journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 10 (2009), Issue 3, Article 67, 10 pp.

ON THE CYCLIC HOMOGENEOUS POLYNOMIAL INEQUALITIES OF DEGREE FOUR

VASILE CIRTOAJE

DEPARTMENT OF AUTOMATIC CONTROL AND COMPUTERS
UNIVERSITY OF PLOIESTI
ROMANIA
vcirtoaje@upg-ploiesti.ro

Received 20 April, 2009; accepted 01 July, 2009
Communicated by N.K. Govil

ABSTRACT. Let $f(x,y,z)$ be a cyclic homogeneous polynomial of degree four with three variables which satisfies $f(1,1,1)=0$. In this paper, we give the necessary and sufficient conditions to have $f(x,y,z)\geq 0$ for any real numbers x,y,z. We also give the necessary and sufficient conditions to have $f(x,y,z)\geq 0$ for the case when f is symmetric and x,y,z are nonnegative real numbers. Finally, some new inequalities with cyclic homogeneous polynomials of degree four are presented.

Key words and phrases: Cyclic inequality, Symmetric inequality, Necessity and sufficiency, Homogeneous polynomial of degree four.

2000 *Mathematics Subject Classification.* 26D05.

1. INTRODUCTION

Let x,y,z be real numbers. The fourth degree Schur's inequality ([3], [5], [7]) is a well-known symmetric homogeneous polynomial inequality which states that

$$\sum x^4 + xyz\sum x \geq \sum xy(x^2+y^2), \tag{1.1}$$

where $\sum$ denotes a cyclic sum over x, y and z. Equality holds for $x=y=z$, and for $x=0$ and $y=z$, or $y=0$ and $z=x$, or $z=0$ and $x=y$.

In [3], the following symmetric homogeneous polynomial inequality was proved

$$\sum x^4 + 8\sum x^2y^2 \geq 3\left(\sum xy\right)\left(\sum x^2\right), \tag{1.2}$$

with equality for $x=y=z$, and for $x/2=y=z$, or $y/2=z=x$, or $z/2=x=y$. In addition, a more general inequality was proved in [3] for any real k,

$$\sum (x-y)(x-ky)(x-z)(x-kz) \geq 0, \tag{1.3}$$

105-09

with equality for $x = y = z$, and again for $x/k = y = z$, or $y/k = z = x$, or $z/k = x = y$. Notice that this inequality is a consequence of the identity

$$\sum(x-y)(x-ky)(x-z)(x-kz) = \frac{1}{2}\sum(y-z)^2(y+z-x-kx)^2.$$

In 1992, we established the following cyclic homogeneous inequality [1]:

$$\left(\sum x^2\right)^2 \geq 3\sum x^3y, \tag{1.4}$$

which holds for any real numbers x, y, z, with equality for $x = y = z$, and for

$$\frac{x}{\sin^2\frac{4\pi}{7}} = \frac{y}{\sin^2\frac{2\pi}{7}} = \frac{z}{\sin^2\frac{\pi}{7}}$$

or any cyclic permutation thereof.

Six years later, we established a similar cyclic homogeneous inequality [2],

$$\sum x^4 + \sum xy^3 \geq 2\sum x^3y, \tag{1.5}$$

which holds for any real numbers x, y, z, with equality for $x = y = z$, and for

$$x\sin\frac{\pi}{9} = y\sin\frac{7\pi}{9} = z\sin\frac{13\pi}{9}$$

or any cyclic permutation thereof.

As shown in [3], substituting $y = x + p$ and $z = x + q$, the inequalities (1.4) and (1.5) can be rewritten in the form

$$(p^2 - pq + q^2)x^2 + f(p,q)x + g(p,q) \geq 0,$$

where the quadratic polynomial of x has the discriminant

$$\delta_1 = -3(p^3 - p^2q - 2pq^2 + q^3)^2 \leq 0,$$

and, respectively,

$$\delta_2 = -3(p^3 - 3pq^2 + q^3)^2 \leq 0.$$

The symmetric inequalities (1.1), (1.2) and (1.3), as well as the cyclic inequalities (1.4) and (1.5), are particular cases of the inequality $f(x,y,z) \geq 0$, where $f(x,y,z)$ is a cyclic homogeneous polynomial of degree four satisfying $f(1,1,1) = 0$. This polynomial has the general form

$$\begin{aligned} f(x,y,z) = w\sum x^4 + r\sum x^2y^2 \\ + (p+q-r-w)xyz\sum x - p\sum x^3y - q\sum xy^3, \end{aligned} \tag{1.6}$$

where p, q, r, w are real numbers. Since the inequality $f(x,y,z) \geq 0$ with $w \leq 0$ does not hold for all real numbers x, y, z, except the trivial case where $w = p = q = 0$ and $r \geq 0$, we will consider $w = 1$ throughout this paper.

2. Main Results

In 2008, we posted, without proof, the following theorem in the Mathlinks Forum [4].

Theorem 2.1. *Let p, q, r be real numbers. The cyclic inequality*

$$\sum x^4 + r\sum x^2y^2 + (p+q-r-1)xyz\sum x \geq p\sum x^3y + q\sum xy^3 \tag{2.1}$$

holds for any real numbers x, y, z if and only if

$$3(1+r) \geq p^2 + pq + q^2. \tag{2.2}$$

For $p = q = 1$ and $r = 0$, we obtain the fourth degree Schur's inequality (1.1). For $p = q = 3$ and $r = 8$ one gets (1.2), while for $p = q = k + 1$ and $r = k(k + 2)$ one obtains (1.3). In addition, for $p = 3$, $q = 0$ and $r = 2$ one gets (1.4), while for $p = 2$, $q = -1$ and $r = 0$ one obtains (1.5).

In the particular cases $r = 0$, $r = p + q - 1$, $q = 0$ and $p = q$, by Theorem 2.1, we have the following corollaries, respectively.

Corollary 2.2. *Let p and q be real numbers. The cyclic inequality*

$$\sum x^4 + (p + q - 1)xyz \sum x \geq p \sum x^3 y + q \sum xy^3 \tag{2.3}$$

holds for any real numbers x, y, z if and only if

$$p^2 + pq + q^2 \leq 3. \tag{2.4}$$

Corollary 2.3. *Let p and q be real numbers. The cyclic inequality*

$$\sum x^4 + (p + q - 1) \sum x^2 y^2 \geq p \sum x^3 y + q \sum xy^3 \tag{2.5}$$

holds for any real numbers x, y, z if and only if

$$3(p + q) \geq p^2 + pq + q^2. \tag{2.6}$$

Corollary 2.4. *Let p and q be real numbers. The cyclic inequality*

$$\sum x^4 + r \sum x^2 y^2 + (p - r - 1)xyz \sum x \geq p \sum x^3 y \tag{2.7}$$

holds for any real numbers x, y, z if and only if

$$3(1 + r) \geq p^2. \tag{2.8}$$

Corollary 2.5. *Let p and q be real numbers. The symmetric inequality*

$$\sum x^4 + r \sum x^2 y^2 + (2p - r - 1)xyz \sum x \geq p \sum xy(x^2 + y^2) \tag{2.9}$$

holds for any real numbers x, y, z if and only if

$$r \geq p^2 - 1. \tag{2.10}$$

Finding necessary and sufficient conditions such that the cyclic inequality (2.1) holds for any nonnegative real numbers x, y, z is a very difficult problem. On the other hand, the approach for nonnegative real numbers is less difficult in the case when the cyclic inequality (2.1) is symmetric. Thus, in 2008, Le Huu Dien Khue posted, without proof, the following theorem on the Mathlinks Forum [4].

Theorem 2.6. *Let p and r be real numbers. The symmetric inequality (2.9) holds for any nonnegative real numbers x, y, z if and only if*

$$r \geq (p - 1) \max\{2, p + 1\}. \tag{2.11}$$

From Theorem 2.1, setting $p = 1 + \sqrt{6}$, $q = 1 - \sqrt{6}$ and $r = 2$, and then $p = 3$, $q = -3$ and $r = 2$, we obtain the inequalities:

$$\left(\sum x^2\right)\left(\sum x^2 - \sum xy\right) \geq \sqrt{6}\left(\sum x^3 y - \sum xy^3\right), \tag{2.12}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 0.4493$ and $\beta \approx -0.1009$ were found using a computer;

$$(x^2 + y^2 + z^2)^2 \geq 3 \sum xy(x^2 - y^2 + z^2), \tag{2.13}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 0.2469$ and $\beta \approx -0.3570$.

From Corollary 2.2, setting $p = \sqrt{3}$ and $q = -\sqrt{3}$ yields

$$\sum x^4 - xyz\sum x \geq \sqrt{3}\left(\sum x^3y - \sum xy^3\right), \tag{2.14}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 0.3767$ and $\beta \approx -0.5327$. Notice that if x, y, z are nonnegative real numbers, then the best constant in inequality (2.14) is $2\sqrt{2}$ (Problem 19, Section 2.3 in [3], by Pham Kim Hung):

$$\sum x^4 - xyz\sum x \geq 2\sqrt{2}\left(\sum x^3y - \sum xy^3\right). \tag{2.15}$$

From Corollary 2.3, setting $p = 1 + \sqrt{3}$ and $q = 1$, and then $p = 1 - \sqrt{3}$ and $q = 1$, we obtain the inequalities:

$$\sum x^4 - \sum xy^3 \geq \left(1 + \sqrt{3}\right)\left(\sum x^3y - \sum x^2y^2\right), \tag{2.16}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 0.7760$ and $\beta \approx 0.5274$;

$$\sum x^4 - \sum xy^3 \geq \left(\sqrt{3} - 1\right)\left(\sum x^2y^2 - \sum x^3y\right), \tag{2.17}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 1.631$ and $\beta \approx -1.065$.

From Corollary 2.4, setting in succession $p = \sqrt{3}$ and $r = 0$, $p = -\sqrt{3}$ and $r = 0$, $p = 6$ and $r = 11$, $p = 2$ and $r = 1/3$, $p = -1$ and $r = -2/3$, $p = r = (3 + \sqrt{21})/2$, $p = 1$ and $r = -2/3$, $p = r = (3 - \sqrt{21})/2$, $p = \sqrt{6}$ and $r = 1$, we obtain the inequalities below, respectively:

$$\sum x^4 + \left(\sqrt{3} - 1\right)xyz\sum x \geq \sqrt{3}\sum x^3y, \tag{2.18}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 0.7349$ and $\beta \approx -0.1336$ (Problem 5.3.10 in [6]);

$$\sum x^4 + \sqrt{3}\sum x^3y \geq \left(1 + \sqrt{3}\right)xyz\sum x, \tag{2.19}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 7.915$ and $\beta \approx -6.668$;

$$\sum x^4 + 11\sum x^2y^2 \geq 6\left(\sum x^3y + xyz\sum x\right), \tag{2.20}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 0.5330$ and $\beta \approx 2.637$;

$$3\sum x^4 + \left(\sum xy\right)^2 \geq 6\sum x^3y, \tag{2.21}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 0.7156$ and $\beta \approx -0.0390$;

$$\sum x^4 + \sum x^3y \geq \frac{2}{3}\left(\sum xy\right)^2, \tag{2.22}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 1.871$ and $\beta \approx -2.053$;

$$\sum x^4 - xyz\sum x \geq \frac{3 + \sqrt{21}}{2}\left(\sum x^3y - \sum x^2y^2\right), \tag{2.23}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 0.570$ and $\beta \approx 0.255$;

$$\sum x^4 - \sum x^3 y \geq \frac{2}{3}\left(\sum x^2y^2 - xyz\sum x\right), \tag{2.24}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 0.8020$ and $\beta \approx -0.4446$;

$$\sum x^4 - xyz\sum x \geq \frac{\sqrt{21}-3}{2}\left(\sum x^2y^2 - \sum x^3y\right), \tag{2.25}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 1.528$ and $\beta \approx -1.718$;

$$\sum (x^2 - yz)^2 \geq \sqrt{6}\sum xy(x-z)^2, \tag{2.26}$$

with equality for $x = y = z$, and for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 0.6845$ and $\beta \approx 0.0918$ (Problem 21, Section 2.3 in [3]).

From either Corollary 2.5 or Theorem 2.6, setting $r = p^2 - 1$ yields

$$\sum x^4 + (p^2-1)\sum x^2y^2 + p(2-p)xyz\sum x \geq p\sum xy(x^2+y^2), \tag{2.27}$$

which holds for any real numbers p and x, y, z. For $p = k+1$, the inequality (2.27) turns into (1.3).

Corollary 2.7. *Let x, y, z be real numbers. If p, q, r, s are real numbers such that*

$$p + q - r - 1 \leq s \leq 2(r+1) + p + q - p^2 - pq - q^2, \tag{2.28}$$

then

$$\sum x^4 + r\sum x^2y^2 + sxyz\sum x \geq p\sum x^3y + q\sum xy^3. \tag{2.29}$$

Let

$$\alpha = \frac{r+s+1-p-q}{3} \geq 0.$$

Since

$$3(1+r-\alpha) \geq p^2 + pq + q^2,$$

by Theorem 2.1 we have

$$\sum x^4 + (r-\alpha)\sum x^2y^2 + (\alpha+p+q-r-1)xyz\sum x \geq p\sum x^3y + q\sum xy^3.$$

Adding this inequality to the obvious inequality

$$\alpha\left(\sum xy\right)^2 \geq 0,$$

we get (2.29).

From Corollary 2.7, setting $p = 1$, $q = r = 0$ and $s = 2$, we get

$$\sum x^4 + 2xyz\sum x \geq \sum x^3y, \tag{2.30}$$

with equality for $x = y/\alpha = z/\beta$ or any cyclic permutation, where $\alpha \approx 0.8020$ and $\beta \approx -0.4451$. Notice that (2.30) is equivalent to

$$\sum (2x^2 - y^2 - z^2 - xy + yz)^2 + 4\left(\sum xy\right)^2 \geq 0. \tag{2.31}$$

3. PROOF OF THEOREM 2.1

Proof of the Sufficiency. Since

$$\sum x^2y^2 - xyx\sum x = \frac{1}{2}\sum x^2(y-z)^2 \geq 0,$$

it suffices to prove the inequality (2.1) for the least value of r, that is

$$r = \frac{p^2+pq+q^2}{3} - 1.$$

On this assumption, (2.1) is equivalent to each of the following inequalities:

(3.1) $$\sum [2x^2 - y^2 - z^2 - pxy + (p+q)yz - qzx]^2 \geq 0,$$

(3.2) $$\sum [3y^2 - 3z^2 - (p+2q)xy - (p-q)yz + (2p+q)zx]^2 \geq 0,$$

(3.3) $$3[2x^2 - y^2 - z^2 - pxy + (p+q)yz - qzx]^2 + [3y^2 - 3z^2 - (p+2q)xy - (p-q)yz + (2p+q)zx]^2 \geq 0.$$

Thus, the conclusion follows. □

Proof of the Necessity. For $p = q = 2$, we need to show that the condition $r \geq 3$ is necessary to have

$$\sum x^4 + r\sum x^2y^2 + (3-r)xyz\sum x \geq 2\sum x^3y + 2\sum xy^3$$

for any real numbers x, y, z. Indeed, setting $y = z = 1$ reduces this inequality to

$$(x-1)^4 + (r-3)(x-1)^2 \geq 0,$$

which holds for any real x if and only if $r \geq 3$.

In the other cases (different from $p = q = 2$), by Lemma 3.1 below it follows that there is a triple $(a, b, c) = (1, b, c) \neq (1, 1, 1)$ such that

$$\sum [2a^2 - b^2 - c^2 - pab + (p+q)bc - qca]^2 = 0.$$

Since

$$\sum a^2b^2 - abc\sum a = \frac{1}{2}\sum a^2(b-c)^2 > 0,$$

we may write this relation as

$$\frac{p\sum a^3b + q\sum ab^3 - \sum a^4 - (p+q-1)abc\sum a}{\sum a^2b^2 - abc\sum a} = \frac{p^2+pq+q^2}{3} - 1.$$

On the other hand, since (2.1) holds for (a, b, c) (by hypothesis), we get

$$r \geq \frac{p\sum a^3b + q\sum ab^3 - \sum a^4 - (p+q-1)abc\sum a}{\sum a^2b^2 - abc\sum a}.$$

Therefore,

$$r \geq \frac{p^2+pq+q^2}{3} - 1,$$

which is the desired necessary condition. □

Lemma 3.1. *Let p and q be real numbers. Excepting the case $p = q = 2$, there is a real triple $(x, y, z) = (1, y, z) \neq (1, 1, 1)$ such that*

(3.4) $$\sum [2x^2 - y^2 - z^2 - pxy + (p+q)yz - qzx]^2 = 0.$$

Proof. We consider two cases: $p = q \neq 2$ and $p \neq q$.

Case 1. $p = q \neq 2$.

It is easy to prove that $(x, y, z) = (1, p-1, 1) \neq (1, 1, 1)$ is a solution of the equation (3.4).

Case 2. $p \neq q$.

The equation (3.4) is equivalent to

$$\begin{cases} 2y^2 - z^2 - x^2 - pyz + (p+q)zx - qxy = 0 \\ 2z^2 - x^2 - y^2 - pzx + (p+q)xy - qyz = 0. \end{cases}$$

For $x = 1$, we get

$$\begin{cases} 2y^2 - z^2 - 1 - pyz + (p+q)z - qy = 0 \\ 2z^2 - 1 - y^2 - pz + (p+q)y - qyz = 0. \end{cases} \tag{3.5}$$

Adding the first equation multiplied by 2 to the second equation yields

$$z[(2p+q)y - p - 2q] = 3y^2 + (p-q)y - 3. \tag{3.6}$$

Under the assumption that $(2p+q)y - p - 2q \neq 0$, substituting z from (3.6) into the first equation, (3.5) yields

$$(y-1)(ay^3 + by^2 + cy - a) = 0, \tag{3.7}$$

where

$$a = 9 - 2p^2 - 5pq - 2q^2,$$
$$b = 9 + 6p - 6q - 3p^2 + 3q^2 + 2p^3 + 3p^2q + 3pq^2 + q^3,$$
$$c = -9 + 6p - 6q - 3p^2 + 3q^2 - p^3 - 3p^2q - 3pq^2 - 2q^3.$$

The equation (3.7) has a real root $y_1 \neq 1$. To prove this claim, it suffices to show that the equation $ay^3 + by^2 + cy - a = 0$ does not have a root of 1; that is to show that $b + c \neq 0$. This is true because

$$\begin{aligned} b + c &= 12(p-q) - 6(p^2 - q^2) + p^3 - q^3 \\ &= (p-q)(12 - 6p - 6q + p^2 + q^2 + pq), \end{aligned}$$

and

$$p - q \neq 0,$$

$$\begin{aligned} 4(12 - 6p - 6q + p^2 + q^2 + pq) &> 48 - 24(p+q) + 3(p+q)^2 \\ &= 3(p+q-4)^2 \\ &\geq 0. \end{aligned}$$

For $y = y_1$ and $(2p+q)y_1 - p - 2q \neq 0$, from (3.6) we get

$$z_1 = \frac{3y_1^2 + (p-q)y_1 - 3}{(2p+q)y_1 - p - 2q},$$

and the conclusion follows. Thus, it remains to consider that $(2p+q)y_1 - p - 2q = 0$. In this case, we have $2p + q \neq 0$ (since $2p + q = 0$ provides $p + 2q = 0$, which contradicts the hypothesis $p \neq q$), and hence

$$y_1 = \frac{p + 2q}{2p + q}.$$

For $y = y_1$, from (3.6) we get $3(y_1^2 - 1) + (p-q)y_1 = 0$, which yields

$$(2p+q)(p+2q) = 9(p+q). \tag{3.8}$$

Substituting y_1 into the first equation (3.5), we get

$$(2p+q)z^2-(p^2+q^2+pq)z+p+2q=0.$$

To complete the proof, it suffices to show that this quadratic equation has real roots. Due to (3.8), we need to prove that

$$(p^2+q^2+pq)^2\geq 36(p+q).$$

For the nontrivial case $p+q>0$, let us denote $s=p+q$, $s>0$, and write the condition (3.8) as $9s-2s^2=pq$. Since $4pq\leq s^2$, we find that $s\geq 4$. Therefore,

$$(p^2+q^2+pq)^2-36(p+q)=9(s^2-3s)^2-36s=9s(s-1)^2(s-4)\geq 0.$$

□

4. PROOF OF THEOREM 2.6

The condition $r\geq(p-1)\max\{2,p+1\}$ is equivalent to $r\geq p^2-1$ for $p\geq 1$, and $r\geq 2(p-1)$ for $p\leq 1$.

Proof of the Sufficiency. By Theorem 2.1, if $r\geq p^2-1$, then the inequality (2.9) is true for any real numbers x,y,z. Thus, it only remains to consider the case when $p\leq 1$ and $r\geq 2(p-1)$. Writing (2.9) as

$$\sum x^4+xyz\sum x-\sum xy(x^2+y^2)+(1-p)\left[\sum xy(x^2+y^2)-2\sum x^2y^2\right]$$
$$+(r-2p+2)\left(\sum x^2y^2-xyz\sum x\right)\geq 0,$$

we see that it is true because

$$\sum x^4+xyz\sum x-\sum xy(x^2+y^2)\geq 0$$

(Schur's inequality of fourth degree),

$$\sum xy(x^2+y^2)-2\sum x^2y^2=\sum xy(x-y)^2\geq 0$$

and

$$\sum x^2y^2-xyz\sum x=\frac{1}{2}\sum x^2(y-z)^2\geq 0.$$

□

Proof of the Necessity. We need to prove that the conditions $r\geq 2(p-1)$ and $r\geq p^2-1$ are necessary such that the inequality (2.9) holds for any nonnegative real numbers x,y,z. Setting $y=z=1$, (2.9) becomes

$$(x-1)^2[x^2+2(1-p)x+2+r-2p]\geq 0.$$

For $x=0$, we get the necessary condition $r\geq 2(p-1)$, while for $x=p-1$, we get

$$(p-2)^2(r+1-p^2)\geq 0.$$

If $p\neq 2$, then this inequality provides the necessary condition $r\geq p^2-1$. Thus, it remains to show that for $p=2$, we have the necessary condition $r\geq 3$. Indeed, setting $p=2$ and $y=z=1$ reduces the inequality (2.9) to

$$(x-1)^2[(x-1)^2+r-3]\geq 0.$$

Clearly, this inequality holds for any nonnegative x if and only if $r\geq 3$. □

5. OTHER RELATED INEQUALITIES

The following theorem establishes other interesting related inequalities with symmetric homogeneous polynomials of degree four.

Theorem 5.1. *Let x, y, z be real numbers, and let*

$$A = \sum x^4 - \sum x^2y^2, \qquad B = \sum x^2y^2 - xyz\sum x,$$

$$C = \sum x^3y - xyz\sum x, \qquad D = \sum xy^3 - xyz\sum x.$$

Then,

$$AB = C^2 - CD + D^2 \geq \frac{C^2 + D^2}{2} \geq \left(\frac{C + D}{2}\right)^2 \geq CD. \tag{5.1}$$

Moreover, if x, y, z are nonnegative real numbers, then

$$CD \geq B^2. \tag{5.2}$$

The equality $AB = CD$ holds for $x + y + z = 0$, and for $x = y$, or $y = z$, or $z = x$, while the equality $CD = B^2$ holds for $x = y = z$, and for $x = 0$, or $y = 0$, or $z = 0$.

Proof. The inequalities in Theorem 5.1 follow from the identities:

$$D - C = (x + y + z)(x - y)(y - z)(z - x),$$

$$AB - CD = (x + y + z)^2(x - y)^2(y - z)^2(z - x)^2,$$

$$AB - \left(\frac{C + D}{2}\right)^2 = \frac{3}{4}(x + y + z)^2(x - y)^2(y - z)^2(z - x)^2,$$

$$AB - \frac{C^2 + D^2}{2} = \frac{1}{2}(x + y + z)^2(x - y)^2(y - z)^2(z - x)^2,$$

$$CD - B^2 = xyz(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)^2.$$

□

Remark 1. We obtained the identity $AB = C^2 - CD + D^2$ in the following way. For $3(r+1) = p^2 + pq + q^2$, by Theorem 2.1 we have

$$A + (1 + r)B - pC - qD \geq 0,$$

which is equivalent to

$$Bp^2 + (Bq - 3C)p + Bq^2 - 3Dq + 3A \geq 0.$$

Since this inequality holds for any real p and $B \geq 0$, the discriminant of the quadratic of p is non-positive; that is

$$(Bq - 3C)^2 - 4B(Bq^2 - 3Dq + 3A) \leq 0,$$

which is equivalent to

$$B^2q^2 + 2B(C - 2D)q + 4AB - 3C^2 \geq 0.$$

Similarly, the discriminant of the quadratic of q is non-positive; that is

$$B^2(C - 2D)^2 - B^2(4AB - 3C^2) \leq 0,$$

which yields $AB \geq C^2 - CD + D^2$. Actually, this inequality is an identity.

Remark 2. The inequality $CD \geq B^2$ is true if

$$k^2C - 2kB + D \geq 0$$

for any real k. This inequality is equivalent to

$$\sum yz(x-ky)^2 \geq (k-1)^2xyz\sum x,$$

which follows immediately from the Cauchy-Schwarz inequality

$$\left(\sum x\right)\left[\sum yz(x-ky)^2\right] \geq (k-1)^2xyz\left(\sum x\right)^2.$$

On the other hand, assuming that $x = \min\{x,y,z\}$ and substituting $y = x+p$ and $z = x+q$, where $p,q \geq 0$, the inequality $CD \geq B^2$ can be rewritten as

$$A_1x^4 + B_1x^3 + C_1x^2 + D_1x \geq 0,$$

with

$$A_1 = 3(p^2 - pq + q^2)^2 \geq 0,$$
$$B_1 = 4(p+q)(p^2 - pq + q^2)^2 \geq 0,$$
$$C_1 = 2pq(p^2 - pq + q^2)^2 + pq(p^2 - q^2)^2 + (p^3 + q^3)^2 - 2p^2q^2(p^2 + q^2) + 5p^3q^3 \geq 0,$$
$$D_1 = pq[p^5 + q^5 - pq(p^3 + q^3) + p^2q^2(p+q)] \geq 0.$$

REFERENCES

[1] V. CIRTOAJE, Problem 22694, *Gazeta Matematica*, **7-8** (1992), 287.

[2] V. CIRTOAJE, Problem O:887, *Gazeta Matematica*, **10** (1998), 434.

[3] V. CIRTOAJE, *Algebraic Inequalities-Old and New Methods*, GIL Publishing House, 2006.

[4] V. CIRTOAJE AND LE HUU DIEN KHUE, *Mathlinks Forum*, February 2008, [ONLINE: `http://www.mathlinks.ro/Forum/viewtopic.php?t=186179`].

[5] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge University Press, 1952.

[6] P.K. HUNG, *Secrets in Inequalities*, Vol. 2, GIL Publishing House, 2008.

[7] D.S. MITRINOVIĆ, J. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.