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ON HYERS-ULAM STABILITY OF GENERALIZED WILSON'S EQUATION

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Abstract

In this paper, we study the Hyers-Ulam stability problem for the following functional equation

$$(E(K)) \quad \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|f(x)g(y), \quad x, y \in G,$$

where G is a locally compact group, K is a compact subgroup of G , ω_K is the normalized Haar measure of K , Φ is a finite group of K -invariant morphisms of G and $f, g : G \rightarrow \mathbb{C}$ are continuous complex-valued functions such that f satisfies the Kannappan type condition, for all $x, y, z \in G$

$$(*) \quad \int_K \int_K f(zkxk^{-1}hyh^{-1})d\omega_K(k)d\omega_K(h) \\ = \int_K \int_K f(zkyk^{-1}hxx^{-1})d\omega_K(k)d\omega_K(h).$$

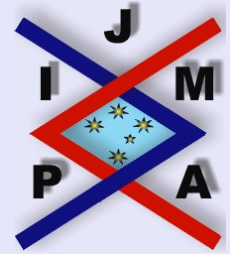
Our results generalize and extend the Hyers-Ulam stability obtained for the Wilson's functional equation.

2000 Mathematics Subject Classification: 39B72.

Key words: Functional equations, Hyers-Ulam stability, Wilson equation, Gelfand pairs.

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1. Introduction

Let G be a locally compact group. Let K be a compact subgroup of G . Let ω_K be the normalized Haar measure of K . A mapping $\varphi : G \rightarrow G$ is a morphism of G if φ is a homeomorphism of G onto itself which is either a group-homomorphism, i.e. $(\varphi(xy) = \varphi(x)\varphi(y), x, y \in G)$, or a group-antihomomorphism, i.e. $(\varphi(xy) = \varphi(y)\varphi(x), x, y \in G)$. We denote by $Mor(G)$ the group of morphism of G and Φ a finite subgroup of $Mor(G)$ of a K -invariant morphisms of G (i.e. $\varphi(K) \subset K$). The number of elements of a finite group Φ will be designated by $|\Phi|$. The Banach algebra of bounded measures on G with complex values is denoted by $M(G)$ and the Banach space of all complex measurable and essentially bounded functions on G by $L_\infty(G)$. $\mathcal{C}(G)$ designates the Banach space of all continuous complex valued functions on G .

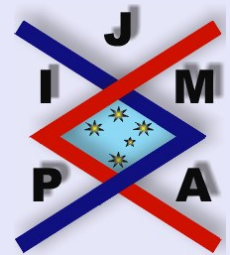
In this paper we are going to generalize the results obtained in [1], [4] and [6] for the integral equation

$$(1.1) \quad \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|f(x)g(y), \quad x, y \in G.$$

This equation may be considered as a common generalization of functional equations of Cauchy and Wilson type

$$(1.2) \quad f(xy) = f(x)g(y), \quad x, y \in G,$$

$$(1.3) \quad f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in G,$$



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where σ is an involution of G . It is also a generalization of the equations

$$(1.4) \quad \int_K f(xkyk^{-1})d\omega_K(k) = f(x)g(y), \quad x, y \in G,$$

$$(1.5) \quad \int_K f(xkyk^{-1})d\omega_K(k) + \int_K f(xk\sigma(y)k^{-1})d\omega_K(k) = 2f(x)g(y), \quad x, y \in G,$$

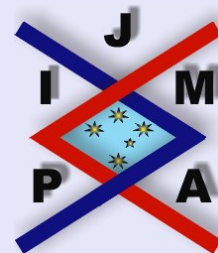
$$(1.6) \quad \int_K f(xky)\bar{\chi}(k)d\omega_K(k) = f(x)g(y), \quad x, y \in G,$$

$$(1.7) \quad \int_K f(xky)\bar{\chi}(k)d\omega_K(k) + \int_K f(xk\sigma(y))\bar{\chi}(k)d\omega_K(k) = 2f(x)g(y), \quad x, y \in G,$$

$$(1.8) \quad \int_K f(xky)d\omega_K(k) = f(x)g(y), \quad x, y \in G,$$

and

$$(1.9) \quad \int_K f(xky)d\omega_K(k) + \int_K f(xk\sigma(y))d\omega_K(k) = 2f(x)g(y), \quad x, y \in G.$$



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If G is a compact group, the equation (1.1) may be considered as a generalization of the equations

$$(1.10) \quad \int_G f(xtyt^{-1})dt = f(x)g(y), \quad x, y \in G,$$

$$(1.11) \quad \int_G f(xtyt^{-1})dt + \int_G f(xt\sigma(y)t^{-1})dt = 2f(x)g(y), \quad x, y \in G,$$

and

$$(1.12) \quad \sum_{\varphi \in \Phi} \int_G f(xt\varphi(y)t^{-1})dt = |\Phi|f(x)g(y), \quad x, y \in G.$$

Furthermore the following equations are also a particular case of (1.1).

$$(1.13) \quad \sum_{\varphi \in \Phi} f(x\varphi(y)) = |\Phi|f(x)g(y), \quad x, y \in G,$$

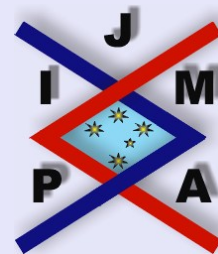
$$(1.14) \quad \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y))d\omega_K(k) = |\Phi|f(x)g(y), \quad x, y \in G,$$

and

$$(1.15) \quad \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y))\bar{\chi}(k)d\omega_K(k) = |\Phi|f(x)g(y), \quad x, y \in G,$$

where χ is a unitary character of K .

In the next section, we note some results for later use.



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2. General Properties

In what follows, we study general properties. Let G, K and Φ given as above

Proposition 2.1 ([4]). For an arbitrary fixed $\tau \in \Phi$, the mapping

$$\begin{aligned}\Phi &\longrightarrow \Phi \\ \varphi &\longmapsto \varphi \circ \tau\end{aligned}$$

is a bijection and for all $x, y \in G$, we have

$$(2.1) \quad \sum_{\varphi \in \Phi} \int_K f(xk\varphi(\tau(y))k^{-1})d\omega_K(k) = \sum_{\psi \in \Phi} \int_K f(xk\psi(y)k^{-1})d\omega_K(k).$$

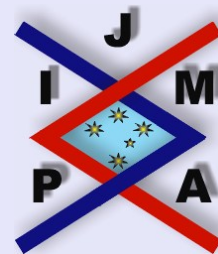
Proposition 2.2. Let $\varphi \in \Phi$ and $f \in \mathcal{C}(G)$, then we have

i)

$$\int_K f(xk\varphi(hy)k^{-1})d\omega_K(k) = \int_K f(xk\varphi(yh)k^{-1})d\omega_K(k), \quad x, y \in G, h \in K.$$

ii) If f satisfies (*), then for all $a, z, y, x \in G$, we have

$$\begin{aligned}\int_K \int_K f(zh\varphi(ykxk^{-1})h^{-1})d\omega_K(h)d\omega_K(k) \\ = \int_K \int_K f(zh\varphi(xkyk^{-1})h^{-1})d\omega_K(h)d\omega_K(k).\end{aligned}$$



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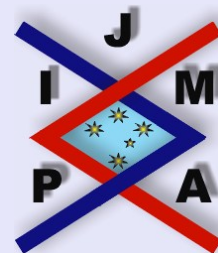
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and

$$\int_K \int_K \int_K f(ah\varphi(zk_1yk_1^{-1}h_1xh_1^{-1})h^{-1})d\omega_K(h)d\omega_K(k_1)d\omega_K(h_1)$$
$$= \int_K \int_K \int_K f(ah\varphi(zk_1xk_1^{-1}h_1yh_1^{-1})h^{-1})d\omega_K(h)d\omega_K(k_1)d\omega_K(h_1).$$

Proof. By easy computations. □



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3. The Main Results

The main result is the following theorem.

Theorem 3.1. Let $\delta > 0$ and let $(f, g) \in \mathcal{C}(G)$ such that f satisfies the condition (*) and the functional inequality

$$(3.1) \quad \left| \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|f(x)g(y) \right| \leq \delta, \quad x, y \in G.$$

Then

i) f, g are bounded or

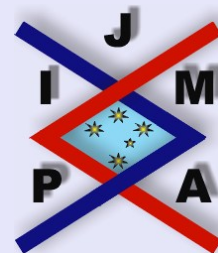
ii) f is unbounded and g satisfies the equation

$$(3.2) \quad \sum_{\varphi \in \Phi} \int_K g(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi|g(x)g(y), \quad x, y \in G.$$

iii) g is unbounded, f satisfies the equation (1.1). Furthermore if $f \neq 0$, then g is a solution of (3.2).

Proof. Let (f, g) be a solution of the inequality (3.1), such that f is unbounded and satisfies the condition (*), then for all $x, y, z \in G$, we get by using Propositions 2.1 and 2.2

$$|\Phi||f(z)| \left| \sum_{\varphi \in \Phi} \int_K g(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|g(x)g(y) \right|$$



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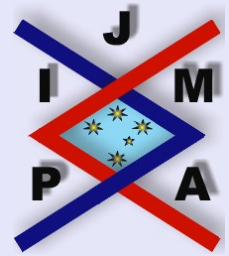
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$$\begin{aligned}
&= \left| \sum_{\varphi \in \Phi} \int_K |\Phi| f(z) g(xk\varphi(y)k^{-1}) d\omega_K(k) - |\Phi|^2 f(z) g(x) g(y) \right| \\
&\leq \left| \sum_{\varphi \in \Phi} \int_K \sum_{\psi \in \Phi} \int_K f(zh\psi(xk\varphi(y)k^{-1})h^{-1}) d\omega_K(h) d\omega_K(k) \right. \\
&\quad \left. - |\Phi| f(z) \sum_{\varphi \in \Phi} \int_K g(xk\varphi(y)k^{-1}) d\omega_K(k) \right| \\
&\quad + \left| \sum_{\psi \in \Phi} \int_K \sum_{\varphi \in \Phi} \int_K f(zh\psi(xk\varphi(y)k^{-1})h^{-1}) d\omega_K(h) d\omega_K(k) \right. \\
&\quad \left. - |\Phi| g(y) \sum_{\psi \in \Phi} \int_K f(zk\psi(x)k^{-1}) d\omega_K(k) \right| \\
&\quad + |\Phi| |g(y)| \left| \sum_{\psi \in \Phi} \int_K f(zh\psi(x)h^{-1}) d\omega_K(h) - |\Phi| f(z) g(x) \right| \\
&\leq \sum_{\varphi \in \Phi} \int_K \left| \sum_{\psi \in \Phi} \int_K f(zh\psi(xk\varphi(y)k^{-1})h^{-1}) d\omega_K(h) - |\Phi| f(z) g(xk\varphi(y)k^{-1}) \right| d\omega_K(k) \\
&\quad + \sum_{\psi \in \Phi} \int_K \left| \sum_{\tau \in \Phi} \int_K f(zh\psi(x)h^{-1}k\tau(y)k^{-1}) d\omega_K(k) - |\Phi| g(y) f(zh\psi(x)h^{-1}) \right| d\omega_K(h) \\
&\quad + |\Phi| |g(y)| \left| \sum_{\psi \in \Phi} \int_K f(zk\psi(x)k^{-1}) d\omega_K(k) - |\Phi| f(z) g(x) \right| \\
&\leq 2|\Phi|\delta + |\Phi| |g(y)| \delta.
\end{aligned}$$



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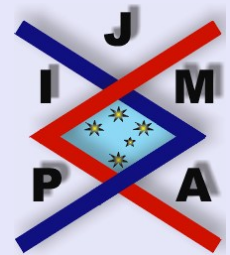
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Since f is unbounded it follows that g is a solution of the functional equation (3.2). For the second case let (f, g) be a solution of the inequality (3.1) such that f satisfies the condition (*) and g is unbounded then for all $x, y, z \in G$, one has

$$\begin{aligned}
 & |\Phi||g(z)| \left| \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|f(x)g(y) \right| \\
 &= \left| \sum_{\varphi \in \Phi} \int_K |\Phi|g(z)f(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|^2g(z)f(x)g(y) \right| \\
 &\leq \left| \sum_{\psi \in \Phi} \int_K \sum_{\varphi \in \Phi} \int_K f(xh\varphi(y)h^{-1}k\psi(z)k^{-1})d\omega_K(h)d\omega_K(k) \right. \\
 &\quad \left. - |\Phi|g(z) \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y)k^{-1})d\omega_K(k) \right| \\
 &+ \left| \sum_{\varphi \in \Phi} \int_K \sum_{\psi \in \Phi} \int_K f(xh\psi(z)h^{-1}k\varphi(y)k^{-1})d\omega_K(h)d\omega_K(k) \right. \\
 &\quad \left. - |\Phi|g(y) \sum_{\psi \in \Phi} \int_K f(xk\psi(z)k^{-1})d\omega_K(k) \right| \\
 &+ |\Phi||g(y)| \left| \sum_{\psi \in \Phi} \int_K f(xk\psi(z)k^{-1})d\omega_K(k) - |\Phi|f(x)g(z) \right|
 \end{aligned}$$



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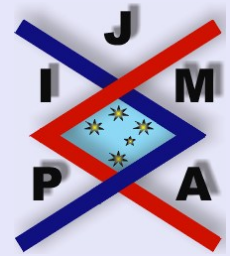
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$$\begin{aligned}
&\leq \sum_{\varphi \in \Phi} \int_K \left| \sum_{\psi \in \Phi} \int_K f(xk\varphi(y)k^{-1}h\psi(z)h^{-1})d\omega_K(h) \right. \\
&\quad \left. - |\Phi|g(z)f(xk\varphi(y)k^{-1}) \right| d\omega_K(k) \\
&\quad + \sum_{\psi \in \Phi} \int_K \left| \sum_{\varphi \in \Phi} \int_K f(xk\psi(z)k^{-1}h\varphi(y)h^{-1})d\omega_K(h) \right. \\
&\quad \left. - |\Phi|g(y)f(xk\psi(z)k^{-1}) \right| d\omega_K(k) \\
&\quad + |\Phi||g(y)| \left| \sum_{\psi \in \Phi} \int_K f(xk\psi(z)k^{-1})d\omega_K(k) - |\Phi|f(x)g(z) \right| \\
&\leq 2|\Phi|\delta + |\Phi||g(y)|\delta.
\end{aligned}$$

Since g is unbounded it follows that f is a solution of (1.1). Now let $f \neq 0$, then there exists $a \in G$ such that $f(a) \neq 0$. Let $\eta = \frac{\delta}{|f(a)|}$ and let

$$F(x) = \frac{1}{|\Phi||f(a)|} \sum_{\varphi \in \Phi} \int_K f(ak\varphi(x)k^{-1})d\omega_K(k).$$

By using Proposition 2.2 it follows that F satisfies the condition (*), and by using the inequality (3.1) one has $|F(x) - g(x)| \leq \frac{\eta}{|\Phi|}$, since g is unbounded it



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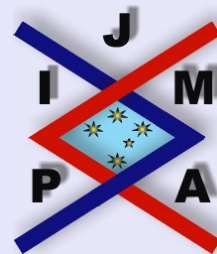
follows that F is unbounded. Furthermore for all $x, y \in G$ we have

$$\begin{aligned}
 & \left| \sum_{\varphi \in \Phi} \int_K F(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|F(x)g(y) \right| \\
 &= \frac{1}{|\Phi|f(a)} \left| \sum_{\varphi \in \Phi} \int_K \sum_{\psi \in \Phi} \int_K f(ah\psi(xk\varphi(y)k^{-1})h^{-1})d\omega_K(h)d\omega_K(k) \right. \\
 & \quad \left. - |\Phi| \frac{1}{|\Phi|f(a)} \sum_{\varphi \in \Phi} \int_K f(ak\varphi(x)k^{-1})d\omega_K(k)g(y) \right| \\
 &\leq \frac{1}{|\Phi|f(a)} \sum_{\varphi \in \Phi} \int_K \left| \sum_{\tau \in \Phi} \int_K f(ah\psi(x)h^{-1}k\tau(y)k^{-1})d\omega_K(k) \right. \\
 & \quad \left. - |\Phi|f(ah\varphi(x)h^{-1})g(y) \right| d\omega_K(k) \\
 &\leq \eta.
 \end{aligned}$$

From the first case it follows that g is a solution of (3.2). □

Corollary 3.2. *Let $\delta > 0$ and let $(f, g) \in \mathcal{C}(G)$ such that f satisfies the condition (*) and the functional inequality*

$$\begin{aligned}
 (3.3) \quad & \left| \int_K f(xkyk^{-1})d\omega_K(k) \right. \\
 & \quad \left. + \int_K f(xk\sigma(y)k^{-1})d\omega_K(k) - 2f(x)g(y) \right| \leq \delta, \quad x, y \in G,
 \end{aligned}$$



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where σ is an involution on G . Then

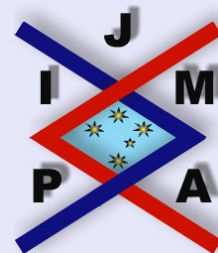
i) f, g are bounded or

ii) f is unbounded and g satisfies the equation

$$(3.4) \quad \int_K g(xkyk^{-1})d\omega_K(k) + \int_K g(xk\sigma(y)k^{-1})d\omega_K(k) = 2g(x)g(y), \quad x, y \in G.$$

iii) g is unbounded, f satisfies the equation (1.5). Furthermore if $f \neq 0$, then g is a solution of (3.4).

Remark 3.1. In the case where $\Phi = \{I\}$, it is not necessary to assume that f satisfies the condition (*) (see [1] and [6]).



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4. Applications

The following theorems are a particular case of Theorem 3.1.

If $K \subset Z(G)$, then we have

Theorem 4.1. Let $\delta > 0$ and let f, g be a complex-valued functions on G such that f satisfies the Kannappan condition ([12])

$$(*) \quad f(zxy) = f(zyx), \quad x, y \in G$$

and the functional inequality

$$(4.1) \quad \left| \sum_{\varphi \in \Phi} f(x\varphi(y)) - |\Phi|f(x)g(y) \right| \leq \delta, \quad x, y \in G.$$

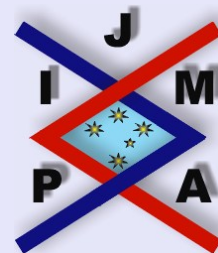
Then

i) f, g are bounded or

ii) f is unbounded and g is a solution of the functional equation

$$(4.2) \quad \sum_{\varphi \in \Phi} g(x\varphi(y)) = |\Phi|g(x)g(y), \quad x, y \in G,$$

iii) g is unbounded and f is a solution of (1.13). Furthermore if $f \neq 0$ then g is a solution of (4.2).



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Corollary 4.2. Let $\delta > 0$ and let f, g be a complex-valued functions on G such that f satisfies the Kannappan condition

$$(*) \quad f(zxy) = f(zyx), \quad x, y \in G$$

and the functional inequality

$$(4.3) \quad |f(xy) + f(x\sigma(y)) - 2f(x)g(y)| \leq \delta, \quad x, y \in G,$$

where σ is an involution on G . Then

i) f, g are bounded or

ii) f is unbounded and g is a solution of the functional equation

$$(4.4) \quad g(xy) + g(x\sigma(y)) = 2g(x)g(y), \quad x, y \in G,$$

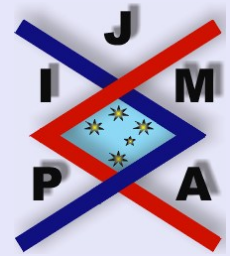
iii) g is unbounded and f is a solution of (1.3). Furthermore if $f \neq 0$ then g is a solution of (4.4).

Remark 4.1. If G is abelian, then the condition $(*)$ holds.

If $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$ and $x \in G$, where χ is a character of K ([13]), then we have

Theorem 4.3. Let $\delta > 0$ and let $(f, g) \in \mathcal{C}(G)$ such that $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$, $x \in G$,

$$(*) \quad \int_K \int_K f(zkxhy)\bar{\chi}(k)\bar{\chi}(h)d\omega_K(k)d\omega_K(h) \\ = \int_K \int_K f(zkyhx)\bar{\chi}(k)\bar{\chi}(h)d\omega_K(k)d\omega_K(h)$$



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and

$$(4.5) \quad \left| \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y))\bar{\chi}(k)d\omega_K(k) - |\Phi|f(x)g(y) \right| \leq \delta, \quad x, y \in G.$$

Then

i) f, g are bounded or

ii) f is unbounded and g is a solution of the functional equation

$$(4.6) \quad \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y))\bar{\chi}(k)d\omega_K(k) = |\Phi|f(x)f(y), \quad x, y \in G,$$

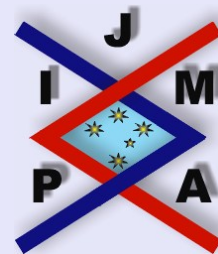
iii) g is unbounded and f is a solution of (1.15). Furthermore if $f \neq 0$ then g is a solution of (4.6).

Corollary 4.4. Let $\delta > 0$ and let $(f, g) \in \mathcal{C}(G)$ such that $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K, x \in G$,

$$(*) \quad \int_K \int_K f(zkxhy)\bar{\chi}(k)\bar{\chi}(h)d\omega_K(k)d\omega_K(h) \\ = \int_K \int_K f(zkyhx)\bar{\chi}(k)\bar{\chi}(h)d\omega_K(k)d\omega_K(h)$$

and

$$(4.7) \quad \left| \int_K f(xky)\bar{\chi}(k)d\omega_K(k) + \int_K f(xk\sigma(y))\bar{\chi}(k)d\omega_K(k) - 2f(x)g(y) \right| \leq \delta, \quad x, y \in G.$$



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where σ is an involution of G . Then

i) f, g are bounded or

ii) f is unbounded and g is a solution of the functional equation

$$(4.8) \quad \int_K g(xky)\overline{\chi}(k)d\omega_K(k) + \int_K g(xk\sigma(y))\overline{\chi}(k)d\omega_K(k) = 2g(x)g(y), \quad x, y \in G.$$

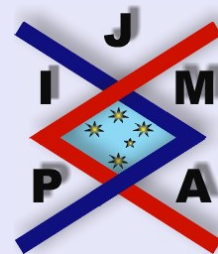
iii) g is unbounded and f is a solution of (1.7). Furthermore if $f \neq 0$ then g is a solution of (4.8).

Remark 4.2. If the algebra $\overline{\chi}\omega_K \star M(G) \star \overline{\chi}\omega_K$ is commutative then the condition (*) holds [4].

In the next theorem we assume that f to be bi- K -invariant (i.e. $f(hxk) = f(x)$, $h, k \in K, x \in G$ ([7], [10]), then we have

Theorem 4.5. Let $\delta > 0$ and let $(f, g) \in \mathcal{C}(G)$ such that $f(kxh) = f(x)$, $k, h \in K, x \in G$,

$$(*) \quad \int_K \int_K f(zkxhy)d\omega_K(k)d\omega_K(h) = \int_K \int_K f(zkyhx)d\omega_K(k)d\omega_K(h), \quad x, y, z \in G$$



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and

$$(4.9) \quad \left| \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y))d\omega_K(k) - |\Phi|f(x)g(y) \right| \leq \delta, \quad x, y \in G.$$

Then

i) f, g are bounded or

ii) f is unbounded and g is a solution of the functional equation

$$(4.10) \quad \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y))d\omega_K(k) = |\Phi|f(x)f(y), \quad x, y \in G,$$

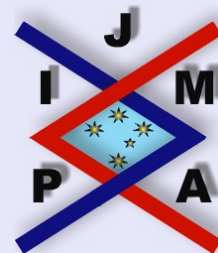
iii) g is unbounded and f is a solution of (1.14). Furthermore if $f \neq 0$ then g is a solution of (4.10).

Corollary 4.6 ([6]). Let $\delta > 0$ and let $(f, g) \in \mathcal{C}(G)$ such that $f(kxh) = f(x)$, $k, h \in K, x \in G$,

$$(*) \quad \int_K \int_K f(zkxhy)d\omega_K(k)d\omega_K(h) = \int_K \int_K f(zkyhx)d\omega_K(k)d\omega_K(h), \quad x, y, z \in G$$

and

$$(4.11) \quad \left| \int_K f(xky)d\omega_K(k) + \int_K f(xk\sigma(y))d\omega_K(k) - 2f(x)g(y) \right| \leq \delta, \quad x, y \in G.$$



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where σ is an involution of G . Then

- i) f, g are bounded or
- ii) f is unbounded and g is a solution of the functional equation

$$(4.12) \quad \int_K g(xky)d\omega_K(k) + \int_K g(xk\sigma(y))d\omega_K(k) = 2g(x)g(y), \quad x, y \in G.$$

- iii) g is unbounded and f is a solution of (1.9). Furthermore if $f \neq 0$ then g is a solution of (4.12).

Remark 4.3. If the algebra $\omega_K \star M(G) \star \omega_K$ is commutative then the condition (*) holds [4].

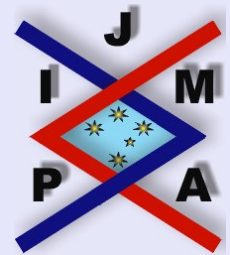
In the next corollary, we assume that $G = K$ is a compact group

Theorem 4.7. Let $\delta > 0$ and let (f, g) be complex measurable and essentially bounded functions on G such that f is a central function and (f, g) satisfy the inequality

$$(4.13) \quad \left| \sum_{\varphi \in \Phi} \int_G f(xt\varphi(y)t^{-1})dt - |\Phi|f(x)g(y) \right| \leq \delta, \quad x, y \in G.$$

Then

- i) f and g are bounded or



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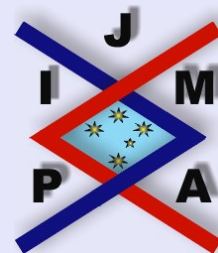
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ii) f is unbounded and g is a solution of the functional equation

$$(4.14) \quad \sum_{\varphi \in \Phi} \int_G g(xt\varphi(y)t^{-1})dt = |\Phi|g(x)g(y), \quad x, y \in G.$$

iii) g is unbounded and $f \equiv 0$.

Proof. Let $(f, g) \in L^\infty(G)$. Since f is central [5], then it satisfies the condition (*) ([4]). For (iii), if $f \neq 0$ then g is a solution of the functional equation (4.14). In view of the proposition in [9] we get the fact that g is continuous. Since G is compact then g is bounded. \square



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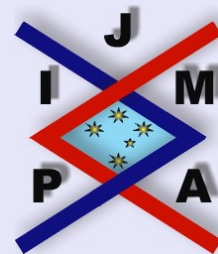
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References

- [1] R. BADORA, On Hyers-Ulam stability of Wilson's functional equation, *Aequationes Math.*, **60** (2000), 211–218.
- [2] J. BAKER, J. LAWRENCE AND F. ZORZITTO, The stability of the equation $f(x + y) = f(x)f(y)$, *Proc. Amer. Math. Soc.*, **74** (1979), 242–246.
- [3] J. BAKER, The stability of the cosine equation, *Proc. Amer. Math. Soc.*, **80**(3) (1980), 411–416.
- [4] B. BOUIKHALENE, On the stability of a class of functional equations, *J. Inequal. in Pure & Appl. Math.*, **4**(5) (2003), Article 104. [ONLINE <http://jipam.vu.edu.au/article.php?sid=345>]
- [5] J.L. CLERC, Les représentations des groupes compacts, *Analyse Harmoniques*, les Cours CIMPA, Université de Nancy I, 1980.
- [6] E. ELQORACHI AND M. AKKOUCHE, On Hyers-Ulam stability of Cauchy and Wilson equations, *Georgian Math. J.*, **11**(1) (2004), 69–82.
- [7] J. FARAUT, *Analyse Harmonique sur les Paires de Guelfand et les Espaces Hyperboliques*, les Cours CIMPA, Université de Nancy I, 1980.
- [8] W. FORG-ROB AND J. SCHWAIGER, The stability of some functional equations for generalized hyperbolic functions and for the generalized hyperbolic functions and for the generalized cosine equation, *Results in Math.*, **26** (1994), 247–280.
- [9] Z. GAJDA, On functional equations associated with characters of unitary representations of groups, *Aequationes Math.*, **44** (1992), 109–121.



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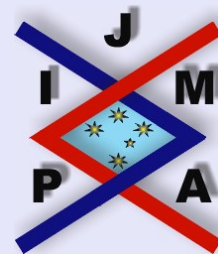
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- [10] S. HELGASON, *Groups and Geometric Analysis*, Academic Press, New York-London 1984.
- [11] E. HEWITT AND K.A. ROSS, *Abstract Harmonic Analysis*, Vol. I and II. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
- [12] P.I. KANNAPPAN, The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$, for groups, *Proc. Amer. Math. Soc.*, **19** (1968), 69–74.
- [13] R. TAKAHASHI, $SL(2, \mathbb{R})$, *Analyse Harmoniques*, les Cours CIMPA, Université de Nancy I, 1980.



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