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SOME RESULTS ON THE UNITARY ANALOGUE OF THE LEHMER PROBLEM

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ABSTRACT. Let M be a positive integer with $M \geq 4$, and let φ^* denote the unitary analogue of Euler's totient function φ. Using Grytczuk-Wójtowicz's techniques from the paper [2] we strengthen considerably the lower estimations of the solutions n of the equation $M\varphi^*(n) = n - 1$. Moreover, we show that the set of positive integers, which do not fulfil this equation for any $M \geq 2$, contains an interesting subset generated by Ramsey's theorem.

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1. INTRODUCTION

Throughout this paper $\mathbb{N}$ denotes the set of positive integers, and the numbers $M, n \in \mathbb{N}$ are fixed with $M \geq 2$. Let φ be the Euler's totient function, and let $\varphi^*(n)$ be the number of all natural numbers $k \leq n$ such that $(k, n)^* = 1$, where $(k, n)^*$ is the greatest divisor d of k, which is also a *unitary divisor of* n (i.e., such that $(d, n/d) = 1$).

A classical (and still unsolved) problem proposed by Lehmer concerns the existence of a composite number n which fulfils the equation

$$M\varphi(n) = n - 1 \tag{1.1}$$

(see e.g. [3, p. 212-215]). Subbarao, Siva Rama Prasad and Dixit studied in [4, 5] an analogous equation for the function φ^*:

$$M\varphi^*(n) = n - 1. \tag{1.2}$$

Let

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \cdots \cdot p_r^{\alpha_r} \tag{1.3}$$

be the prime factorization of n, where $p_1 < p_2 < \cdots < p_r$ and $\alpha_1, \dots, \alpha_r \in \mathbb{N}$. Put $\omega(n) = r$. It is known (and easy to verify), that every solution n of the equation (1.1), must be odd and squarefree. Moreover, since for n of the form (1.3) we have

$$\varphi^*(n) = (p_1^{\alpha_1} - 1) \cdot (p_2^{\alpha_2} - 1) \cdot \cdots \cdot (p_r^{\alpha_r} - 1)$$

108-08

(see [4]), *no solution* n *of the equation (1.2) can be the power of a prime number*.

Put $\mathcal{S}_M^* := \{n \in \mathbb{N} : M\varphi^*(n) = n-1\}$, and $\mathcal{S}^* := \bigcup_{M\geq 2} \mathcal{S}_M^*$. In the papers [4, 5] the authors obtained the following estimations of $n \in \mathcal{S}^*$:

$$n < (r-2.3)^{2^r-1}, \text{ where } r = \omega(n), \tag{1.4}$$

$$\text{if } 3 \nmid n, \text{ then } \omega(n) \geq 11 \text{ if } 5|n, \text{ and } \omega(n) \geq 17 \text{ if } 5 \nmid n, \tag{1.5}$$

$$\omega(n) \geq 1850 \text{ when } 3|n, \tag{1.6}$$

$$\omega(n) \geq 17 \text{ when the number } 455 \text{ is not a unitary divisor of } n, \tag{1.7}$$

$$\omega(n) \geq 33 \text{ for } M = 3, 4 \text{ or } 5. \tag{1.8}$$

In this paper, we show that the techniques of [2] allow us to obtain lower estimations for the elements of $\mathcal{S}_M^*$, where $M \geq 4$, which are considerably stronger than cited in (1.5) – (1.8) and *unconditional*.

Our main result reads as follows.

Theorem 1.1. *Let* $M \geq 4$ *and let* $n \in \mathcal{S}_M^*$ *be of the form* (1.3).

(a) *If* $p_1 = 3$, *then* $\omega(n) \geq 3049^{M/4} - 1509$.
(b) *If* $p_1 > 3$, *then* $\omega(n) \geq 143^{M/4} - 1$.

Thus, for $n \in \mathcal{S}_M^*$, where $M \geq 4$, we have (in general): $\omega(n) \geq 1540$ when $3|n$ (for $M = 4$ this result is slightly weaker than (1.6)), and $\omega(n) \geq 142$ when $3 \nmid n$ (for $M = 4$ this result is stronger than (1.8)). Moreover,

- $\omega(n) \geq 21147$ when $3|n$, and $\omega(n) \geq 493$ when $3 \nmid n$ — for $M = 5$;
- $\omega(n) \geq 166849$ when $3|n$, and $\omega(n) \geq 1709$ when $3 \nmid n$ — for $M = 6$; and
- $\omega(n) > 1249543$ when $3|n$, and $\omega(n) \geq 5912$ when $3 \nmid n$ — for $M \geq 7$.

Further, by an argument similar to that of [2, Proof of corollary], we obtain

Corollary 1.2. *Let* $M \geq 4$, *and let* $n \in \mathcal{S}_M^*$ *be of the form* (1.3).

(a) *If* $p_1 = 3$, *then* $n > (cM6^M)^{6^M}$, *where* $c = 0.597... = \frac{\log 6}{3}$.
(b) *If* $p_1 > 3$, *then* $n > (dM3^M)^{3^M}$, *where* $d = 0.366... = \frac{\log 3}{3}$.

Using estimation (1.4) we obtain the following analogue of [2, Theorem 2].

Theorem 1.3. *Let* $\mathcal{P} = \{P_1, P_2, \dots\}$, *where* $P_i < P_{i+1}$ *for all* $i \geq 1$, *denote the set of all prime numbers. For every integer* $k \geq 2$ *there exists an infinite subset* $\mathcal{P}(k)$ *of the set* $\mathcal{P}$ *such that*

(a) *for every pairwise distinct primes* $p_1, p_2, \dots, p_k \in \mathcal{P}(k)$ *and* $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{N}$ *the number* $n = p_1^{\alpha_1} p_2^{\alpha 2} p_3^{\alpha 3} \cdots p_k^{\alpha_k}$ *does not fulfil equation* (1.2);
(b) $\mathcal{P}(k)$ *is maximal with respect to inclusion.*

$($*Notice that, by the general inequality* $\omega(n) \geq 11$ $($*see* $(1.4))$, *we have* $\mathcal{P}(k) = \mathcal{P}$ *for* $k \leq 10$.$)$

2. PROOFS

Proof of Theorem 1.1. We give here only an outline of the proof of Theorem 1.1, in which we essentially use the technique used in the proof of [2, Theorem 1].

Let n be of the form (1.3), and let n' be the squarefree kernel of n, i.e., $n' = p_1 \cdot p_2 \cdot \dots \cdot p_r$. Notice first that

$$\frac{\varphi(n)}{n} = \frac{\varphi(n')}{n'}. \tag{2.1}$$

The first step of the proof of [2, Theorem 1] is the inequality $4 \le M < n/\varphi(n)$ for n odd and squarefree ($n = n'$). An exact analysis of this proof shows that, by equality (2.1) the following result is true:

Lemma 2.1. *Let $M \ge 4$ be an integer, let n be of the form (1.3) with $p_1 \ge 3$, and suppose that*

$$M < \frac{n}{\varphi(n)}. \tag{2.2}$$

Then

(a) $\omega(n) \ge 3049^{M/4} - 1509$ *if* $p_1 = 3$ *and* $p_j \equiv 5 (\bmod 6)$ *for* $2 \le j \le \omega(n)$,
(b) $\omega(n) \ge 143^{M/4} - 1$ *if* $p_1 > 3$.

Since $n \in \mathcal{S}^*_M$ and $M \ge 4$, by equation (1.2) and the forms of φ^* and φ, we obtain:

$$\begin{aligned} M < \frac{n}{\varphi^*(n)} &= \prod_{i=1}^{r} \frac{p_i^{\alpha_i}}{p_i^{\alpha_i} - 1} \\ &= \prod_{i=1}^{r} \left(1 + \frac{1}{p_i^{\alpha_i} - 1}\right) \\ &\le \prod_{i=1}^{r} \left(1 + \frac{1}{p_i - 1}\right) \\ &= \prod_{i=1}^{r} \frac{p_i}{p_i - 1} = \frac{n'}{\varphi(n')} = \frac{n}{\varphi(n)}. \end{aligned}$$

Therefore every element $n \in \mathcal{S}^*_M$ fulfils inequality (2.2).

Further, if $3|n$ (i.e. $p_1 = 3$), then from (1.2) and the form of φ^*, we obtain that $3 \nmid (p_j^{\alpha_j} - 1)$, whence $3 \nmid (p_j - 1)$ for $j \ge 2$; thus $p_j \equiv 5 (\bmod 6)$. Now we can apply condition (a) of Lemma 2.1, which finishes the proof of case (a) of our theorem.

Case (b) of our theorem follows from case (b) of Lemma 2.1. □

Proof of Theorem 1.3. We will use here the idea and symbols used in the proof of [2, Theorem 2]. Let $[\mathbb{N}]^k$ be the set of k-element increasing sequences of $\mathbb{N}$, where $k \ge 2$.

Consider the function $f : [\mathbb{N}]^k \to \{0, 1\}$ of the form $f(i_1, i_2, \dots, i_k) = 0$ iff the number $P_{i_1}^{\alpha_1} P_{i_2}^{\alpha_2} \cdots P_{i_k}^{\alpha_k}$ fulfils equation (1.2) for some $\alpha_1, \dots, \alpha_k \in \mathbb{N}$.

By the Ramsey Theorem [1], there is an infinite subset $\mathbb{N}(k)$ of the set $\mathbb{N}$ such that

$$f([\mathbb{N}(k)]^k) = \{0\} \quad \text{or} \quad f([\mathbb{N}(k)]^k) = \{1\}.$$

Respectively, there is an infinite subset $\mathcal{P}(k)$ of $\mathcal{P}$ such that

$$P_{i_1}^{\alpha_1} P_{i_2}^{\alpha_2} \dots P_{i_k}^{\alpha_k} \in \mathcal{S}^* \quad \text{for some} \quad \alpha_1, \dots, \alpha_k \in \mathbb{N}, \tag{*}$$

or

$$P_{i_1}^{\alpha_1} P_{i_2}^{\alpha_2} \dots P_{i_k}^{\alpha_k} \notin \mathcal{S}^* \quad \text{for all} \quad \alpha_1, \dots, \alpha_n \in \mathbb{N}, \tag{**}$$

for all pairwise distinct elements $P_{i_1}, \dots, P_{i_k} \in \mathcal{P}(k)$. From inequality (1.4) we obtain that, for every $k \geq 2$ the number $\#\{n \in \mathbb{N} : \omega(n) \leq k\}$ is finite, and thus case $(*)$ is impossible. Hence case $(**)$ takes place, which implies that the set $\mathcal{P}(k)$ fulfils condition (a) of Theorem 1.3.

The existence of a maximal (with respect to inclusion) set $\mathcal{P}(k)$ follows from Kuratowski-Zorn's Lemma. □

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