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ON A NEW MULTIPLE EXTENSION OF HILBERT'S INTEGRAL INEQUALITY

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ABSTRACT. This paper gives a new multiple extension of Hilbert's integral inequality with a best constant factor, by introducing a parameter λ and the Γ function. Some particular results are obtained.

Key words and phrases: Hilbert's integral inequality; Weight coefficient , Γ function.

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1. INTRODUCTION

If $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^2(x)dx < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible (cf. Hardy et al. [2]). Inequality (1.1) is well known as Hilbert's integral inequality, which had been extended by Hardy [1] as:

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(x)dx < \infty,$$

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then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.2) is called Hardy- Hilbert's integral inequality, and is important in analysis and its applications (cf. Mitrinović et al.[6]).

Recently, by introducing a parameter λ, Yang [9] gave an extension of (1.2) as:

If $\lambda > 2 - \min\{p,q\}$, $f, g \geq 0$ satisfy

$$0 < \int_0^\infty x^{1-\lambda} f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty x^{1-\lambda} g^q(x)dx < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < k_\lambda(p) \left\{\int_0^\infty x^{1-\lambda} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{1-\lambda} g^q(x)dx\right\}^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible ($B(u,v)$ is the β function). For $\lambda = 1$, inequality (1.3) reduces to (1.2).

On the problem for multiple extension of (1.1), [3, 4] gave some new results and Yang [8] gave an improvement of their works as:

If $n \in \mathbb{N}\backslash\{1\}$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\lambda > n - \min\limits_{1\leq i\leq n}\{p_i\}$, $f_i \geq 0$, satisfy

$$0 < \int_0^\infty x^{n-1-\lambda} f_i^{p_i}(x)dx < \infty \quad (i = 1, 2, \ldots, n),$$

then

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n f_i(x_i)dx_1 \ldots dx_n < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{p_i+\lambda-n}{p_i}\right) \left\{\int_0^\infty x^{n-1-\lambda} f_i^{p_i}(x)dx\right\}^{\frac{1}{p_i}}, \tag{1.4}$$

where the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{p_i+\lambda-n}{p_i}\right)$ is the best possible. For $n = 2$, inequality (1.4) reduces to (1.3). It follows that (1.4) is a multiple extension of (1.3), (1.2) and (1.1).

In 2003, Yang et. al [11] provided an extensive account of the above results.

The main objective of this paper is to build a new extension of (1.1) with a best constant factor other than (1.4), and give some new particular results. That is

Theorem 1.1. *If* $n \in \mathbb{N}\backslash\{1\}$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\lambda > 0$, $f_i \geq 0$, *satisfy*

$$0 < \int_0^\infty x^{p_i-1-\lambda} f_i^{p_i}(x)dx < \infty \quad (i = 1, 2, \ldots, n),$$

then

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n f_i(x_i)dx_1 \ldots dx_n < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right) \left\{\int_0^\infty x^{p_i-1-\lambda} f_i^{p_i}(x)dx\right\}^{\frac{1}{p_i}}, \tag{1.5}$$

where the constant factor $\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(\frac{\lambda}{p_i}\right)$ *is the best possible. In particular,*

(a) *for* $\lambda = 1$, *we have*

$$\int_0^{\infty}\cdots\int_0^{\infty}\frac{\prod_{i=1}^{n} f_i(x_i)}{\sum_{j=1}^{n} x_j}dx_1\ldots dx_n < \prod_{i=1}^{n}\Gamma\left(\frac{1}{p_i}\right)\left\{\int_0^{\infty} x^{p_i-2} f_i^{p_i}(x)dx\right\}^{\frac{1}{p_i}}; \tag{1.6}$$

(b) *for* $n = 2$, *using the symbol of (1.3) and setting* $\widetilde{k}_{\lambda}(p) = B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$, *we have*

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{(x+y)^{\lambda}}dxdy < \widetilde{k}_{\lambda}(p)\left\{\int_0^{\infty} x^{p-1-\lambda} f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^{\infty} x^{q-1-\lambda} g^q(x)dx\right\}^{\frac{1}{q}}, \tag{1.7}$$

where the constant factors in (1.6) and (1.7) are still the best possible.

In order to prove the theorem, we introduce some lemmas.

2. Some Lemmas

Lemma 2.1. *If* $k \in \mathbb{N}$, $r_i > 1$ $(i = 1, 2, \ldots, k+1)$, *and* $\sum_{i=1}^{k+1} r_i = \lambda(k)$, *then*

$$\int_0^{\infty}\cdots\int_0^{\infty}\frac{1}{\left(1+\sum_{j=1}^{k} u_j\right)^{\lambda(k)}}\prod_{j=1}^{k} u_j^{r_j-1}du_1\ldots du_k = \frac{\prod_{i=1}^{k+1}\Gamma(r_i)}{\Gamma(\lambda(k))}. \tag{2.1}$$

Proof. We establish (2.1) by mathematical induction. For $k = 1$, since $r_1 + r_2 = \lambda(1)$, and (see [7])

$$B(p,q) = \int_0^{\infty}\frac{u^{p-1}}{(1+u)^{p+q}}du = B(q,p) \qquad (p, q > 0), \tag{2.2}$$

we have (2.1). Suppose for $k \in \mathbb{N}$, that (2.1) is valid. Then for $k+1$, since $r_1 + \sum_{i=2}^{k+1} r_i = \lambda(k+1)$, by setting $v = u_1 \Big/ \left(1+\sum_{j=2}^{k+1} u_j\right)$, we obtain

$$\begin{aligned} &\int_0^{\infty}\cdots\int_0^{\infty}\frac{1}{\left(1+\sum_{j=1}^{k+1} u_j\right)^{\lambda(k+1)}}\prod_{j=1}^{k+1} u_j^{r_j-1}du_1\ldots du_{k+1} \\ &= \int_0^{\infty}\cdots\int_0^{\infty}\frac{v^{r_1-1}\left(1+\sum_{j=2}^{k+1} u_j\right)^{r_1}\prod_{j=2}^{k+1} u_j^{r_j-1}}{\left(1+\sum_{j=2}^{k+1} u_j\right)^{\lambda(k+1)}(1+v)^{\lambda(k+1)}}dvdu_2\ldots du_{k+1} \\ &= \int_0^{\infty}\cdots\int_0^{\infty}\frac{\prod_{j=2}^{k+1} u_j^{r_j-1}}{\left(1+\sum_{j=2}^{k+1} u_j\right)^{\lambda(k+1)-r_1}}du_2\ldots du_{k+1}\int_0^{\infty}\frac{v^{r_1-1}}{(1+v)^{\lambda(k+1)}}dv. \end{aligned} \tag{2.3}$$

In view of (2.2) and the assumption of k, we have

$$\int_0^{\infty}\frac{v^{r_1-1}}{(1+v)^{\lambda(k+1)}}dv = \frac{1}{\Gamma(\lambda(k+1))}\Gamma\left(\sum_{i=2}^{k+1} r_i\right)\Gamma(r_1); \tag{2.4}$$

$$\int_0^{\infty}\cdots\int_0^{\infty}\frac{\prod_{j=2}^{k+1} u_j^{r_j-1}}{\left(1+\sum_{j=2}^{k+1} u_j\right)^{\lambda(k+1)-r_1}}du_2\ldots du_{k+1} = \frac{\prod_{i=2}^{k+2}\Gamma(r_i)}{\Gamma\left(\sum_{i=2}^{k+1} r_i\right)}. \tag{2.5}$$

Then, by (2.5), (2.4) and (2.3), we find

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{j=1}^{k+1} u_j^{r_j-1}}{\left(1+\sum_{j=1}^{k+1} u_j\right)^{\lambda(k+1)}} du_1 \ldots du_{k+1} = \frac{\prod_{i=1}^{k+2} \Gamma(r_i)}{\Gamma(\lambda(k+1))}.$$

Hence (2.1) is valid for $k \in \mathbb{N}$ by induction. The lemma is proved. □

Lemma 2.2. *If* $n \in \mathbb{N}\backslash\{1\},\ p_i > 1\ (i = 1, 2, \ldots, n),\ \sum_{i=1}^n \frac{1}{p_i} = 1$ *and* $\lambda > 0$*, set the weight coefficient* $\omega(x_i)$ *as*

$$\omega(x_i) := x_i^{\frac{\lambda}{p_j}} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{j=1(j\neq i)}^{n} x_j^{(\lambda-p_j)/p_j}}{\left(\sum_{j=1}^n x_j\right)^{\lambda}} dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n. \tag{2.6}$$

Then, each $\omega(x_i)$ *is constant, that is*

$$\omega(x_i) = \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma\left(\frac{\lambda}{p_j}\right), \qquad (i = 1, 2, \ldots, n). \tag{2.7}$$

Proof. Fix i. Setting $\widetilde{p_n} = p_i$, and $\widetilde{p_j} = p_j, u_j = x_j/x_i$, for $j = 1, 2, \ldots, i-1$; $\widetilde{p_j} = p_{j+1}, u_j = x_{j+1}/x_i$, for $j = i, i+1, \ldots, n-1$ in (2.6), by simplification, we have

$$\omega(x_i) = \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(1+\sum_{j=1}^{n-1} u_j\right)^{\lambda}} \prod_{j=1}^{n-1} u_j^{-1+\frac{\lambda}{p_j}} du_1 \ldots du_{n-1}. \tag{2.8}$$

Substitution of $n-1$ for k, λ for $\lambda(k)$ and $\lambda/\widetilde{p_j}$ for r_j $(j = 1, 2, \ldots, n)$ into (2.1), in view of (2.8), we have

$$\omega(x_i) = \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma\left(\frac{\lambda}{\widetilde{p_j}}\right) = \frac{1}{\Gamma(\lambda)} \prod_{j=1}^n \Gamma\left(\frac{\lambda}{p_j}\right).$$

Hence, (2.7) is valid. The lemma is proved. □

Lemma 2.3. *As in the assumption of Lemma 2.2, for* $0 < \varepsilon < \lambda$*, we have*

$$\begin{aligned} I &:= \varepsilon \int_1^\infty \cdots \int_1^\infty \frac{\prod_{i=1}^n x_i^{(\lambda-p_i-\varepsilon)/p_i}}{\left(\sum_{j=1}^n x_j\right)^{\lambda}} dx_1 \ldots dx_n \\ &\geq \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right) \qquad (\varepsilon \to 0^+). \end{aligned} \tag{2.9}$$

Proof. Setting $u_i = x_i/x_n$ $(i = 1, 2, \ldots, n-1)$ in the following, we find

$$\begin{aligned} I &= \varepsilon \int_1^\infty x_n^{-1-\varepsilon} \left[\int_{x_n^{-1}}^\infty \cdots \int_{x_n^{-1}}^\infty \frac{\prod_{i=1}^{n-1} u_i^{(\lambda-p_i-\varepsilon)/p_i}}{\left(1+\sum_{j=1}^{n-1} u_j\right)^{\lambda}} du_1 \ldots du_{n-1} \right] dx_n \\ &\geq \varepsilon \int_1^\infty x_n^{-1-\varepsilon} \left[\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{n-1} u_i^{(\lambda-p_i-\varepsilon)/p_i}}{\left(1+\sum_{j=1}^{n-1} u_j\right)^{\lambda}} du_1 \ldots du_{n-1} \right] dx_n \\ &\quad - \varepsilon \int_1^\infty x_n^{-1} \sum_{j=1}^{n-1} A_j(x_n) dx_n, \end{aligned} \tag{2.10}$$

where, for $j = 1, 2, \ldots, n-1$, $A_j(x_n)$ is defined by

$$A_j(x_n) := \int \cdots \int_{D_j} \frac{\prod_{i=1}^{n-1} u_i^{(\lambda - p_i - \varepsilon)/p_i}}{(1 + \sum_{j=1}^{n-1} u_j)^{\lambda}} du_1 \ldots du_{n-1}, \tag{2.11}$$

satisfying $D_j = \{(u_1, u_2, \ldots, u_{n-1}) | 0 < u_j \le x_n^{-1},\ 0 < u_k < \infty\ (k \ne j)\}$.

Without loss of generality, we estimate the integral $A_j(x_n)$ for $j = 1$.

(a) For $n = 2$, we have

$$\begin{aligned} A_1(x_n) &= \int_0^{x_n^{-1}} \frac{1}{(1+u_1)^{\lambda}} u_1^{(\lambda - p_1 - \varepsilon)/p_1} du_1 \\ &\le \int_0^{x_n^{-1}} u_1^{(\lambda - p_1 - \varepsilon)/p_1} du_1 = \frac{p_1}{\lambda - \varepsilon} x_n^{-(\lambda - \varepsilon)/p_1}; \end{aligned}$$

(b) For $n \in \mathbb{N} \backslash \{1, 2\}$, by (2.1), we have

$$\begin{aligned} A_1(x_n) &\le \int_0^{\infty} \cdots \int_0^{\infty} \frac{\prod_{i=2}^{n-1} u_i^{-1 + \frac{\lambda - \varepsilon}{p_i}}}{\left(1 + \sum_{j=2}^{n-1} u_j\right)^{\lambda}} du_1 \ldots du_{n-1} \int_0^{x_n^{-1}} u_1^{\frac{\lambda - p_1 - \varepsilon}{p_1}} du_1 \\ &\le \frac{p_1 x_n^{-\frac{\lambda - \varepsilon}{p_1}}}{\lambda - \varepsilon} \int_0^{\infty} \cdots \int_0^{\infty} \frac{\prod_{i=2}^{n-1} u_i^{-1 + (\lambda - \varepsilon)/p_i}}{\left(1 + \sum_{j=2}^{n-1} u_j\right)^{(\lambda - \varepsilon)(1 - p_1^{-1})}} du_1 \ldots du_{n-1} \\ &= \frac{p_1 x_n^{-\frac{\lambda - \varepsilon}{p_1}}}{\lambda - \varepsilon} \cdot \frac{\prod_{i=2}^{n} \Gamma(\frac{\lambda - \varepsilon}{p_i})}{\Gamma((\lambda - \varepsilon)(1 - p_1^{-1}))}. \end{aligned}$$

By virtue of the results of (a) and (b), for $j = 1, 2, \ldots, n-1$,we have

$$A_j(x_n) \le \frac{p_j}{\lambda - \varepsilon} x_n^{-(\lambda - \varepsilon)/p_j} O_j(1) \quad (\varepsilon \to 0^+,\ n \in \mathbb{N} \backslash \{1\}). \tag{2.12}$$

By (2.11), since for $\varepsilon \to 0^+$,

$$\int_0^{\infty} \cdots \int_0^{\infty} \frac{\prod_{i=1}^{n-1} u_i^{-1 + (\lambda - \varepsilon)/p_i}}{\left(1 + \sum_{j=1}^{n-1} u_j\right)^{\lambda}} du_1 \ldots du_{n-1} = \frac{\prod_{i=1}^{n} \Gamma\left(\frac{\lambda}{p_i}\right)}{\Gamma(\lambda)} + o(1);$$

$$\begin{aligned} \int_1^{\infty} x_n^{-1} \sum_{j=1}^{n-1} A_j(x_n) dx_n &= \sum_{j=1}^{n-1} \int_1^{\infty} x_n^{-1} A_j(x_n) dx_n \\ &\le \sum_{j=1}^{n-1} \frac{p_j}{\lambda - \varepsilon} O_j(1) \int_1^{\infty} x_n^{-1 - (\lambda - \varepsilon)/p_j} dx_n \\ &= \sum_{j=1}^{n-1} \left(\frac{p_j}{\lambda - \varepsilon}\right)^2 O_j(1), \end{aligned}$$

then by (2.10) , we find

$$I \geq \left(\frac{\prod_{i=1}^{n} \Gamma\left(\frac{\lambda}{p_i}\right)}{\Gamma(\lambda)} + o(1) \right) - \varepsilon \sum_{j=1}^{n-1} \left(\frac{p_j}{\lambda - \varepsilon} \right)^2 O_j(1)$$
$$\to \frac{\prod_{i=1}^{n} \Gamma\left(\frac{\lambda}{p_i}\right)}{\Gamma(\lambda)} \qquad (\varepsilon \to 0^+).$$

Thereby, (2.9) is valid and the lemma is proved. □

3. PROOF OF THE THEOREM AND REMARKS

Proof of Theorem 1.1. By Hölder's inequality, we have

$$J := \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n f_i(x_i) dx_1 \dots dx_n$$
$$= \int_0^\infty \cdots \int_0^\infty \left\{ \prod_{i=1}^n \frac{f_i(x_i)}{\left(\sum_{j=1}^n x_j\right)^{\lambda/p_i}} \left[x_i^{(p_i-\lambda)(1-p_i^{-1})} \prod_{\substack{j=1 \\ (j \neq i)}}^n x_j^{\frac{\lambda - p_j}{p_j}} \right]^{\frac{1}{p_i}} \right\} dx_1 \dots dx_n$$

$$\leq \prod_{i=1}^n \left\{ \int_0^\infty \cdots \int_0^\infty \frac{f_i^{p_i}(x_i)}{\left(\sum_{j=1}^n x_j\right)^\lambda} x_i^{(p_i-\lambda)(1-p_i^{-1})} \prod_{\substack{j=1 \\ (j \neq i)}}^n x_j^{\frac{\lambda - p_j}{p_j}} dx_1 \dots dx_n \right\}^{\frac{1}{p_i}}. \tag{3.1}$$

If (3.1) takes the form of equality, then there exists constants $C_1, C_2, \dots, C_n$, such that they are not all zero and for any $i \neq k \in \{1, 2, \dots, n\}$ (see [5]),

$$C_i \frac{f_i^{p_i}(x_i) x_i^{(p_i-\lambda)(1-p_i^{-1})}}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{\substack{j=1 \\ (j \neq i)}}^n x_j^{\frac{\lambda - p_j}{p_j}} = C_k \frac{f_k^{p_k}(x_k) x_k^{(p_k-\lambda)(1-p_k^{-1})}}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{\substack{j=1 \\ (j \neq k)}}^n x_j^{\frac{\lambda - p_j}{p_j}},$$
$$\text{a.e. in } (0, \infty) \times \cdots \times (0, \infty). \tag{3.2}$$

Assume that $C_i \neq 0$. By simplification of (3.2), we find

$$x_i^{p_i - \lambda} f_i^{p_i}(x_i) = F(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$
$$= \text{constant a.e. in } (0, \infty) \times \cdots \times (0, \infty),$$

which contradicts the fact that $0 < \int_0^\infty x_i^{p_i - \lambda - 1} f_i^{p_i}(x) dx < \infty$. Hence by (2.6) and (3.1), we conclude

$$J < \prod_{i=1}^n \left\{ \int_0^\infty \omega(x_i) x_i^{p_i - 1 - \lambda} f_i^{p_i}(x_i) dx_i \right\}^{\frac{1}{p_i}}. \tag{3.3}$$

Then by (2.7), we have (1.5).

For $0 < \varepsilon < \lambda$, setting $\widetilde{f}_i(x_i)$ as: $\widetilde{f}_i(x_i) = 0$, for $x_i \in (0, 1)$;

$$\widetilde{f}_i(x_i) = x_i^{(\lambda - p_i - \varepsilon)/p_i}, \qquad \text{for } x_i \in [1, \infty) \ (i = 1, 2, \dots, n),$$

then we find

$$\varepsilon \prod_{i=1}^{n} \left\{ \int_0^\infty x_i^{p_i-1-\lambda} \widetilde{f}_i^{p_i}(x_i) dx_i \right\}^{\frac{1}{p_i}} = 1. \tag{3.4}$$

By (2.9), we find

$$\varepsilon \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n \widetilde{f}_i(x_i) dx_1 \dots dx_n = I \geq \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right) \qquad (\varepsilon \to 0^+). \tag{3.5}$$

If the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$ in (1.5) is not the best possible, then there exists a positive constant $K < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$, such that (1.5) is still valid if one replaces $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$ by K. In particular, one has

$$\varepsilon \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n \widetilde{f}_i(x_i) dx_1 \dots dx_n < \varepsilon K \prod_{i=1}^n \left\{ \int_0^\infty x^{p_i-1-\lambda} \widetilde{f}_i^{p_i}(x) dx \right\}^{\frac{1}{p_i}},$$

and in view of (3.4) and (3.5), it follows that $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right) \leq K$ $(\varepsilon \to 0^+)$. This contradicts the fact $K < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$. Hence the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$ in (1.5) is the best possible.

The theorem is proved. □

Remark 3.1. For $\lambda = 1$, inequality (1.7) reduces to (see [10])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^\infty x^{p-2} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q-2} g^q(x) dx \right\}^{\frac{1}{q}}. \tag{3.6}$$

For $p = q = 2$, both (3.6) and (1.2) reduce to (1.1). It follows that inequalities (3.6) and (1.2) are different extensions of (1.1). Hence, inequality (1.5) is a new multiple extension of (1.1). Since all the constant factors in the obtained inequalities are the best possible, we have obtained new results.

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