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NOTE ON INEQUALITIES INVOLVING INTEGRAL TAYLOR'S REMAINDER

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ABSTRACT. In this paper, some inequalities involving the integral Taylor's remainder are obtained by using various well-known methods.

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1. INTRODUCTION

In [4] – [5], H. Gauchman has derived some new types of inequalities involving Taylor's remainder.

In [1], L. Bougoffa continued to create several integral inequalities involving Taylor's remainder.

The purpose of this paper is to give some supplements and improvements for the results obtained in [1] – [3].

In [1], two notations $R_{n,f}(c,x)$ and $r_{n,f}(a,b)$ have been adopted to denote the nth Taylor's remainder of function f with center c and the integral Taylor's remainder respectively, i.e.,

$$R_{n,f}(c,x) = f(x) - \sum_{k=0}^{n} \frac{f^{(n)}(c)}{n!}(x-c)^k,$$

and

$$r_{n,f}(a,b) = \int_a^b \frac{(b-x)^n}{n!} f^{(n+1)}(x)dx.$$

However, it is evident that

$$R_{n,f}(a,b) = \int_a^b \frac{(b-x)^n}{n!} f^{(n+1)}(x)\, dx = r_{n,f}(a,b),$$

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and

$$(-1)^n R_{n,f}(b,a) = \int_a^b \frac{(x-a)^n}{n!} f^{(n+1)}(x)\,dx = (-1)^n r_{n,f}(b,a).$$

So, we would like only to keep the notation $R_{n,f}(\cdot,\cdot)$ in what follows.

We start by changing the order of integration to give a simple different proof of Lemma 1.1 and Lemma 1.2 in [5] and [1]. i.e.,

$$\begin{aligned}\int_a^b R_{n,f}(a,x)\,dx &= \int_a^b \left(\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt\right) dx \\ &= \int_a^b \left(\int_t^b \frac{(x-t)^n}{n!} f^{(n+1)}(t)dx\right) dt \\ &= \int_a^b \frac{(b-t)^{n+1}}{(n+1)!} f^{(n+1)}(t)dt.\end{aligned}$$

and

$$\begin{aligned}(-1)^{n+1}\int_a^b R_{n,f}(b,x)dx &= \int_a^b \left(\int_x^b \frac{(t-x)^n}{n!} f^{(n+1)}(t)dt\right) dx \\ &= \int_a^b \left(\int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(t)dx\right) dt \\ &= \int_a^b \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(t)dt.\end{aligned}$$

2. Results Obtained via the Leibniz Formula

We prove the following theorem by using the Leibniz formula.

Theorem 2.1. *Let f be a function defined on $[a,b]$. Assume that $f \in C^{n+1}([a,b])$. Then*

$$\left|\sum_{k=0}^{p}(-1)^k C_p^k R_{n-k,f}(a,b)\right| \le \sum_{k=0}^{p-1} C_{p-1}^k \left|f^{(n-k)}(a)\right| \frac{(b-a)^{n-k}}{(n-k)!}, \tag{2.1}$$

$$\left|\sum_{k=0}^{p}(-1)^{n-k+1} C_p^k R_{n-k,f}(b,a)\right| \le \sum_{k=0}^{p-1} C_{p-1}^k \left|f^{(n-k)}(b)\right| \frac{(b-a)^{n-k}}{(n-k)!}, \tag{2.2}$$

$$\left|\sum_{k=0}^{p}(-1)^k C_p^k \int_a^b R_{n-k,f}(a,x)dx\right| \le \sum_{k=0}^{p-1} C_{p-1}^k \left|f^{(n-k)}(a)\right| \frac{(b-a)^{n-k+1}}{(n-k+1)!}, \tag{2.3}$$

$$\left|\sum_{k=0}^{p}(-1)^{n-k+1} C_p^k \int_a^b R_{n-k,f}(b,x)dx\right| \le \sum_{k=0}^{p-1} C_{p-1}^k \left|f^{(n-k)}(b)\right| \frac{(b-a)^{n-k+1}}{(n-k+1)!}, \tag{2.4}$$

where $C_p^k = \frac{p!}{(p-k)!k!}$.

Proof. We apply the following Leibniz formula

$$(FG)^{(p)} = F^{(p)}G + C_p^1 F^{(p-1)}G^{(1)} + \cdots + C_p^{p-1}F^{(1)}G^{(p-1)} + FG^{(P)},$$

provided the functions $F, G \in C^p([a,b])$.

Let $F(x) = f^{(n-p+1)}(x)$, $G(x) = \frac{(b-x)^n}{n!}$. Then

$$\left(f^{(n-p+1)}(x)\frac{(b-x)^n}{n!}\right)^{(p)} = \sum_{k=0}^{p}(-1)^k C_p^k f^{(n-k+1)}(x)\frac{(b-x)^{n-k}}{(n-k)!}.$$

Integrating both sides of the preceding equation with respect to x from a to b gives us

$$\left[\left(f^{(n-p+1)}(x)\frac{(b-x)^n}{n!}\right)^{(p-1)}\right]_{x=a}^{x=b} = \sum_{k=0}^{p}(-1)^k C_p^k \int_a^b f^{(n-k+1)}(x)\frac{(b-x)^{n-k}}{(n-k)!}\,dx.$$

The integral on the right is $R_{n-k,f}(a,x)$, and to evaluate the term on the left hand side, we must again apply the Leibniz formula, obtaining

$$-\sum_{k=0}^{p-1}(-1)^k C_{p-1}^k f^{(n-k)}(a)\frac{(b-a)^{n-k}}{(n-k)!} = \sum_{k=0}^{p}(-1)^k C_p^k R_{n-k,f}(a,b).$$

Consequently,

$$\left|\sum_{k=0}^{p}(-1)^k C_p^k R_{n-k,f}(a,b)\right| \le \sum_{k=0}^{p-1} C_{p-1}^k \left|f^{(n-k)}(a)\right| \frac{(b-a)^{n-k}}{(n-k)!},$$

which proves (2.1).

For the proof of (2.2), we take

$$F(x) = f^{(n-p+1)}(x), \quad G(x) = \frac{(x-a)^n}{n!}.$$

For the proof of (2.3), we take

$$F(x) = f^{(n-p+1)}(x), \quad G(x) = \frac{(b-x)^{n+1}}{(n+1)!}.$$

For the proof of (2.4), we take

$$F(x) = f^{(n-p+1)}(x), \quad G(x) = \frac{(x-a)^{n+1}}{(n+1)!}.$$

□

Remark 2.2. It should be noticed that (2.3) and (2.4) have been mentioned and proved in [1] with some misprints in the conclusion.

3. RESULTS OBTAINED BY A VARIANT OF THE GRÜSS INEQUALITY

The following is a variant of the Grüss inequality which has been proved almost at the same time by X.L. Cheng and J. Sun in [3] as well as M. Matić in [6] respectively.

Let $h, g : [a,b] \to \mathbb{R}$ be two integrable functions such that $\gamma \le g(x) \le \Gamma$ for some constants γ, Γ for all $x \in [a,b]$. Then

$$(3.1)\quad \left|\int_a^b h(x)g(x)\,dx - \frac{1}{b-a}\int_a^b h(x)\,dx\int_a^b g(x)dx\right| \le \frac{1}{2}\left(\int_a^b \left|h(x) - \frac{1}{b-a}\int_a^b h(y)dy\right| dx\right)(\Gamma - \gamma).$$

Theorem 3.1. *Let $f(x)$ be a function defined on $[a,b]$ such that $f \in C^{n+1}([a,b])$ and $m \leq f^{(n+1)}(x) \leq M$ for each $x \in [a,b]$, where m and M are constants. Then*

$$\left| R_{n,f}(a,b) - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+1)!}(b-a)^n \right| \leq \frac{n(b-a)^{n+1}(M-m)}{(n+1)!(n+1)\sqrt[n]{n+1}}, \tag{3.2}$$

$$\left| (-1)^{n+1} R_{n,f}(b,a) - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+1)!}(b-a)^n \right| \leq \frac{n(b-a)^{n+1}(M-m)}{(n+1)!(n+1)\sqrt[n]{n+1}}, \tag{3.3}$$

$$\left| \int_a^b R_{n,f}(a,x)\,dx - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \right| \leq \frac{(n+1)(b-a)^{n+2}(M-m)}{(n+2)!(n+2)\sqrt[n+1]{n+2}} \tag{3.4}$$

and

$$\left| (-1)^{n+1} \int_a^b R_{n,f}(b,x)\,dx - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+2)!}(b-a)^{n+1} \right| \leq \frac{(n+1)(b-a)^{n+2}(M-m)}{(n+2)!(n+2)\sqrt[n+1]{n+2}}. \tag{3.5}$$

Proof. To prove (3.2), setting $g(x) = f^{(n+1)}(x)$ and $h(x) = \frac{(b-x)^n}{n!}$ in (3.1), we obtain

$$\begin{aligned} \left| R_{n,f}(a,b) - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+1)!}(b-a)^n \right| &\leq \frac{M-m}{2} \int_a^b \left| \frac{(b-x)^n}{n!} - \frac{(b-a)^n}{(n+1)!} \right| dx \\ &= \frac{n(b-a)^{n+1}(M-m)}{(n+1)!(n+1)\sqrt[n]{n+1}}. \end{aligned}$$

The proofs of (3.3), (3.4) and (3.5) are similar and so are omitted. □

Remark 3.2. It should be noticed that Theorem 3.1 improves Theorem 3.1 in [1] and Theorem 2.1 in [5].

4. Results Obtained via the Steffensen Inequality

In [2] we can find a general version of the well-known Steffensen inequality as follows: Let $h : [a,b] \to \mathbb{R}$ be a nonincreasing mapping on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ be an integrable mapping on $[a,b]$ with

$$\phi \leq g(x) \leq \Phi, \text{ for all } x \in [a,b],$$

then

$$\phi \int_a^{b-\lambda} h(x)dx + \Phi \int_{b-\lambda}^b h(x)dx \leq \int_a^b h(x)g(x)dx \leq \Phi \int_a^{a+\lambda} h(x)dx + \phi \int_{a+\lambda}^b h(x)dx, \tag{4.1}$$

where

$$\lambda = \int_a^b G(x)\,dx, \quad G(x) = \frac{g(x)-\phi}{\Phi-\phi}, \quad \Phi \neq \phi. \tag{4.2}$$

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f(x) \in C^{n+1}([a,b])$ and $m \leq f^{(n+1)}(x) \leq M$ for each $x \in [a,b]$, where m and M are constants. Then*

$$\begin{aligned} \frac{m(b-a)^{n+1} + (M-m)\lambda^{n+1}}{(n+1)!} &\leq R_{n,f}(a,b) \\ &\leq \frac{M(b-a)^{n+1} - (M-m)(b-a-\lambda)^{n+1}}{(n+1)!}, \end{aligned} \tag{4.3}$$

$$\frac{m(b-a)^{n+1}+(M-m)\lambda^{n+1}}{(n+1)!} \le (-1)^{n+1}R_{n,f}(b,a) \le \frac{M(b-a)^{n+1}-(M-m)(b-a-\lambda)^{n+1}}{(n+1)!}, \tag{4.4}$$

$$\frac{m(b-a)^{n+2}+(M-m)\lambda^{n+2}}{(n+2)!} \le \int_a^b R_{n,f}(a,x)dx \le \frac{M(b-a)^{n+2}-(M-m)(b-a-\lambda)^{n+2}}{(n+2)!}, \tag{4.5}$$

and

$$\frac{m(b-a)^{n+2}+(M-m)\lambda^{n+2}}{(n+2)!} \le (-1)^{n+1}\int_a^b R_{n,f}(b,x)dx \le \frac{M(b-a)^{n+2}-(M-m)(b-a-\lambda)^{n+2}}{(n+2)!}, \tag{4.6}$$

where $\lambda = \frac{f(b)-f(a)-m(b-a)}{M-m}$.

Proof. Observe that $\frac{(b-x)^n}{n!}$ is a decreasing function of x on $[a,b]$, then by (4.1) and (4.2) we have

$$m\int_a^{b-\lambda}\frac{(b-x)^n}{n!}dx + M\int_{b-\lambda}^b \frac{(b-x)^n}{n!}dx \le \int_a^b \frac{(b-x)^n}{n!}f^{(n+1)}(x)dx \le M\int_a^{a+\lambda}\frac{(b-x)^n}{n!}dx + m\int_{a+\lambda}^b\frac{(b-x)^n}{n!}dx$$

with

$$\lambda = \int_a^b \frac{f^{(n+1)}(x)-m}{M-m}dx = \frac{f^{(n)}(b)-f^{(n)}(a)-m(b-a)}{M-m},$$

and (4.3) follows.

Since $\frac{(x-a)^n}{n!}$ is a increasing function of x on $[a,b]$, then

$$M\int_a^{a+\lambda}\frac{(x-a)^n}{n!}dx + m\int_{a+\lambda}^b\frac{(x-a)^n}{n!}dx \le \int_a^b\frac{(x-a)^n}{n!}f^{(n+1)}(x)dx \le m\int_a^{b-\lambda}\frac{(x-a)^n}{n!}dx + M\int_{b-\lambda}^b\frac{(x-a)^n}{n!}dx,$$

and (4.4) follows.

The proofs of (4.5) and (4.6) are similar and so are omitted. □

Remark 4.2. It should be mentioned that (4.5) and (4.6) have also been proved in [4]

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