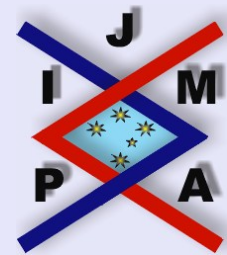


## ON ESTIMATES OF NORMAL STRUCTURE COEFFICIENTS OF BANACH SPACES

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## Abstract

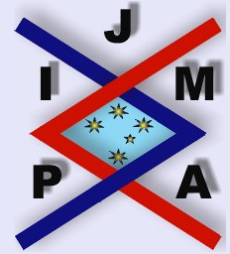
We obtained the estimates of Normal structure coefficient  $N(X)$  by Neumann-Jordan constant  $C_{NJ}(X)$  of a Banach space  $X$  and found that  $X$  has uniform normal structure if  $C_{NJ}(X) < (3 + \sqrt{5})/4$ . These results improved both Prus' [6] and Kato, Maligranda and Takahashi's [4] work.

*2000 Mathematics Subject Classification:* 46B20, 46E30.

*Key words:* Normal structure coefficient, Neumann-Jordan constant, Non-square constants, Banach space

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# 1. Introduction

Let  $X = (X, \|\cdot\|)$  be a real Banach space. Geometrical properties of a Banach space  $X$  are determined by its unit ball  $B_X = \{x \in X : \|x\| \leq 1\}$  or its unit sphere  $S_X = \{x \in X : \|x\| = 1\}$ . A Banach space  $X$  is called uniformly non-square if there exists a  $\delta \in (0, 1)$  such that for any  $x, y \in S_X$  either  $\|x+y\|/2 \leq 1 - \delta$  or  $\|x-y\|/2 \leq 1 - \delta$ . The constant

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S_X\}$$

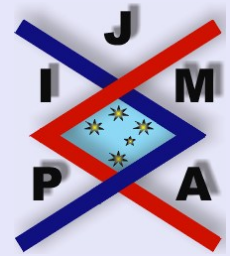
is called the non-square constant of  $X$  in the sense of James. It is well-known that  $\sqrt{2} \leq J(X) \leq 2$  if  $\dim X \geq 2$ . The Neumann-Jordan constant  $C_{NJ}(X)$  of a Banach space  $X$  is defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, \text{ not both zero}\right\}.$$

Clearly,  $1 \leq C_{NJ}(X) \leq 2$ , and  $X$  is a Hilbert space if and only if  $C_{NJ}(X) = 1$ . Kato, Maligranda and Takahashi [4] proved that for any non-trivial Banach space  $X$  ( $\dim X \geq 2$ ),

$$(1.1) \quad \frac{1}{2}J(X)^2 \leq C_{NJ}(X) \leq \frac{J(X)^2}{(J(X) - 1)^2 + 1}.$$

A Banach space  $X$  is said to have normal structure if  $r(K) < \text{diam}(K)$  for every non-singleton closed bounded convex subset  $K$  of  $X$ , where  $\text{diam}(K) = \sup\{\|x-y\| : x, y \in K\}$  is the diameter of  $K$  and  $r(K) = \inf\{\sup\{\|x -$



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$\|y\| : y \in K\} : x \in K\}$  is the Chebyshev radius of  $K$ . The normal structure coefficient of  $X$  is the number

$$N(X) = \inf\{\text{diam}(K)/r(K) : K \subset X \text{ bounded and convex, } \text{diam}(K) > 0\}.$$

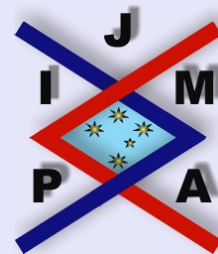
Obviously,  $1 \leq N(X) \leq 2$ . It is known [5], [2] that if the space  $X$  is reflexive, then the infimum in the definition of  $N(X)$  can be taken over all convex hulls of finite subsets of  $X$ . The space  $X$  is said to have uniform normal structure if  $N(X) > 1$ . If  $X$  has uniform normal structure, then  $X$  is reflexive and hence  $X$  has fixed point property. Gao and Lau [3] showed that if  $J(X) < 3/2$ , then  $X$  has uniform normal structure. Prus [6] gave an estimate of  $N(X)$  by  $J(X)$  which contains Gao-Lau's [3] and Bynum's [1] results: For any non-trivial Banach space  $X$ ,

$$(1.2) \quad N(X) \geq J(X) + 1 - \{(J(X) + 1)^2 - 4\}^{\frac{1}{2}}.$$

Kato, Maligranda and Takahashi [4] proved

$$(1.3) \quad N(X) \geq \left(C_{NJ}(X) - \frac{1}{4}\right)^{-\frac{1}{2}},$$

which implies that if  $C_{NJ}(X) < 5/4$  then  $X$  has uniform normal structure. This result is a little finer than Gao-Lau's condition by  $J(X)$ . This paper is devoted to improving the above results.




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## 2. Main Results

Our proofs are based on the idea due to Prus [6], who estimated  $N(X)$  by modulus of convexity of  $X$ . Let  $C$  be a convex hull of a finite subset of a Banach space  $X$ . Since  $C$  is compact, there exists an element  $y \in C$  such that  $\sup\{\|x - y\| : x \in C\} = r(C)$ . Translating the set  $C$  we can assume that  $y = 0$ . Prus [6] gave the following

**Proposition 2.1.** *Let  $C$  be a convex hull of a finite subset of a Banach space  $X$  such that  $\sup\{\|x\| : x \in C\} = r(C)$ . Then there exist points  $x_1, \dots, x_n \in C$ , norm-one functionals  $x_1^*, \dots, x_n^* \in X^*$  and nonnegative number  $\lambda_1, \dots, \lambda_n$  such that  $\sum_{i=1}^n \lambda_i = 1$ ,*

$$x_i^*(x_i) = \|x_i\| = r(C)$$

for  $i = 1, \dots, n$  and  $\sum_{i=1}^n \lambda_i x_i^*(x) = 0$  whenever  $\lambda x \in C$  for some  $\lambda > 0$ .

Without loss of generality, we assume  $r(C) = 1$  therefore  $C \subset B_X$ .

**Theorem 2.2.** *Let  $X$  be a non-trivial Banach space with the Neumann-Jordan constant  $C_{NJ}(X)$ . Then*

$$(2.1) \quad N(X) \geq \frac{2}{\sqrt{8C_{NJ}(X) - 1} - 1}.$$

*Proof.* Let  $C$  be a convex hull of a finite subset of  $X$  such that  $\sup\{\|x\| : x \in C\} = r(C) = 1$  and  $\text{diam}C = d$ . By Proposition 2.1 we obtain elements  $x_1, \dots, x_n \in C$ , norm-one functionals  $x_1^*, \dots, x_n^* \in X^*$  and nonnegative numbers  $\lambda_1, \dots, \lambda_n$  such that  $\sum_{i=1}^n \lambda_i = 1$ ,  $x_i^*(x_i) = 1$  and  $\sum_{j=1}^n \lambda_j x_j^*(x_i) = 0$  for  $i = 1, \dots, n$ .



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Define

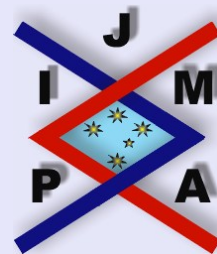
$$(2.2) \quad x_{i,j} = \frac{1}{d}(x_i - x_j), y_{i,j} = x_i$$

$i, j = 1, \dots, n$ . Clearly  $x_{i,j}, y_{i,j} \in B_X$  and  $x_{i,j} + y_{i,j} = (1 + 1/d)x_i - (1/d)x_j$ ,  $x_{i,j} - y_{i,j} = (-1 + 1/d)x_i - (1/d)x_j$ . It follows that

$$\begin{aligned} & \sum_{i,j=1}^n \lambda_i \lambda_j [\|x_{i,j} + y_{i,j}\|^2 + \|x_{i,j} - y_{i,j}\|^2] \\ & \geq \sum_{j=1}^n \lambda_j \sum_{i=1}^n \lambda_i [x_i^*(x_{i,j} + y_{i,j})]^2 + \sum_{i=1}^n \lambda_i \sum_{j=1}^n \lambda_j [x_j^*(x_{i,j} - y_{i,j})]^2 \\ & = \sum_{j=1}^n \lambda_j \sum_{i=1}^n \lambda_i \left[1 + \frac{1}{d} - \frac{1}{d}x_i^*(x_j)\right]^2 + \sum_{i=1}^n \lambda_i \sum_{j=1}^n \lambda_j \left[\frac{1}{d} + \left(1 - \frac{1}{d}\right)x_j^*(x_i)\right]^2 \\ & = \left(1 - \frac{1}{d}\right)^2 - 2\left(1 - \frac{1}{d}\right)\frac{1}{d} \sum_{j=1}^n \lambda_j \sum_{i=1}^n \lambda_i x_i^*(x_j) + \frac{1}{d^2} \sum_{j=1}^n \lambda_j \sum_{i=1}^n \lambda_i [x_i^*(x_j)]^2 \\ & \quad + \frac{1}{d^2} + 2\left(1 - \frac{1}{d}\right)\frac{1}{d} \sum_{i=1}^n \lambda_i \sum_{j=1}^n \lambda_j x_j^*(x_i) + \left(1 - \frac{1}{d}\right)^2 \sum_{i=1}^n \lambda_i \sum_{j=1}^n \lambda_j [x_j^*(x_i)]^2 \\ & \geq \left(1 + \frac{1}{d}\right)^2 + \frac{1}{d^2}. \end{aligned}$$

Therefore there exist  $i, j$  such that

$$\|x_{i,j} + y_{i,j}\|^2 + \|x_{i,j} - y_{i,j}\|^2 \geq \left(1 + \frac{1}{d}\right)^2 + \frac{1}{d^2}.$$



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From the definition of Neumann-Jordan constant we see that

$$(2.3) \quad C_{NJ}(X) \geq \frac{\|x_{i,j} + y_{i,j}\|^2 + \|x_{i,j} - y_{i,j}\|^2}{4} \geq \frac{1}{4} \left[ \left(1 + \frac{1}{d}\right)^2 + \frac{1}{d^2} \right].$$

This inequality is equivalent to the following one

$$(2.4) \quad d \geq \frac{2}{\sqrt{8C_{NJ}(X) - 1} - 1}.$$

Therefore, we obtain the desired estimate (2.1) since  $C \subset X$  is arbitrary. The proof is finished.  $\square$

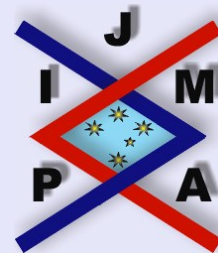
It is easy to check that

$$\frac{1}{\sqrt{C_{NJ}(X) - \frac{1}{4}}} < \frac{2}{\sqrt{8C_{NJ}(X) - 1} - 1}$$

when  $1 < C_{NJ}(X) < 5/4$ . Therefore, the estimate of the above theorem improves (1.3). It is also not difficult to check that

$$(2.5) \quad \sqrt{2C_{NJ}(X) + 1} - \left( (\sqrt{2C_{NJ}(X) + 1})^2 - 4 \right)^{\frac{1}{2}} < \frac{2}{\sqrt{8C_{NJ}(X) - 1} - 1}$$

when  $1 < C_{NJ}(X) < 5/4$ . Since  $J(X) \leq \sqrt{2C_{NJ}(X)}$ , and the function  $x + 1 - ((x + 1)^2 - 4)^{1/2}$  is decreasing, we have (1.2) from (2.1) and (2.5). So (1.2) becomes a corollary of (2.1).



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Prus [6] gave the result that if  $J(X) < 4/3$ , then  $N(X) > 1$ . Gao and Lau [3] gave a condition that if  $J(X) < 3/2$  then  $N(X) > 1$ . Then they asked whether the estimate  $J(X) < 3/2$  is sharp for  $X$  to have uniform normal structure. Kato, Maligranda and Takahashi [4] found that if  $C_{NJ}(X) < 5/4$ , which implies  $J(X) < \sqrt{10}/2$ , then  $N(X) > 1$ . The following theorem will give a wider interval of  $C_{NJ}(X)$  for  $X$  to have uniform normal structure.

**Theorem 2.3.** *Let  $X$  be a non-trivial Banach space with the Neumann-Jordan constant  $C_{NJ}(X)$  and normal structure coefficient  $N(X)$ . Then*

$$(2.6) \quad C_{NJ}(X) \geq \frac{\left(\sqrt{\frac{N^2(X)}{4} + \frac{1}{N^2(X)}} + N(X) - \frac{1}{N(X)}\right)^2 + \frac{1}{N^2(X)}}{2 \left[1 + \left(\sqrt{\frac{N^2(X)}{4} + \frac{1}{N^2(X)}} + N(X) - \frac{2}{N(X)}\right)^2\right]}.$$

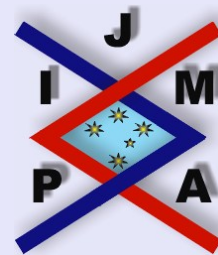
Moreover, if  $C_{NJ}(X) < (3 + \sqrt{5})/4$  or  $J(X) < (1 + \sqrt{5})/2$ , then  $N(X) > 1$  and hence  $X$  has uniform normal structure.

*Proof.* We modify the first step in the proof of Theorem 2.2. In (2.2), let

$$(2.7) \quad x_{i,j} = \frac{1}{d}(x_i - x_j), y_{i,j} = tx_i$$

with  $t > 0$ . Then  $\|x_{i,j}\| \leq 1$ ,  $\|y_{i,j}\| = t$ . Similar to (2.3), we obtain

$$(2.8) \quad C_{NJ}(X) \geq \frac{\left(t + \frac{1}{d}\right)^2 + \frac{1}{d^2}}{2(1 + t^2)}$$




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for any  $t > 0$ . The function

$$f(t) = \frac{\left(t + \frac{1}{d}\right)^2 + \frac{1}{d^2}}{2(1 + t^2)}$$

reach the maximum at the point

$$t_0 = \sqrt{\frac{d^2}{4} + \frac{1}{d^2}} + d - \frac{2}{d}.$$

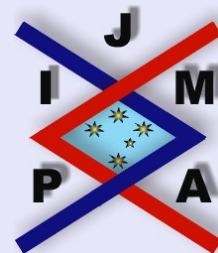
It is decreasing on  $t > t_0$  and increasing on  $0 < t < t_0$ . Therefore, we have

$$(2.9) \quad C_{NJ}(X) \geq \frac{\left(\sqrt{\frac{d^2}{4} + \frac{1}{d^2}} + d - \frac{1}{d}\right)^2 + \frac{1}{d^2}}{2 \left[1 + \left(\sqrt{\frac{d^2}{4} + \frac{1}{d^2}} + d - \frac{2}{d}\right)^2\right]}.$$

Since the function

$$c = g(d) := \frac{\left(\sqrt{\frac{d^2}{4} + \frac{1}{d^2}} + d - \frac{1}{d}\right)^2 + \frac{1}{d^2}}{2 \left[1 + \left(\sqrt{\frac{d^2}{4} + \frac{1}{d^2}} + d - \frac{2}{d}\right)^2\right]}$$

is strictly decreasing and continuous on  $1 \leq d \leq 2$ , we know that the inverse function  $d = g^{-1}(c)$  exists and must also be decreasing. Thus, we have from (2.9) that  $d \geq g^{-1}(C_{NJ}(X))$ . It follows by take the infimum of  $d$  that



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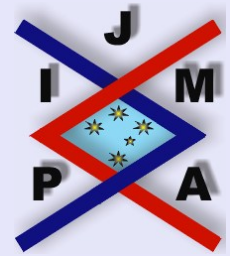
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$N(X) \geq g^{-1}(C_{NJ}(X))$ . Equivalently, we have (2.6). From the above statements of monotony property, we deduce that  $N(X) = 1$  is corresponding to  $C_{NJ}(X) = (3 + \sqrt{5})/4$ . Therefore, if  $C_{NJ}(X) < (3 + \sqrt{5})/4$ , then  $N(X) > 1$ . Since the non-square constant  $J(X) \leq \sqrt{2C_{NX}}$ , we have in other word that if  $J(X) < (1 + \sqrt{5})/2$ , then  $N(X) > 1$ .  $\square$




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