1}^{n-1} u_j\right)^\lambda} du_1 \dots du_{n-1},$$

satisfying  $D_j = \{(u_1, u_2, \dots, u_{n-1}) | 0 < u_j \leq x_n^{-1}, 0 < u_k < \infty (k \neq j)\}$ .

Without loss of generality, we estimate the integral  $A_j(x_n)$  for  $j = 1$ .

(a) For  $n = 2$ , we have

$$\begin{aligned}
 A_1(x_n) &= \int_0^{x_n^{-1}} \frac{1}{(1 + u_1)^\lambda} u_1^{(\lambda-p_1-\varepsilon)/p_1} du_1 \\
 &\leq \int_0^{x_n^{-1}} u_1^{(\lambda-p_1-\varepsilon)/p_1} du_1 = \frac{p_1}{\lambda - \varepsilon} x_n^{-(\lambda-\varepsilon)/p_1};
 \end{aligned}$$



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(b) For  $n \in \mathbb{N} \setminus \{1, 2\}$ , by (2.1), we have

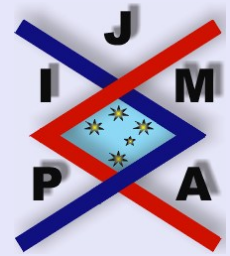
$$\begin{aligned} A_1(x_n) &\leq \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=2}^{n-1} u_i^{-1+\frac{\lambda-\varepsilon}{p_i}}}{\left(1 + \sum_{j=2}^{n-1} u_j\right)^\lambda} du_1 \cdots du_{n-1} \int_0^{x_n^{-1}} u_1^{\frac{\lambda-p_1-\varepsilon}{p_1}} du_1 \\ &\leq \frac{p_1 x_n^{-\frac{\lambda-\varepsilon}{p_1}}}{\lambda - \varepsilon} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=2}^{n-1} u_i^{-1+(\lambda-\varepsilon)/p_i}}{\left(1 + \sum_{j=2}^{n-1} u_j\right)^{(\lambda-\varepsilon)(1-p_1^{-1})}} du_1 \cdots du_{n-1} \\ &= \frac{p_1 x_n^{-\frac{\lambda-\varepsilon}{p_1}}}{\lambda - \varepsilon} \cdot \frac{\prod_{i=2}^n \Gamma\left(\frac{\lambda-\varepsilon}{p_i}\right)}{\Gamma((\lambda - \varepsilon)(1 - p_1^{-1}))}. \end{aligned}$$

By virtue of the results of (a) and (b), for  $j = 1, 2, \dots, n - 1$ , we have

$$(2.12) \quad A_j(x_n) \leq \frac{p_j}{\lambda - \varepsilon} x_n^{-(\lambda-\varepsilon)/p_j} O_j(1) \quad (\varepsilon \rightarrow 0^+, n \in \mathbb{N} \setminus \{1\}).$$

By (2.11), since for  $\varepsilon \rightarrow 0^+$ ,

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{n-1} u_i^{-1+(\lambda-\varepsilon)/p_i}}{\left(1 + \sum_{j=1}^{n-1} u_j\right)^\lambda} du_1 \cdots du_{n-1} = \frac{\prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)}{\Gamma(\lambda)} + o(1);$$



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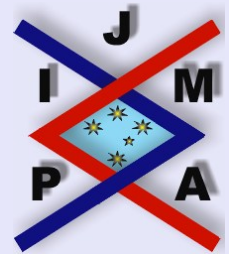
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$$\begin{aligned} \int_1^\infty x_n^{-1} \sum_{j=1}^{n-1} A_j(x_n) dx_n &= \sum_{j=1}^{n-1} \int_1^\infty x_n^{-1} A_j(x_n) dx_n \\ &\leq \sum_{j=1}^{n-1} \frac{p_j}{\lambda - \varepsilon} O_j(1) \int_1^\infty x_n^{-1 - (\lambda - \varepsilon)/p_j} dx_n \\ &= \sum_{j=1}^{n-1} \left( \frac{p_j}{\lambda - \varepsilon} \right)^2 O_j(1), \end{aligned}$$

then by (2.10), we find

$$\begin{aligned} I &\geq \left( \frac{\prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)}{\Gamma(\lambda)} + o(1) \right) - \varepsilon \sum_{j=1}^{n-1} \left( \frac{p_j}{\lambda - \varepsilon} \right)^2 O_j(1) \\ &\rightarrow \frac{\prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)}{\Gamma(\lambda)} \quad (\varepsilon \rightarrow 0^+). \end{aligned}$$

Thereby, (2.9) is valid and the lemma is proved. □



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### 3. Proof of the Theorem and Remarks

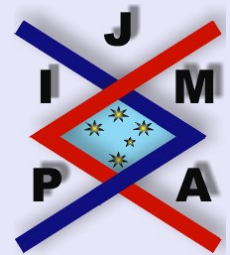
*Proof of Theorem 1.1.* By Hölder's inequality, we have

$$\begin{aligned}
 (3.1) \quad J &:= \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n \\
 &= \int_0^\infty \cdots \int_0^\infty \left\{ \prod_{i=1}^n \frac{f_i(x_i)}{\left(\sum_{j=1}^n x_j\right)^{\lambda/p_i}} \left[ x_i^{(p_i-\lambda)(1-p_i^{-1})} \prod_{\substack{j=1 \\ (j \neq i)}}^n x_j^{\frac{\lambda-p_j}{p_j}} \right]^{\frac{1}{p_i}} \right\} dx_1 \cdots dx_n \\
 &\leq \prod_{i=1}^n \left\{ \int_0^\infty \cdots \int_0^\infty \frac{f_i^{p_i}(x_i)}{\left(\sum_{j=1}^n x_j\right)^\lambda} x_i^{(p_i-\lambda)(1-p_i^{-1})} \prod_{\substack{j=1 \\ (j \neq i)}}^n x_j^{\frac{\lambda-p_j}{p_j}} dx_1 \cdots dx_n \right\}^{\frac{1}{p_i}}.
 \end{aligned}$$

If (3.1) takes the form of equality, then there exists constants  $C_1, C_2, \dots, C_n$ , such that they are not all zero and for any  $i \neq k \in \{1, 2, \dots, n\}$  (see [5]),

$$(3.2) \quad C_i \frac{f_i^{p_i}(x_i) x_i^{(p_i-\lambda)(1-p_i^{-1})}}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{\substack{j=1 \\ (j \neq i)}}^n x_j^{\frac{\lambda-p_j}{p_j}} = C_k \frac{f_k^{p_k}(x_k) x_k^{(p_k-\lambda)(1-p_k^{-1})}}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{\substack{j=1 \\ (j \neq k)}}^n x_j^{\frac{\lambda-p_j}{p_j}},$$

a.e. in  $(0, \infty) \times \cdots \times (0, \infty)$ .



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Assume that  $C_i \neq 0$ . By simplification of (3.2), we find

$$\begin{aligned} x_i^{p_i-\lambda} f_i^{p_i}(x_i) &= F(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \\ &= \text{constant a.e. in } (0, \infty) \times \dots \times (0, \infty), \end{aligned}$$

which contradicts the fact that  $0 < \int_0^\infty x_i^{p_i-\lambda-1} f_i^{p_i}(x) dx < \infty$ . Hence by (2.6) and (3.1), we conclude

$$(3.3) \quad J < \prod_{i=1}^n \left\{ \int_0^\infty \omega(x_i) x_i^{p_i-1-\lambda} f_i^{p_i}(x_i) dx_i \right\}^{\frac{1}{p_i}}.$$

Then by (2.7), we have (1.5).

For  $0 < \varepsilon < \lambda$ , setting  $\tilde{f}_i(x_i)$  as:  $\tilde{f}_i(x_i) = 0$ , for  $x_i \in (0, 1)$ ;

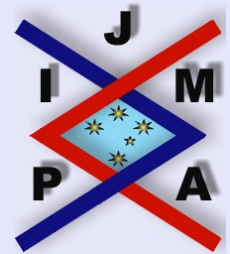
$$\tilde{f}_i(x_i) = x_i^{(\lambda-p_i-\varepsilon)/p_i}, \quad \text{for } x_i \in [1, \infty) \quad (i = 1, 2, \dots, n),$$

then we find

$$(3.4) \quad \varepsilon \prod_{i=1}^n \left\{ \int_0^\infty x_i^{p_i-1-\lambda} \tilde{f}_i^{p_i}(x_i) dx_i \right\}^{\frac{1}{p_i}} = 1.$$

By (2.9), we find

$$\begin{aligned} (3.5) \quad \varepsilon \int_0^\infty \dots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n \tilde{f}_i(x_i) dx_1 \dots dx_n \\ = I \geq \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right) \quad (\varepsilon \rightarrow 0^+). \end{aligned}$$



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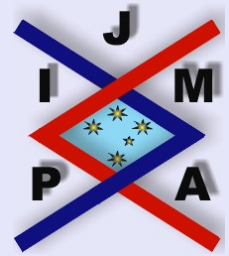


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If the constant factor  $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$  in (1.5) is not the best possible, then there exists a positive constant  $K < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$ , such that (1.5) is still valid if one replaces  $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$  by  $K$ . In particular, one has

$$\varepsilon \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n \tilde{f}_i(x_i) dx_1 \cdots dx_n < \varepsilon K \prod_{i=1}^n \left\{ \int_0^\infty x^{p_i-1-\lambda} \tilde{f}_i^{p_i}(x) dx \right\}^{\frac{1}{p_i}},$$

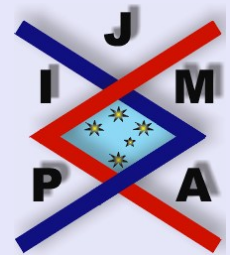
and in view of (3.4) and (3.5), it follows that  $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right) \leq K$  ( $\varepsilon \rightarrow 0^+$ ). This contradicts the fact  $K < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$ . Hence the constant factor  $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$  in (1.5) is the best possible.

The theorem is proved.  $\square$

**Remark 1.** For  $\lambda = 1$ , inequality (1.7) reduces to (see [10])

$$(3.6) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^\infty x^{p-2} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q-2} g^q(x) dx \right\}^{\frac{1}{q}}.$$

For  $p = q = 2$ , both (3.6) and (1.2) reduce to (1.1). It follows that inequalities (3.6) and (1.2) are different extensions of (1.1). Hence, inequality (1.5) is a new multiple extension of (1.1). Since all the constant factors in the obtained inequalities are the best possible, we have obtained new results.



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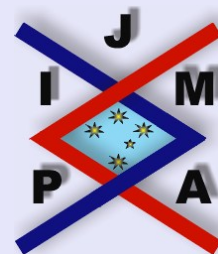
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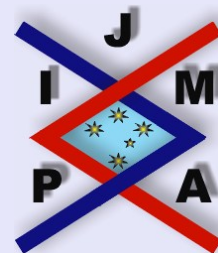
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