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SOME INEQUALITIES FOR KUREPA'S FUNCTION

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ABSTRACT. In this paper we consider Kurepa's function $K(z)$ [3]. We give some recurrent relations for Kurepa's function via appropriate sequences of rational functions and gamma function. Also, we give some inequalities for Kurepa's function $K(x)$ for positive values of x.

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1. KUREPA'S FUNCTION $K(z)$

Đuro Kurepa considered, in the article [3], the function of left factorial $!n$ as a sum of factorials $!n = 0! + 1! + 2! + \cdots + (n-1)!$. Let us use the standard notation:

$$K(n) = \sum_{i=0}^{n-1} i!. \tag{1.1}$$

Sum (1.1) corresponds to the sequence $A003422$ in [5]. Analytical extension of the function (1.1) over the set of complex numbers is determined by the integral:

$$K(z) = \int_0^\infty e^{-t} \frac{t^z - 1}{t - 1} dt, \tag{1.2}$$

which converges for $\operatorname{Re} z > 0$ [4]. For function $K(z)$ we use the term *Kurepa's function*. It is easily verified that Kurepa's function $K(z)$ is a solution of the functional equation:

$$K(z) - K(z-1) = \Gamma(z). \tag{1.3}$$

Let us observe that since $K(z-1) = K(z) - \Gamma(z)$, it is possible to make the analytic continuation of Kurepa's function $K(z)$ for $\operatorname{Re} z \leq 0$. In that way, the Kurepa's function $K(z)$ is a meromorphic function with simple poles at $z = -1$ and $z = -n$ $(n \geq 3)$ [4]. Let us emphasize

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that in the following consideration, in Sections 2 and 3, it is sufficient to use only the fact that function $K(z)$ is a solution of the functional equation (1.3).

2. REPRESENTATION OF THE KUREPA'S FUNCTION VIA SEQUENCES OF POLYNOMIALS AND THE GAMMA FUNCTION

Đuro Kurepa considered, in article [4], the sequences of following polynomials:

$$P_n(z) = (z-n)P_{n-1}(z) + 1, \tag{2.1}$$

with an initial member $P_0(z) = 1$. On the basis of [4] we can conclude that the following statements are true:

Lemma 2.1. *For each* $n \in \mathbb{N}$ *and* $z \in \mathbb{C}$ *we have explicitly*:

$$P_n(z) = 1 + \sum_{j=0}^{n-1} \prod_{i=0}^{j} (z - n + i). \tag{2.2}$$

Theorem 2.2. *For each* $n \in \mathbb{N}$ *and* $z \in \mathbb{C} \backslash (\mathbb{Z}^- \cup \{0, 1, \ldots, n\})$ *is valid:*

$$K(z) = K(z-n) + \big(P_n(z) - 1\big) \cdot \Gamma(z-n). \tag{2.3}$$

3. REPRESENTATION OF THE KUREPA'S FUNCTION VIA SEQUENCES OF RATIONAL FUNCTIONS AND THE GAMMA FUNCTION

Let us observe that on the basis of a functional equation for the gamma function $\Gamma(z+1) = z\Gamma(z)$, it follows that the Kurepa function is the solution of the following functional equation:

$$K(z+1) - (z+1)K(z) + zK(z-1) = 0. \tag{3.1}$$

For $z \in \mathbb{C}\backslash\{0\}$, based on (3.1), we have:

$$K(z-1) = \frac{z+1}{z}K(z) - \frac{1}{z}K(z+1) = Q_1(z)K(z) - R_1(z)K(z+1), \tag{3.2}$$

for rational functions $Q_1(z) = \frac{z+1}{z}$, $R_1(z) = \frac{1}{z}$ over $\mathbb{C}\backslash\{0\}$. Next, for $z \in \mathbb{C}\backslash\{0, 1\}$, based on (3.1), we obtain

$$\begin{aligned} K(z-2) &= \frac{z}{z-1}K(z-1) - \frac{1}{z-1}K(z) \\ &\underset{(3.2)}{=} \frac{z}{z-1}\Big(\frac{z+1}{z}K(z) - \frac{1}{z}K(z+1)\Big) - \frac{1}{z-1}K(z) \\ &= \frac{z}{z-1}K(z) - \frac{1}{z-1}K(z+1) = Q_2(z)K(z) - R_2(z)K(z+1), \end{aligned} \tag{3.3}$$

for rational functions $Q_2(z) = \frac{z}{z-1}$, $R_2(z) = \frac{1}{z-1}$ over $\mathbb{C}\backslash\{0, 1\}$. Thus, for values $z \in \mathbb{C}\backslash\{0, 1, \ldots, n-1\}$, based on (3.1), by mathematical induction we have:

$$K(z-n) = Q_n(z)K(z) - R_n(z)K(z+1), \tag{3.4}$$

for rational functions $Q_n(z)$, $R_n(z)$ over $\mathbb{C}\backslash\{0, 1, \ldots, n-1\}$, which fulfill the same recurrent relations:

$$Q_n(z) = \frac{z-n+2}{z-n+1}Q_{n-1}(z) - \frac{1}{z-n+1}Q_{n-2}(z) \tag{3.5}$$

and

$$R_n(z) = \frac{z-n+2}{z-n+1}R_{n-1}(z) - \frac{1}{z-n+1}R_{n-2}(z), \tag{3.6}$$

with different initial functions $Q_{1,2}(z)$ and $R_{1,2}(z)$.

Based on the previous consideration we can conclude:

Lemma 3.1. *For each* $n \in \mathbb{N}$ *and* $z \in \mathbb{C}\backslash\{0,1,\dots,n-1\}$ *let the rational function* $Q_n(z)$ *be determined by the recurrent relation (3.5) with initial functions* $Q_1(z) = \frac{z+1}{z}$ *and* $Q_2(z) = \frac{z}{z-1}$*. Thus the sequence* $Q_n(z)$ *has an explicit form*:

$$Q_n(z) = 1 + \sum_{j=0}^{n-1} \prod_{i=0}^{j} \frac{1}{z-i}. \tag{3.7}$$

Lemma 3.2. *For each* $n \in \mathbb{N}$ *and* $z \in \mathbb{C}\backslash\{0,1,\dots,n-1\}$ *let the rational function* $R_n(z)$ *be determined by the recurrent relation (3.6) with initial functions* $R_1(z) = \frac{1}{z}$ *and* $R_2(z) = \frac{1}{z-1}$*. Thus the sequence* $R_n(z)$ *has an explicit form*:

$$R_n(z) = \sum_{j=0}^{n-1} \prod_{i=0}^{j} \frac{1}{z-i}. \tag{3.8}$$

Theorem 3.3. *For each* $n \in \mathbb{N}$ *and* $z \in \mathbb{C}\backslash\left(\mathbb{Z}^- \cup \{0,1,\dots,n-1\}\right)$ *we have*

$$K(z) = K(z-n) + \big(Q_n(z) - 1\big) \cdot \Gamma(z+1) \tag{3.9}$$

and

$$K(z) = K(z-n) + R_n(z) \cdot \Gamma(z+1). \tag{3.10}$$

4. SOME INEQUALITIES FOR KUREPA'S FUNCTION

In this section we consider the Kurepa function $K(x)$, given by an integral representation (1.2), for positive values of x. Thus the Kurepa function is positive and in the following consideration we give some inequalities for the Kurepa function.

Lemma 4.1. *For* $x \in [0,1]$ *the following inequalities are true:*

$$\Gamma\left(x + \frac{1}{2}\right) < x^2 - \frac{7}{4}x + \frac{9}{5} \tag{4.1}$$

and

$$(x+2)\Gamma(x+1) > \frac{9}{5}. \tag{4.2}$$

Proof. It is sufficient to use an approximation formula for the function $\Gamma(x+1)$ with a polynomial of the fifth degree:

$$P_5(x) = -0.1010678\,x^5 + 0.4245549\,x^4 - 0.6998588\,x^3 + 0.9512363\,x^2 - 0.5748646\,x + 1$$

which has an absolute error $|\varepsilon(x)| < 5 \cdot 10^{-5}$ for values of argument $x \in [0,1]$ [1] (formula 6.1.35, page 257). To prove the first inequality, for values $x \in [0,1/2]$, it is necessary to consider an equivalent inequality obtained by the following substitution $t = x + 1/2$ (thus $\Gamma(x+1/2) = \Gamma(t+1)/t$). To prove the first inequality, for values $x \in (1/2, 1]$, it is necessary to consider an equivalent inequality by the following substitution $t = x - 1/2$ (thus $\Gamma(x+1/2) = \Gamma(t+1)$). ☐

Remark 4.2. We note that for a proof of the previous inequalities it is possible to use other polynomial approximations (of a lower degree) of functions $\Gamma(x+1/2)$ and $\Gamma(x+1)$ for values $x \in [0,1]$.

Lemma 4.3. *For* $x \in [0,1]$ *the following inequality is true*:

$$K(x) \le \frac{9}{5}x. \tag{4.3}$$

Proof. Let us note that the first derivation of Kurepa's function $K(x)$, for values $x \in [0,1]$, is given by the following integral [4]:

$$K^{'}(x) = \int_0^\infty e^{-t}t^x \frac{\log t}{t-1} dt. \tag{4.4}$$

For $t \in (0,\infty)\backslash\{1\}$ Karamata's inequality is true: $\frac{\log t}{t-1} \leq \frac{1}{\sqrt{t}}$ [2]. Hence, for $x \in [0,1]$ the following inequality is true:

$$K^{'}(x) = \int_0^\infty e^{-t}t^x \frac{\log t}{t-1} dt \leq \int_0^\infty e^{-t}t^{x-1/2} dt = \Gamma\left(x+\frac{1}{2}\right). \tag{4.5}$$

Next, on the basis of Lemma 4.1 and inequality (4.5), for $x \in [0,1]$, the following inequalities are true:

$$K(x) \leq \int_0^x \Gamma\left(t+\frac{1}{2}\right) dt \leq \int_0^x \left(t^2 - \frac{7}{4}t + \frac{9}{5}\right) dt \leq \frac{9}{5}x. \tag{4.6}$$

□

Theorem 4.4. *For $x \geq 3$ the following inequality is true*:

$$K(x-1) \leq \Gamma(x), \tag{4.7}$$

while the equality is true for $x = 3$.

Proof. Based on the functional equation (1.3) the inequality (4.7), for $x \geq 3$, is equivalent to the following inequality:

$$K(x) \leq 2\Gamma(x). \tag{4.8}$$

Let us represent $[3,\infty) = \bigcup_{n=3}^{\infty}[n, n+1)$. Then, we prove that the inequality (4.8) is true, by mathematical induction over intervals $[n, n+1)$ ($n \geq 3$).

(i) Let $x \in [3,4)$. Then the following decomposition holds: $K(x) = K(x-3) + \Gamma(x-2) + \Gamma(x-1) + \Gamma(x)$. Hence, by Lemma 4.3, the following inequality is true:

$$K(x) \leq \frac{9}{5}(x-3) + \Gamma(x-2) + \Gamma(x-1) + \Gamma(x), \tag{4.9}$$

because $x - 3 \in [0,1)$. Next, by Lemma 4.1, the following inequality is true:

$$\frac{9}{5}(x-3) \leq (x-1)(x-3)\Gamma(x-2), \tag{4.10}$$

because $x - 3 \in [0,1)$. Now, based on (4.9) and (4.10) we conclude that the inequality is true:

$$K(x) \leq (x-1)(x-3)\Gamma(x-2) + \Gamma(x-2) + \Gamma(x-1) + \Gamma(x) = 2\Gamma(x). \tag{4.11}$$

(ii) Let the inequality (4.8) be true for $x \in [n, n+1)$ $(n \geq 3)$.

(iii) For $x \in [n+1, n+2)$ $(n \geq 3)$, based on the inductive hypothesis, the following inequality is true:

$$K(x) = K(x-1) + \Gamma(x) \leq 2\Gamma(x-1) + \Gamma(x) \leq 2\Gamma(x). \tag{4.12}$$

□

Remark 4.5. The inequality (4.8) is an improvement of the inequalities of Arandjelović: $K(x) \leq 1 + 2\Gamma(x)$, given in [4], with respect to the interval $[3,\infty)$.

Corollary 4.6. *For each* $k \in \mathbb{N}$ *and* $x \geq k+2$ *the following inequality is true*:

$$\frac{K(x-k)}{\Gamma(x-k+1)} \leq 1, \tag{4.13}$$

while the equality is true for $x = k+2$*.*

Theorem 4.7. *For each* $k \in \mathbb{N}$ *and* $x \geq k+2$ *the following double inequality is true:*

$$R_k(x) < \frac{K(x)}{\Gamma(x+1)} \leq \frac{P_{k-1}(x)+1}{P_{k-1}(x)} \cdot R_k(x), \tag{4.14}$$

while the equality is true for $x = k+2$*.*

Proof. For each $k \in \mathbb{N}$ and $x > k$ let us introduce the following function $G_k(x) = \sum_{i=0}^{k-1} \Gamma(x-i)$. Thus, the following relations:

$$G_k(x) = \Gamma(x+1) \cdot R_k(x) \tag{4.15}$$

and

$$G_k(x) = \Gamma(x-k) \cdot (P_k(x) - 1) \tag{4.16}$$

are true. The inequality $G_k(x) < K(x)$ is true for $x > k$. Hence, based on (4.15), the left inequality in (4.14) is true for all $x \geq k+2$. On the other hand, based on (4.16) and (4.13), for $x \geq k+2$, the following inequality is true:

$$\begin{aligned} \frac{K(x)}{G_k(x)} &= 1 + \frac{K(x-k)}{G_k(x)} = 1 + \frac{K(x-k)}{\Gamma(x-k)(P_k(x)-1)} \\ &= 1 + \frac{K(x-k)/\Gamma(x-k+1)}{P_{k-1}(x)} \leq 1 + \frac{1}{P_{k-1}(x)} = \frac{P_{k-1}(x)+1}{P_{k-1}(x)}. \end{aligned} \tag{4.17}$$

Hence, based on (4.15), the right inequality in (4.14) holds for all $x \geq k+2$. □

Corollary 4.8. *If for each* $k \in \mathbb{N}$ *we mark*:

$$A_k(x) = R_k(x) \quad \textit{and} \quad B_k(x) = \frac{P_{k-1}(x)+1}{P_{k-1}(x)} \cdot R_k(x), \tag{4.18}$$

thus, the following is true:

$$A_k(x) < A_{k+1}(x) < \frac{K(x)}{\Gamma(x+1)} \leq B_{k+1}(x) < B_k(x) \quad (x \geq k+3) \tag{4.19}$$

and

$$A_k(x), B_k(x) \sim \frac{1}{x} \ \wedge \ B_k(x) - A_k(x) = \frac{R_k(x)}{P_{k-1}(x)} \sim \frac{1}{x^k} \quad (x \to \infty). \tag{4.20}$$

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