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THE GENERALIZED SINE LAW AND SOME INEQUALITIES FOR SIMPLICES

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Abstract

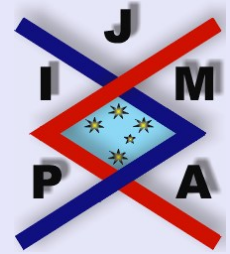
The sines of k -dimensional vertex angles of an n -simplex is defined and the law of sines for k -dimensional vertex angles of an n -simplex is established. Using the generalized sine law for n -simplex, we obtain some inequalities for the sines of k -dimensional vertex angles of an n -simplex. Besides, we obtain inequalities for volumes of n -simplices. As corollaries, the generalizations to several dimensions of the Neuberg-Pedoe inequality and P. Chiakuei inequality, and an inequality for pedal simplex are given.

2000 Mathematics Subject Classification: 52A40.

Key words: Simplex, k -dimensional vertex angle, Volume, Inequality.

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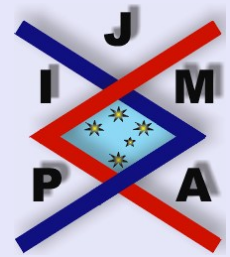
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1. Introduction

The law of sines for triangles in E^2 has natural analogues in higher dimensions. In 1978, F. Eriksson [1] defined the n -dimensional sines of the n -dimensional corners of an n -simplex in n -dimensional Euclidean space E^n and obtained the law of sines for the n -dimensional corners of an n -simplex. In this paper, the sines of k -dimensional vertex angles of an n -simplex will be defined, and the law of sines for k -dimensional vertex angles of an n -simplex will be established. Using the generalized sine law for simplices and a known inequality in [2], we get some inequalities for the sines of k -dimensional vertex angles of an n -simplex.

Recently, Yang Lu and Zhang Jingzhong [2, 3], Yang Shiguo [4], Leng Gangson [5, 6] and D. Veljan [7] and V. Volenec et al. [9] have obtained some important inequalities for volumes of n -simplices. In this paper, some interesting new inequalities for volumes of n -simplices will be established. As corollaries, we will obtain an inequality for pedal simplex and a generalization to several dimensions of the Neuberg-Pedoe inequality, which differs from the results in [4], [5] and [6].



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2. The Generalized Sine Law for Simplices

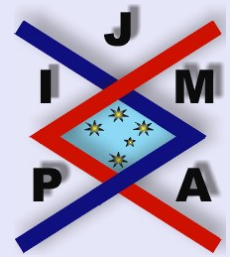
Let A_i ($i = 1, 2, \dots, n + 1$) be the vertices of an n -dimensional simplex Ω_n in the n -dimensional Euclidean space E^n , V the volume of the simplex Ω_n and F_i $(n - 1)$ -dimensional content of Ω_n . F. Eriksson defined the n -dimensional sines of the n -dimensional corners α_i of the n -simplex Ω_n and obtained the law of sines for n -simplices as follows [1]

$$(2.1) \quad \frac{F_i}{n \sin \alpha_i} = \frac{(n - 1)! \prod_{j=1}^{n+1} F_j}{(nV)^{n-1}} \quad (i = 1, 2, \dots, n + 1).$$

In this paper, we will define the sines of the k -dimensional vertex angles of an n -dimensional simplex and establish the law of sines for the k -dimensional vertex angles of an n -simplex. Let $V_{i_1 i_2 \dots i_k}$ be the $(k - 1)$ -dimensional content of the $(k - 1)$ -dimensional face $A_{i_1} A_{i_2} \dots A_{i_k}$ ($(k - 1)$ -simplex) of the simplex Ω_n for $k \in \{2, 3, \dots, n + 1\}$ and $i_1, i_2, \dots, i_k \in \{1, 2, \dots, n + 1\}$, O and R denote the circumcenter and circumradius of the simplex Ω_n respectively. $\overrightarrow{OA_i} = R\mathbf{u}_i$ ($i = 1, 2, \dots, n + 1$), \mathbf{u}_i is the unit vector. The sines of the k -dimensional vertex angles of the simplex Ω_n are defined as follows.

Definition 2.1. Let α_{ij} denote the angle formed by the vectors \mathbf{u}_i and \mathbf{u}_j . The sine of a k -dimensional vertex angle $\varphi_{i_1 i_2 \dots i_k}$ of the simplex Ω_n corresponding the $(k - 1)$ -dimensional face $A_{i_1} A_{i_2} \dots A_{i_k}$ is defined as

$$(2.2) \quad \sin \varphi_{i_1 i_2 \dots i_k} = (-D_{i_1 i_2 \dots i_k})^{\frac{1}{2}},$$



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where

$$(2.3) \quad D_{i_1 i_2 \dots i_k} = \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & & & \\ \vdots & & \boxed{-\frac{1}{2} \sin^2 \frac{\alpha_{i_1 i_m}}{2}} & \\ 1 & & & \end{vmatrix} \quad (l, m = 1, 2, \dots, k).$$

We will prove that

$$(2.4) \quad 0 < (-D_{i_1 i_2 \dots i_k})^{\frac{1}{2}} \leq 1.$$

If $n = 2$, the sine of the 2-dimensional vertex angle φ_{ij} of the triangle $A_1 A_2 A_3$ is the sine of the angle formed by two edges $A_k A_i$ and $A_k A_j$.

With the notation introduced above, we establish the law of sines for the k -dimensional vertex angles of an n -simplex as follows.

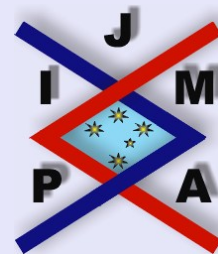
Theorem 2.1. *For an n -dimensional simplex Ω_n in E^n and $k \in \{2, 3, \dots, n + 1\}$, we have*

$$(2.5) \quad \frac{V_{i_1 i_2 \dots i_k}}{\sin \varphi_{i_1 i_2 \dots i_k}} = \frac{(2R)^{k-1}}{(k-1)!} \quad (1 \leq i_1 < i_2 < \dots < i_k \leq n + 1).$$

Put $\varphi_{12 \dots i-1, i+1, \dots, n+1} = \theta_i$, $V_{12 \dots i-1, i+1, \dots, n+1} = F_i$ ($i = 1, 2, \dots, n + 1$), by Theorem 2.1 we obtain the law of sines for the n -dimensional vertex angles of n -simplices as follows.

Corollary 2.2.

$$(2.6) \quad \frac{F_1}{\sin \theta_1} = \frac{F_2}{\sin \theta_2} = \dots = \frac{F_{n+1}}{\sin \theta_{n+1}} = \frac{(2R)^{n-1}}{(n-1)!}.$$



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If we take $n = 2$ in Theorem 2.1 or Corollary 2.2, we obtain the law of sines for a triangle $A_1A_2A_3$ in the form

$$(2.7) \quad \frac{a_1}{\sin A_1} = \frac{a_2}{\sin A_2} = \frac{a_3}{\sin A_3} = 2R.$$

Proof of Theorem 2.1. Let $a_{ij} = |A_iA_j|$ ($i, j = 1, 2, \dots, n + 1$), then

$$a_{ij} = 2R \sin \frac{\alpha_{ij}}{2},$$

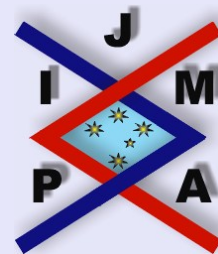
$$(2.8) \quad \sin^2 \varphi_{i_1 i_2 \dots i_k}$$

$$= -D_{i_1 i_2 \dots i_k} = - \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & & & \\ \vdots & & \boxed{-\frac{1}{8R^2} a_{i_l i_m}^2} & \\ 1 & & & \end{vmatrix}$$

$$= (-1)^k (8R^2)^{-(k-1)} \cdot \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & & & \\ \vdots & & \boxed{a_{i_l i_m}^2} & \\ 1 & & & \end{vmatrix} \quad (l, m = 1, 2, \dots, k).$$

By the formula for the volume of a simplex, we have

$$(2.9) \quad \begin{aligned} \sin^2 \varphi_{i_1 i_2 \dots i_k} &= -D_{i_1 i_2 \dots i_k} \\ &= (-1)^k (8R^2)^{-(k-1)} (-1)^k 2^{k-1} (k-1)!^2 V_{i_1 i_2 \dots i_k}^2 \\ &= \frac{(k-1)!^2}{(2R)^{2(k-1)}} V_{i_1 i_2 \dots i_k}^2. \end{aligned}$$



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From this equality (2.5) follows.

Now we prove that inequality (2.4) holds. When $k \geq 2$, we have $k \leq 2^{k-1}$. Using the Voljan-Korchmaros inequality [3], we have

$$(2.10) \quad V_{i_1 i_2 \dots i_k} \leq \frac{1}{(k-1)!} \left(\frac{k}{2^{k-1}} \right)^{\frac{1}{2}} \left(\prod_{1 \leq l < m \leq k} a_{i_l i_m} \right)^{\frac{2}{k}}.$$

Equality holds if and only if the simplex $A_{i_1} A_{i_2} \dots A_{i_k}$ is regular.

Combining inequality (2.10) with equality (2.5), we get

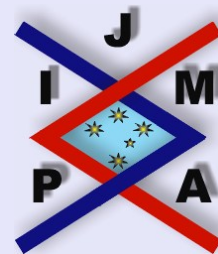
$$(2.11) \quad \begin{aligned} V_{i_1 i_2 \dots i_k} &\leq \frac{(2R)^{k-1}}{(k-1)!} \cdot \left(\frac{k}{2^{k-1}} \right)^{\frac{1}{2}} \left(\prod_{1 \leq l < m \leq k} \sin \frac{\alpha_{i_l i_m}}{2} \right)^{\frac{2}{k}} \\ &\leq \frac{(2R)^{k-1}}{(k-1)!} \cdot \left(\frac{k}{2^{k-1}} \right)^{\frac{1}{2}} \\ &\leq \frac{(2R)^{k-1}}{(k-1)!}. \end{aligned}$$

Using equality (2.5) and inequality (2.11), we get

$$0 < (-D_{i_1 i_2 \dots i_k})^{\frac{1}{2}} = \sin \varphi_{i_1 i_2 \dots i_k} = \frac{(k-1)!}{(2R)^{k-1}} V_{i_1 i_2 \dots i_k} \leq 1.$$

□

For the sines of the k -dimensional vertex angles of an n -simplex, we obtain an inequality as follows.



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Theorem 2.3. Let $\varphi_{i_1 i_2 \dots i_k}$ ($1 \leq i_1 < i_2 < \dots < i_k \leq n + 1$) denote the k -dimensional vertex angles of an n -simplex Ω_n in E^n , and $\lambda_i > 0$ ($i = 1, 2, \dots, n + 1$) be arbitrary real numbers, then we have

$$(2.12) \quad \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n+1} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k} \sin^2 \varphi_{i_1 i_2 \dots i_k} \leq \frac{n! \cdot \left(\sum_{i=1}^{n+1} \lambda_i\right)^k}{(n-k+1)!(k-1)!(4n)^{k-1}}.$$

Equality holds if $\lambda_1 = \lambda_2 = \dots = \lambda_{n+1}$ and the simplex Ω_n is regular.

By taking $\lambda_1 = \lambda_2 = \dots = \lambda_{n+1}$ in the inequality (2.12), we get:

Corollary 2.4.

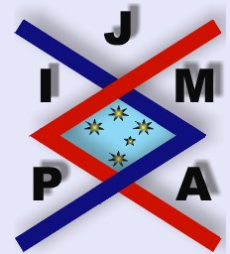
$$(2.13) \quad \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n+1} \sin^2 \varphi_{i_1 i_2 \dots i_k} \leq \frac{n! \cdot (n+1)^k}{(n-k+1)!(k-1)!(4n)^{k-1}}.$$

Equality holds if the simplex Ω_n is regular.

To prove Theorem 2.3, we need a lemma as follows.

Lemma 2.5. Let Ω_n be an n -simplex in E^n , $x_i > 0$ ($i = 1, 2, \dots, n + 1$) be real numbers, $V_{i_1 i_2 \dots i_{s+1}}$ be the s -dimensional volume of the s -dimensional simplex $A_{i_1} A_{i_2} \dots A_{i_{s+1}}$ for $i_1, i_2, \dots, i_{s+1} \in \{1, 2, \dots, n + 1\}$. Put

$$M_s = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n+1} x_{i_1} x_{i_2} \dots x_{i_{s+1}} V_{i_1 i_2 \dots i_{s+1}}^2, \quad M_0 = \sum_{i=1}^{n+1} x_i,$$



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then we have

$$(2.14) \quad M_s^l \geq \frac{[(n-l)!(l!)^3]^s}{[(n-s)!(s!)^3]^l} (n! \cdot M_0)^{l-s} M_l^s \quad (1 \leq s < l \leq n).$$

Equality holds if and only if the inertial ellipsoid of the points A_1, A_2, \dots, A_{n+1} with masses x_1, x_2, \dots, x_{n+1} is a sphere.

For the proof of Lemma 2.5, the reader is referred to [2] or [9].

Proof of Theorem 2.3. By putting $s = 1, l = k-1$ and $x_i = \lambda_i$ ($i = 1, 2, \dots, n+1$) in the inequality (2.14), we have

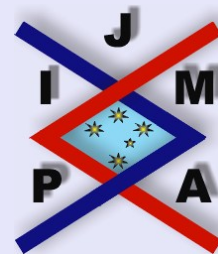
$$(2.15) \quad \left(\sum_{1 \leq i < j \leq n+1} \lambda_i \lambda_j a_{ij}^2 \right)^{k-1} \geq \frac{(n-k+1)! \cdot (k-1)!^3}{[(n-1)!]^{k-1}} \left(n! \cdot \sum_{i=1}^{n+1} \lambda_i \right)^{k-2} \times \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n+1} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k} V_{i_1 i_2 \dots i_k}^2.$$

By Theorem 2.1, we have

$$(2.16) \quad V_{i_1 i_2 \dots i_k} = \frac{(2R)^{k-1}}{(k-1)!} \sin \varphi_{i_1 i_2 \dots i_k}.$$

Using the known inequality [3]

$$(2.17) \quad \sum_{1 \leq i < j \leq n+1} \lambda_i \lambda_j a_{ij}^2 \leq \left(\sum_{i=1}^{n+1} \lambda_i \right)^2 R^2,$$



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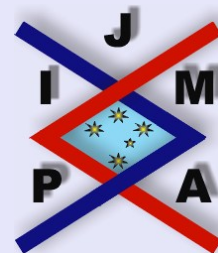
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with equality if and only if the point $P = \sum_{i=1}^{n+1} \lambda_i A_i$ is the circumcenter of simplex Ω_n .

Combining (2.15) with (2.16) and (2.17), we obtain inequality (2.12). It is easy to see that equality holds in (2.12) if $\lambda_1 = \lambda_2 = \dots = \lambda_{n+1}$ and simplex Ω_n is regular. \square



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3. Some Inequalities for Volumes of Simplices

Let P be an arbitrary point inside the simplex Ω_n and B_i the orthogonal projection of the point P on the $(n - 1)$ -dimensional plane σ_i containing $(n - 1)$ -simplex $f_i = A_1 \cdots A_{i-1} A_{i+1} \cdots A_{n+1}$. Simplex $\bar{\Omega}_n = B_1 B_2 \cdots B_{n+1}$ is called the pedal simplex of the point P with respect to the simplex Ω_n . Let $r_i = |PB_i|$ ($i = 1, 2, \dots, n + 1$), \bar{V} be the volume of the pedal simplex $\bar{\Omega}_n$, $V(i)$ and $\bar{V}(i)$ denote the volumes of two n -dimensional simplices $\Omega_n(i) = A_1 \cdots A_{i-1} P A_{i+1} \cdots A_{n+1}$ and $\bar{\Omega}_n(i) = B_1 \cdots B_{i-1} P B_{i+1} \cdots B_{n+1}$, respectively. Then we have an inequality for volumes of just defined n -simplices as follows.

Theorem 3.1. *Let P be an arbitrary point inside n -dimensional simplex Ω_n and λ_i ($i = 1, 2, \dots, n + 1$) positive real numbers, then we have*

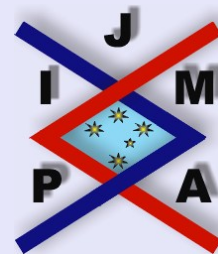
$$(3.1) \quad \sum_{i=1}^{n+1} \lambda_1 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_{n+1} \bar{V}(i) \leq \frac{1}{n^n} \left[\sum_{i=1}^n \lambda_i V(i) \right]^n V^{1-n},$$

with equality if the simplex Ω_n is regular, P is the circumcenter of Ω_n and $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$.

Now we state some applications of Theorem 3.1.

If taking $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$ in inequality (3.1), we have

$$(3.2) \quad \sum_{i=1}^{n+1} \bar{V}(i) \leq \frac{1}{n^n} \left[\sum_{i=1}^{n+1} V(i) \right]^n \cdot V^{1-n}.$$



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Since the point P is in the interior of the simplex Ω_n , then

$$(3.3) \quad \sum_{i=1}^{n+1} \bar{V}(i) = \bar{V}, \quad \sum_{i=1}^{n+1} V(i) = V.$$

Using (3.2) and (3.3) we obtain an inequality for the volume of the pedal simplex $\bar{\Omega}_n$ of the point P with respect to the simplex Ω_n as follows.

Corollary 3.2. *Let P be an arbitrary point inside the n -simplex Ω_n , then we have*

$$(3.4) \quad \bar{V} \leq \frac{1}{n^n} V,$$

with equality if simplex Ω_n is regular and P is the circumcenter of Ω_n .

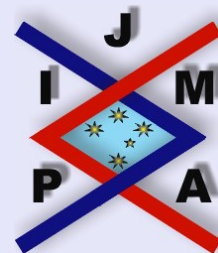
Corollary 3.3. *Let P be an arbitrary point inside the n -simplex Ω_n , then we have*

$$(3.5) \quad \sum_{i=1}^{n+1} V(i) \cdot \bar{V}(i) \leq \frac{1}{(n+1)n^n} V^2,$$

with equality if the simplex Ω_n is regular and P is the circumcenter of Ω_n .

Proof. Let $\lambda_i = [V(i)]^{-1}$ ($i = 1, 2, \dots, n+1$) in inequality (3.1); we get

$$(3.6) \quad \sum_{i=1}^{n+1} V(i) \cdot \bar{V}(i) \leq \left(\frac{n+1}{n}\right)^n V^{1-n} \prod_{j=1}^{n+1} V(j).$$



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Using the arithmetic-geometric mean inequality and equality (3.3), we have

$$\sum_{i=1}^{n+1} V(i) \cdot \bar{V}(i) \leq \left(\frac{n+1}{n}\right)^n V^{1-n} \left[\frac{1}{n+1} \sum_{j=1}^{n+1} V(j)\right]^{n+1} = \frac{1}{(n+1)n^n} V^2.$$

It is easy to see that equality in (3.5) holds if the simplex Ω_n is regular and the point P is the circumcenter of Ω_n . \square

Proof of Theorem 3.1. Let h_i be the altitude of simplex Ω_n from vertex A_i , $\overrightarrow{PB_i} = r_i \mathbf{e}_i$, where \mathbf{e}_i is the unit outer normal vector of the i -th side face $f_i = A_1 \cdots A_{i-1} A_{i+1} \cdots A_{n+1}$ of the simplex Ω_n , and ${}^n \sin \alpha_k$ be the n -dimensional sine of the k -th corner α_k of the simplex Ω_n . Wang and Yang [8] proved that

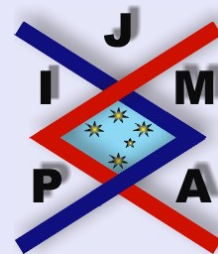
$$(3.7) \quad {}^n \sin \alpha_n = [\det(e_i \cdot e_j)_{ij \neq k}]^{\frac{1}{2}} \quad (k = 1, 2, \dots, n+1).$$

By the formula for the volume of an n -simplex and (3.7), we have

$$(3.8) \quad \bar{V}(i) = \frac{1}{n!} [\det(r_l r_k \mathbf{e}_l \cdot \mathbf{e}_k)_{l,k \neq i}]^{\frac{1}{2}} = \frac{1}{n!} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} r_j \right) \cdot {}^n \sin \alpha_i.$$

Using (3.8), (2.1) and $nV = h_i F_i$, we get that

$$\begin{aligned} & \sum_{i=1}^{n+1} \lambda_1 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_{n+1} \bar{V}(i) \\ &= \frac{1}{n!} \sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j r_j \right) \cdot {}^n \sin \alpha_i \end{aligned}$$



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$$\begin{aligned}
&= \frac{1}{n!} \sum_{i=1}^{n+1} \left\{ \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j r_j \right) (nV)^{n-1} \left[(n-1)! \cdot \prod_{\substack{j=1 \\ j \neq i}}^{n+1} F_j \right]^{-1} \right\} \\
&= [(n!)^2 \cdot V]^{-1} \sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j r_j h_j \right),
\end{aligned}$$

i.e.

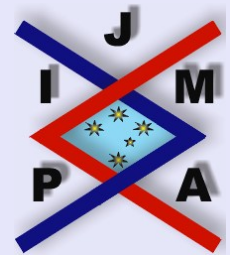
$$(3.9) \quad (n!)^2 V \sum_{i=1}^{n+1} \lambda_1 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_{n+1} \bar{V}(i) = \sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j r_j h_j \right).$$

Taking $s = n - 1$, $l = n$ in inequality (2.14), we get

$$(3.10) \quad \left[\sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} x_j \right) F_i^2 \right]^n \geq \frac{n^{3n}}{(n!)^2} \left(\sum_{i=1}^{n+1} x_i \right) \left(\prod_{i=1}^{n+1} x_i \right)^{n-1} V^{2(n-1)}.$$

Let $x_i = (\lambda_i r_i h_i)^{-1}$ ($i = 1, 2, \dots, n + 1$) in inequality (3.10). Then we have

$$(3.11) \quad \left(\sum_{i=1}^{n+1} \lambda_i r_i h_i F_i^2 \right)^n \geq \frac{n^{3n}}{(n!)^2} \left[\sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j r_j h_j \right) F_i^2 \right] \cdot V^{2(n-1)}.$$



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Using inequality (3.11) and $r_i F_i = nV(i)$, $h_i F_i = nV$, we get

$$(3.12) \quad V^n \left[\sum_{i=1}^{n+1} \lambda_i V(i) \right]^n \geq \frac{n^n}{(n!)^2} V^{2(n-1)} \sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j r_j h_j \right).$$

Substituting equality (3.9) into inequality (3.12) we get inequality (3.1). It is easy to prove that equality in (3.1) holds if simplex Ω_n is regular, P is the circumcenter of Ω_n and $\lambda_1 = \lambda_2 = \dots = \lambda_{n+1}$. Theorem 3.1 is proved. \square

Finally, we shall establish some inequalities for volumes of two n -simplices. As corollaries, the generalizations to several dimensions of the Neuberg-Pedoe inequality and P.Chiakui inequality will be given.

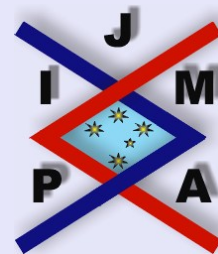
Let a_i ($i = 1, 2, 3$) denote the sides of the triangle $A_1A_2A_3$ with area Δ , and a'_i ($i = 1, 2, 3$) denote the sides of the triangle $A'_1A'_2A'_3$ with area Δ' , then

$$(3.13) \quad \sum_{i=1}^3 a_i^2 \left(\sum_{j=1}^3 (a'_j)^2 - 2(a'_i)^2 \right) \geq 16\Delta\Delta',$$

with equality if and only if $\Delta A_1A_2A_3$ is similar to $\Delta A'_1A'_2A'_3$.

Inequality (3.13) is the well-known Neuberg-Pedoe inequality.

In 1984, P. Chiakui [9] proved the following sharpening of the Neuberg-



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Pedoe inequality:

$$(3.14) \quad \sum_{i=1}^3 a_i^2 \left(\sum_{j=1}^3 (a'_j)^2 - 2(a'_i)^2 \right) \geq 8 \left(\frac{(a'_1)^2 + (a'_2)^2 + (a'_3)^2}{a_1^2 + a_2^2 + a_3^2} \Delta^2 + \frac{a_1^2 + a_2^2 + a_3^2}{(a'_1)^2 + (a'_2)^2 + (a'_3)^2} (\Delta')^2 \right),$$

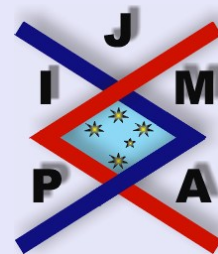
with equality if and only if $\Delta A_1 A_2 A_3$ is similar to $\Delta A'_1 A'_2 A'_3$.

Recently, Leng Gangson [5] has extended inequality (3.14) to the edge lengths and volumes of two n -simplices. In this paper, we shall extend inequality (3.14) to the volumes of two n -simplices and the contents of their side faces. As a corollary, we get a generalization to several dimensions of the Neuberg-Pedoe inequality. Let A_i ($i = 1, 2, \dots, n + 1$) be the vertices of n -simplex Ω_n in E^n , V the volume of the simplex Ω_n and F_i ($n - 1$)-dimensional content of the $(n - 1)$ -dimensional face $f_i = A_1 \cdots A_{i-1} A_{i+1} \cdots A_{n+1}$ of Ω_n . For two n -simplices Ω_n and Ω'_n and real numbers $\alpha, \beta \in (0, 1]$, we put

$$(3.15) \quad \sigma_n(\alpha) = \sum_{i=1}^{n+1} F_i^\alpha, \quad \sigma_n(\beta) = \sum_{i=1}^{n+1} (F'_i)^\beta, \quad b_n = \frac{n^3}{n+1} \left(\frac{n+1}{n!^2} \right)^{\frac{1}{n}}.$$

We obtain an inequality for volumes of two n -simplices as follows.

Theorem 3.4. *For any two n -dimensional simplices Ω_n and Ω'_n and two arbitrary*



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bitrary real numbers $\alpha, \beta \in (0, 1]$, we have

$$(3.16) \quad \sum_{i=1}^{n+1} F_i^\alpha \left(\sum_{j=1}^{n+1} (F'_j)^\beta - 2(F'_i)^\beta \right) \geq \frac{(n-1)^2}{2} \left[b_n^\alpha \frac{\sigma_n(\beta)}{\sigma_n(\alpha)} V^{2(n-1)\alpha/n} + b_n^\beta \frac{\sigma_n(\alpha)}{\sigma_n(\beta)} (V')^{2(n-1)\beta/n} \right].$$

Equality holds if and only if simplices Ω_n and Ω'_n are regular.

Using inequality (3.16) and the arithmetic-geometric mean inequality, we get the following corollary.

Corollary 3.5. For any two n -dimensional simplices Ω_n and Ω'_n and two arbitrary real numbers $\alpha, \beta \in (0, 1]$, we have

$$(3.17) \quad \sum_{i=1}^{n+1} F_i^\alpha \left(\sum_{j=1}^{n+1} (F'_j)^\beta - 2(F'_i)^\beta \right) \geq b_n^{(\alpha+\beta)/2} (n^2 - 1) (V^\alpha (V')^\beta)^{(n-1)/n}.$$

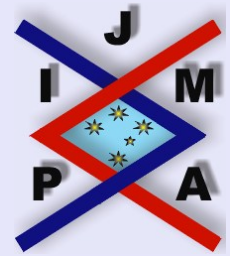
Equality holds if and only if simplices Ω_n and Ω'_n are regular.

If we let $\alpha = \beta$ in Corollary 3.5, we get Leng Gangson's inequality [6] as follows. For any $\theta \in (0, 1]$ we have

$$(3.18) \quad \sum_{i=1}^{n+1} F_i^\theta \left(\sum_{j=1}^{n+1} (F'_j)^\theta - 2(F'_i)^\theta \right) \geq b_n^\theta (n^2 - 1) (VV')^{(n-1)\theta/n},$$

with equality if and only if simplices Ω_n and Ω'_n are regular.

To prove Theorem 3.4, we need some lemmas as follows.



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Lemma 3.6. For an n -simplex Ω_n and arbitrary number $\alpha \in (0, 1]$, we have

$$(3.19) \quad \frac{\prod_{i=1}^{n+1} F_i^{2\alpha}}{\sum_{i=1}^{n+1} F_i^{2\alpha}} \geq \frac{1}{(n+1)^{(n-1)\alpha+1}} \left[\frac{n^{3n}}{(n!)^2} \right]^\alpha V^{2(n-1)\alpha},$$

with equality if and only if simplex Ω_n is regular.

Proof. If taking $l = n$, $s = n - 1$ and $x_i = F_i^2$ ($i = 1, 2, \dots, n + 1$) in inequality (2.14), we get an inequality as follows

$$\frac{(n+1)^n (n!)^2}{n^{3n}} \prod_{i=1}^{n+1} F_i^2 \geq V^{2(n-1)} \sum_{i=1}^{n+1} F_i^2,$$

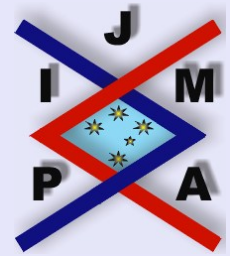
or

$$(3.20) \quad \frac{(n+1)^{n\alpha} (n!)^{2\alpha}}{n^{3n\alpha}} \prod_{i=1}^{n+1} F_i^{2\alpha} \geq V^{2(n-1)\alpha} \left(\sum_{i=1}^{n+1} F_i^2 \right)^\alpha.$$

It is easy to prove that equality in (3.20) holds if and only if simplex Ω_n is regular. From inequality (3.20) we know that inequality (3.19) holds for $\alpha = 1$. For $\alpha \in (0, 1)$, using inequality (3.20) and the well-known inequality

$$(3.21) \quad \sum_{i=1}^{n+1} F_i^2 \geq (n+1) \left(\frac{1}{n+1} \sum_{i=1}^{n+1} F_i^{2\alpha} \right)^{\frac{1}{\alpha}},$$

we get inequality (3.19). It is easy to see that equality in (3.19) holds if and only if the simplex Ω_n is regular. \square



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Lemma 3.7. For an n -simplex $\Omega_n (n \geq 3)$ and an arbitrary number $\alpha \in (0, 1]$, we have

$$(3.22) \quad \left(\sum_{i=1}^{n+1} F_i^\alpha \right)^2 - 2 \sum_{i=1}^{n+1} F_i^{2\alpha} \geq b_n^\alpha (n^2 - 1) V^{2(n-1)\alpha/n},$$

with equality if and only if the simplex Ω_n is regular.

For the proof of Lemma 3.7, the reader is referred to [6].

Lemma 3.8. Let $a_i (i = 1, 2, 3)$ and Δ denote the sides and the area of the triangle $(A_1 A_2 A_3)$, respectively. For arbitrary number $\alpha \in (0, 1]$, denote by Δ_α the area of the triangle $(A_1 A_2 A_3)_\alpha$ with sides $a_i^\alpha (i = 1, 2, 3)$, then the following inequality holds

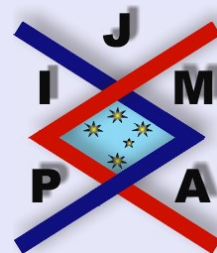
$$(3.23) \quad \Delta_\alpha^2 \geq \frac{3}{16} \left(\frac{16}{3} \Delta^2 \right)^\alpha.$$

For $\alpha \neq 1$, equality holds if and only if $a_1 = a_2 = a_3$.

For the proof of Lemma 3.8, the reader is referred to [9].

Lemma 3.9. Let numbers $x_i > 0, y_i > 0 (i = 1, 2, \dots, n + 1)$, $\sigma_n = \sum_{i=1}^{n+1} x_i$, $\sigma'_n = \sum_{i=1}^{n+1} y_i$, then

$$(3.24) \quad \sigma_n \sigma'_n - 2 \sum_{i=1}^{n+1} x_i y_i \geq \frac{1}{2} \left[\frac{\sigma'_n}{\sigma_n} \left(\sigma_n^2 - 2 \sum_{i=1}^{n+1} x_i^2 \right) + \frac{\sigma_n}{\sigma'_n} \left((\sigma'_n)^2 - 2 \sum_{i=1}^{n+1} y_i^2 \right) \right],$$



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with equality if and only if

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \dots = \frac{y_{n+1}}{x_{n+1}}.$$

Proof. Inequality (3.24) is

$$(3.25) \quad \frac{\sigma'_n}{\sigma_n} \sum_{i=1}^{n+1} x_i^2 + \frac{\sigma_n}{\sigma'_n} \sum_{i=1}^{n+1} y_i^2 \geq 2 \sum_{i=1}^{n+1} x_i y_i.$$

Now we prove that inequality (3.25) holds. Using the arithmetic-geometric mean inequality, we have

$$\frac{\sigma'_n}{\sigma_n} x_i^2 + \frac{\sigma_n}{\sigma'_n} y_i^2 \geq 2x_i y_i \quad (i = 1, 2, \dots, n + 1).$$

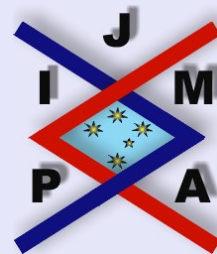
Adding up those $n + 1$ inequalities, we get inequality (3.25). Equality in (3.25) holds if and only if $\frac{\sigma'_n}{\sigma_n} x_i^2 = \frac{\sigma_n}{\sigma'_n} y_i^2$ ($i = 1, 2, \dots, n + 1$), i.e.

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \dots = \frac{y_{n+1}}{x_{n+1}} = \frac{\sigma'_n}{\sigma_n}.$$

□

Proof of Theorem 3.4. For $n = 2$, consider two triangles $(A_1 A_2 A_3)_\alpha$ and $(A'_1 A'_2 A'_3)_\beta$. Using inequality (3.14) and Lemma 3.8, we have

$$(3.26) \quad \sum_{i=1}^3 a_i^\alpha \left(\sum_{j=1}^3 (a'_j)^\beta - 2(a'_i)^\beta \right) \geq \frac{1}{2} \left[b_2^\alpha \frac{\sigma'_2(\beta)}{\sigma_2(\alpha)} \Delta^\alpha + b_2^\beta \frac{\sigma_2(\alpha)}{\sigma'_2(\beta)} (\Delta')^\beta \right].$$



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Equality in (3.26) holds if and only if $a_1 = a_2 = a_3$ and $a'_1 = a'_2 = a'_3$. Hence, inequality (3.16) holds for $n = 2$.

For $n \geq 3$, taking $x_i = F_i^\alpha$, $y_i = (F'_i)^\beta$ ($i = 1, 2, \dots, n + 1$) in inequality (3.24), we get

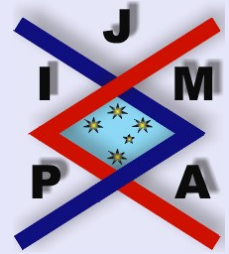
$$\begin{aligned}
 (3.27) \quad & \sum_{i=1}^{n+1} F_i^\alpha \left(\sum_{j=1}^{n+1} (F'_j)^\beta - 2(F'_i)^\beta \right) \\
 &= \left(\sum_{i=1}^{n+1} F_i^\alpha \right) \left(\sum_{i=1}^{n+1} (F'_i)^\beta \right) - 2 \sum_{i=1}^{n+1} F_i^\alpha (F'_i)^\beta \\
 &\geq \frac{1}{2} \left\{ \frac{\sigma'_n(\beta)}{\sigma_n(\alpha)} \left[\left(\sum_{i=1}^{n+1} F_i^\alpha \right)^2 - 2 \sum_{i=1}^{n+1} F_i^{2\alpha} \right] \right. \\
 &\quad \left. + \frac{\sigma_n(\alpha)}{\sigma'_n(\beta)} \left[\left(\sum_{i=1}^{n+1} (F'_i)^\beta \right)^2 - 2 \sum_{i=1}^{n+1} F_i^{2\beta} \right] \right\}.
 \end{aligned}$$

Using inequality (3.27) and Lemma 3.7, we get

$$\sum_{i=1}^{n+1} F_i^\alpha \left(\sum_{i=1}^{n+1} (F'_j)^\beta \right) \geq \frac{n^2 - 1}{2} \left[b_n^\alpha \frac{\sigma'_n(\beta)}{\sigma_n(\alpha)} V^{2(n-1)\alpha/n} + b_n^\beta \frac{\sigma_n(\alpha)}{\sigma'_n(\beta)} V^{2(n-1)\beta/n} \right].$$

Hence, inequality (3.16) is true for $n \geq 3$. For $n \geq 3$, it is easy to see that equality in (3.16) holds if and only if two simplices Ω_n and σ'_n are regular.

Theorem 3.4 is proved. \square



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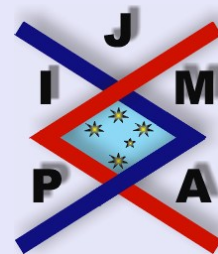
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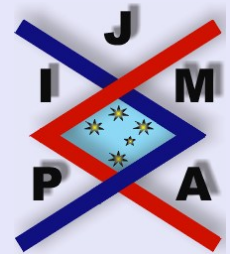
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