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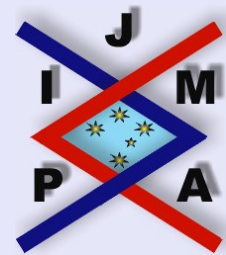
COEFFICIENT ESTIMATES FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract

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Abstract

For some real α ($\alpha > 1$), two subclasses $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ of analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ in \mathbb{U} are introduced. The object of the present paper is to discuss the coefficient estimates for functions $f(z)$ belonging to the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$.

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1. Introduction and Definitions

Let \mathcal{A} denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{M}(\alpha)$ be the subclass of \mathcal{A} consisting of functions $f(z)$ which satisfy the inequality:

$$\operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} < \alpha \quad (z \in \mathbb{U})$$

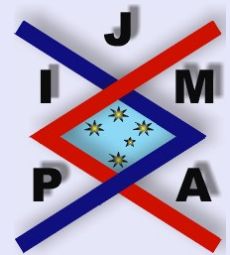
for some α ($\alpha > 1$). And let $\mathcal{N}(\alpha)$ be the subclass of \mathcal{A} consisting of functions $f(z)$ which satisfy the inequality:

$$\operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} < \alpha \quad (z \in \mathbb{U})$$

for some α ($\alpha > 1$). Then, we see that $f(z) \in \mathcal{N}(\alpha)$ if and only if $z f'(z) \in \mathcal{M}(\alpha)$.

Remark 1.1. For $1 < \alpha \leq \frac{4}{3}$, the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were introduced by Uralegaddi et al. [3].

Remark 1.2. The classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ correspond to the case $k = 2$ of the classes $\mathcal{M}_k(\alpha)$ and $\mathcal{N}_k(\alpha)$, respectively, which were investigated recently by Owa and Srivastava [1].



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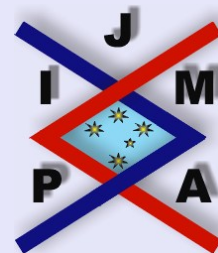
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We easily see that

Example 1.1.

$$(i) f(z) = z(1 - z)^{2(\alpha-1)} \in \mathcal{M}(\alpha).$$

$$(ii) g(z) = \frac{1}{2\alpha-1} \{1 - (1 - z)^{2\alpha-1}\} \in \mathcal{N}(\alpha).$$



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2. Inclusion Theorems Involving Coefficient Inequalities

In this section we derive sufficient conditions for $f(z)$ to belong to the aforementioned function classes, which are obtained by using coefficient inequalities.

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\} |a_n| \leq 2(\alpha-1)$$

for some k ($0 \leq k \leq 1$) and some α ($\alpha > 1$), then $f(z) \in \mathcal{M}(\alpha)$.

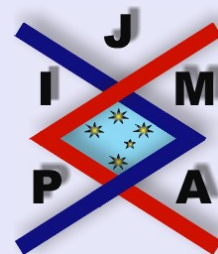
Proof. Let us suppose that

$$(2.1) \quad \sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\} |a_n| \leq 2(\alpha-1)$$

for $f(z) \in \mathcal{A}$.

It suffices to show that

$$\left| \frac{\frac{zf'(z)}{f(z)} - k}{\frac{zf'(z)}{f(z)} - (2\alpha - k)} \right| < 1 \quad (z \in \mathbb{U}).$$



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We note that

$$\begin{aligned} \left| \frac{\frac{zf'(z)}{f(z)} - k}{\frac{zf'(z)}{f(z)} - (2\alpha - k)} \right| &= \left| \frac{1 - k + \sum_{n=2}^{\infty} (n - k)a_n z^{n-1}}{1 + k - 2\alpha + \sum_{n=2}^{\infty} (n + k - 2\alpha)a_n z^{n-1}} \right| \\ &\leq \frac{1 - k + \sum_{n=2}^{\infty} (n - k)|a_n||z|^{n-1}}{2\alpha - 1 - k - \sum_{n=2}^{\infty} |n + k - 2\alpha| |a_n||z|^{n-1}} \\ &< \frac{1 - k + \sum_{n=2}^{\infty} (n - k)|a_n|}{2\alpha - 1 - k - \sum_{n=2}^{\infty} |n + k - 2\alpha| |a_n|}. \end{aligned}$$

The last expression is bounded above by 1 if

$$1 - k + \sum_{n=2}^{\infty} (n - k)|a_n| \leq 2\alpha - 1 - k - \sum_{n=2}^{\infty} |n + k - 2\alpha| |a_n|$$

which is equivalent to our condition:

$$\sum_{n=2}^{\infty} \{(n - k) + |n + k - 2\alpha|\} |a_n| \leq 2(\alpha - 1)$$

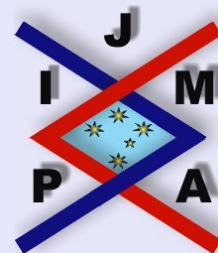
of the theorem. This completes the proof of the theorem. \square

If we take $k = 1$ and some α ($1 < \alpha \leq \frac{3}{2}$) in Theorem 2.1, then we have

Corollary 2.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} (n - \alpha)|a_n| \leq \alpha - 1$$

for some α ($1 < \alpha \leq \frac{3}{2}$), then $f(z) \in \mathcal{M}(\alpha)$.



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Example 2.1. The function $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{4(\alpha - 1)}{n(n+1)(n-k+|n+k-2\alpha|)} z^n$$

belongs to the class $\mathcal{M}(\alpha)$.

For the class $\mathcal{N}(\alpha)$, we have

Theorem 2.3. If $f(z) \in \mathcal{A}$ satisfies

$$(2.2) \quad \sum_{n=2}^{\infty} n(n-k+1+|n+k-2\alpha|)|a_n| \leq 2(\alpha-1)$$

for some k ($0 \leq k \leq 1$) and some α ($\alpha > 1$), then $f(z)$ belongs to the class $\mathcal{N}(\alpha)$.

Corollary 2.4. If $f(z) \in \mathcal{A}$ satisfies

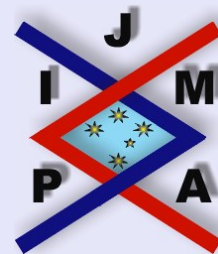
$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \alpha-1$$

for some α ($1 < \alpha \leq \frac{3}{2}$), then $f(z) \in \mathcal{N}(\alpha)$.

Example 2.2. The function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{4(\alpha-1)}{n^2(n+1)(n-k+|n+k-2\alpha|)} z^n$$

belongs to the class $\mathcal{N}(\alpha)$.



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Further, denoting by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of \mathcal{A} consisting of all starlike functions of order α , and of all convex functions of order α , respectively (see [2]), we derive

Theorem 2.5. *If $f(z) \in \mathcal{A}$ satisfies the coefficient inequality (2.1) for some α ($1 < \alpha \leq \frac{k+2}{2} \leq \frac{3}{2}$), then $f(z) \in \mathcal{S}^*\left(\frac{4-3\alpha}{3-2\alpha}\right)$. If $f(z) \in \mathcal{A}$ satisfies the coefficient inequality (2.2) for some α ($1 < \alpha \leq \frac{k-2}{2} \leq \frac{3}{2}$) then $f(z) \in \mathcal{K}\left(\frac{4-3\alpha}{3-2\alpha}\right)$.*

Proof. For some α ($1 < \alpha \leq \frac{k+2}{2} \leq \frac{3}{2}$), we see that the coefficient inequality (2.1) implies that

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq \alpha - 1.$$

It is well-known that if $f(z) \in \mathcal{A}$ satisfies

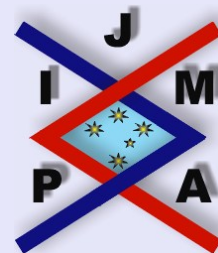
$$\sum_{n=2}^{\infty} \frac{n - \beta}{1 - \beta} |a_n| \leq 1$$

for some β ($0 \leq \beta < 1$), then $f(z) \in \mathcal{S}^*(\beta)$ by Silverman [2]. Therefore, we have to find the smallest positive β such that

$$\sum_{n=2}^{\infty} \frac{n - \beta}{1 - \beta} |a_n| \leq \sum_{n=2}^{\infty} \frac{n - \alpha}{\alpha - 1} |a_n| \leq 1.$$

This gives that

$$(2.3) \quad \beta \leq \frac{(2 - \alpha)n - \alpha}{n - 2\alpha + 1}$$



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for all $n = 2, 3, 4, \dots$. Noting that the right-hand side of the inequality (2.3) is increasing for n , we conclude that

$$\beta \leq \frac{4 - 3\alpha}{3 - 2\alpha},$$

which proves that $f(z) \in \mathcal{S}^* \left(\frac{4-3\alpha}{3-2\alpha} \right)$. Similarly, we can show that if $f(z) \in \mathcal{A}$ satisfies (2.2), then $f(z) \in \mathcal{K} \left(\frac{4-3\alpha}{3-2\alpha} \right)$. \square

Our result for the coefficient estimates of functions $f(z) \in \mathcal{M}(\alpha)$ is contained in

Theorem 2.6. *If $f(z) \in \mathcal{M}(\alpha)$, then*

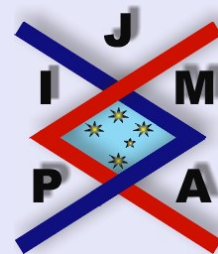
$$(2.4) \quad |a_n| \leq \frac{\prod_{j=2}^n (j + 2\alpha - 4)}{(n-1)!} \quad (n \geq 2).$$

Proof. Let us define the function $p(z)$ by

$$p(z) = \frac{\alpha - \frac{zf'(z)}{f(z)}}{\alpha - 1}$$

for $f(z) \in \mathcal{M}(\alpha)$. Then $p(z)$ is analytic in \mathbb{U} , $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$ ($z \in \mathbb{U}$). Therefore, if we write

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$



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then $|p_n| \leq 2$ ($n \geq 1$). Since

$$\alpha f(z) - z f'(z) = (\alpha - 1)p(z)f(z),$$

we obtain that

$$(1 - n)a_n = (\alpha - 1)(p_{n-1} + a_2 p_{n-2} + a_3 p_{n-3} + \cdots + a_{n-1} p_1).$$

If $n = 2$, then $-a_2 = (\alpha - 1)p_1$ implies that

$$|a_2| = (\alpha - 1)|p_1| \leq 2\alpha - 2.$$

Thus the coefficient estimate (2.4) holds true for $n = 2$. Next, suppose that the coefficient estimate

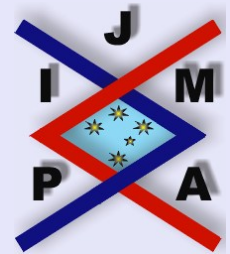
$$|a_k| \leq \frac{\prod_{j=2}^k (j + 2\alpha - 4)}{(k - 1)!}$$

is true for all $k = 2, 3, 4, \dots, n$. Then we have that

$$-n a_{n+1} = (\alpha - 1)(p_n + a_2 p_{n-1} + a_3 p_{n-2} + \cdots + a_n p_1),$$

so that

$$\begin{aligned} n|a_{n+1}| &\leq (2\alpha - 2)(1 + |a_2| + |a_3| + \cdots + |a_n|) \\ &\leq (2\alpha - 2) \left(1 + (2\alpha - 2) + \frac{(2\alpha - 2)(2\alpha - 1)}{2!} + \cdots \right. \\ &\quad \left. + \frac{\prod_{j=2}^n (j + 2\alpha - 4)}{(n - 1)!} \right) \end{aligned}$$



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$$\begin{aligned}
&= (2\alpha - 2) \left(\frac{(2\alpha - 1)2\alpha(2\alpha + 1) \cdots (2\alpha + n - 4)}{(n - 2)!} \right. \\
&\quad \left. + \frac{(2\alpha - 2)(2\alpha - 1)2\alpha \cdots (2\alpha + n - 4)}{(n - 1)!} \right) \\
&= \frac{\prod_{j=2}^{n+1} (j + 2\alpha - 4)}{(n - 1)!}.
\end{aligned}$$

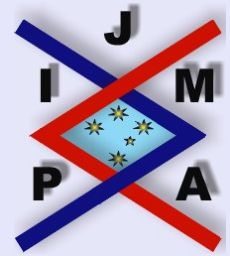
Thus, the coefficient estimate (2.4) holds true for the case of $k = n + 1$. Applying the mathematical induction for the coefficient estimate (2.4), we complete the proof of Theorem 2.6. \square

For the functions $f(z)$ belonging to the class $\mathcal{N}(\alpha)$, we also have

Theorem 2.7. *If $f(z) \in \mathcal{N}(\alpha)$, then*

$$|a_n| \leq \frac{\prod_{j=2}^n (j + 2\alpha - 4)}{n!} \quad (n \geq 2).$$

Remark 2.1. *We can not show that Theorem 2.6 and Theorem 2.7 are sharp. If we can prove that Theorem 2.6 is sharp, then the sharpness of Theorem 2.7 follows.*



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