ON SOME INEQUALITIES FOR THE SKEW LAPLACIAN ENERGY OF DIGRAPHS

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Abstract: In this paper we introduce and investigate the skew Laplacian energy of a digraph. We establish upper and lower bounds for the skew Laplacian energy of a digraph.

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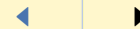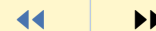
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1. Introduction

In this paper we are concerned with simple directed graphs. A directed graph (or just digraph) G consists of a non-empty finite set $V(G) = \{v_1, v_2, \ldots, v_n\}$ of elements called vertices and a finite set $\Gamma(G)$ of ordered pairs of distinct vertices called arcs. Two vertices are called adjacent if they are connected by an arc. The skew-adjacency matrix of G is the $n \times n$ matrix $S(G) = [a_{ij}]$ where $a_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(G)$, $a_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(G)$, $a_{ij} = 0$ otherwise. Hence $S(G)$ is a skew symmetric matrix of order n and all its eigenvalues are of the form $i\lambda$ where $i = \sqrt{-1}$ and $\lambda \in \mathbb{R}$. The skew energy of G is the sum of the absolute value of the eigenvalues of $S(G)$. For additional information on the skew energy of digraphs we refer to [1]. The degree of a vertex in a digraph G is the degree of the corresponding vertex of the underlying graph of G. Let $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, the diagonal matrix with vertex degrees $d_1, d_2, \ldots, d_n$ of $v_1, v_2, \ldots, v_n$. Then $L(G) = D(G) - S(G)$ is called the Laplacian matrix of the digraph G. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of $L(G)$. Then the set $\sigma_{SL}(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$ is called the skew Laplacian spectrum of the digraph G. The Laplacian matrix of a simple, undirected (n, m) graph G_1 is $L(G_1) = D(G_1) - A(G_1)$, where $A(G_1)$ is the adjacency matrix of G_1. It is symmetric, singular, positive semi-definite and all its eigenvalues are real and non negative. It is well known that the smallest eigenvalue is zero and its multiplicity is equal to the number of connected components of G_1. The Laplacian spectrum of the graph G_1, consisting of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ is the spectrum of its Laplacian matrix $L(G_1)$ [3, 4]. The spectrum of the graph G_1, consisting of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the spectrum of its adjacency matrix $A(G_1)$. The ordinary and Laplacian eigenvalues obey the following well-known relations:

$$(1.1) \qquad \sum_{i=1}^{n} \lambda_i = 0; \qquad \sum_{i=1}^{n} \lambda_i^2 = 2m,$$

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(1.2)
$$\sum_{i=1}^{n} \alpha_i = 2m; \qquad \sum_{i=1}^{n} \alpha_i^2 = 2m + \sum_{i=1}^{n} d_i^2.$$

The energy of the graph G_1 is defined as

$$E(G_1) = \sum_{i=1}^{n} |\lambda_i|.$$

For a survey of the mathematical properties of the energy we refer to [5]. In order to define the Laplacian energy of G_1, Gutman and Zhou [6] introduced auxiliary "eigenvalues" β_i, $i = 1, 2, \ldots, n$, defined by

$$\beta_i = \alpha_i - \frac{2m}{n}.$$

Then it follows that

$$\sum_{i=1}^{n} \beta_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \beta_i^2 = 2M$$

where $M = m + \frac{1}{2}\sum_{i=1}^{n}(d_i - \frac{2m}{n})^2$.

If G_1 is an (n, m)-graph and its Laplacian eigenvalues are $\alpha_1, \alpha_2, \ldots, \alpha_n$, then the Laplacian energy of G_1 [6] is defined by

$$LE(G_1) = \sum_{i=1}^{n} |\beta_i| = \sum_{i=1}^{n} \left| \alpha_i - \frac{2m}{n} \right|.$$

Gutman and Zhou [6] have shown a great deal of analogy between the properties of $E(G_1)$ and $LE(G_1)$. Among others they proved the following two inequalities:

(1.3)
$$LE(G_1) \leq \sqrt{2Mn}$$

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and

(1.4) $$2\sqrt{M} \le LE(G_1) \le 2M.$$

Various bounds for the Laplacian energy of a graph can be found in [8, 9].

The main purpose of this paper is to introduce the concept of the skew Laplacian energy $SLE(G)$ of a simple, connected digraph G, and to establish upper and lower bounds for $SLE(G)$ which are similar to (1.3) and (1.4). We may mention here that the skew Laplacian energy of a digraph considered in [2] was actually the second spectral moment.
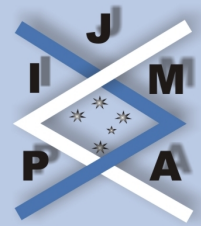
2. Bounds for the Skew Laplacian Energy of a Digraph

We begin by giving the formal definition of the skew Laplacian energy of a digraph.

Definition 2.1. *Let $S(G)$ be the skew adjacency matrix of a simple digraph G, possessing n vertices and m edges. Then the skew Laplacian energy of the digraph G is defined as*

$$SLE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the Laplacian matrix $L(G) = D(G) - S(G)$.

In analogy with (1.2), Adiga and Smitha [2] have proved that

$$\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} d_i = 2m \tag{2.1}$$

and

$$\sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} d_i(d_i - 1). \tag{2.2}$$

We may observe that equations (2.1) and (2.2) are evident as (1.1) and (1.2), which follow from the trace equality.

Define $\gamma_i = \mu_i - \frac{2m}{n}$ for $i = 1, 2, \ldots, n$. On using (2.1) and (2.2) we see that

$$\sum_{i=1}^{n} \gamma_i = 0 \tag{2.3}$$

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and

$$(2.4) \qquad \sum_{i=1}^{n} \gamma_i^2 = 2M,$$

where

$$M = -m + \frac{1}{2} \sum_{i=1}^{n} \left(d_i - \frac{2m}{n} \right)^2.$$

Since $2m/n$ is the average vertex degree, we have $M + m = 0$ if and only if G is regular.

Theorem 2.2. *Let G be an (n, m)-digraph and let d_i be the degree of the ith vertex of G, $i = 1, 2, \ldots, n$. If $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the Laplacian matrix $L(G) = D(G) - S(G)$, where $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix and $S(G) = [a_{ij}]$ is the skew-adjacency matrix of G, then*

$$SLE(G) \le \sqrt{2M_1 n}.$$

Here $M_1 = M + 2m = m + \frac{1}{2}\sum_{i=1}^{n}(d_i - \frac{2m}{n})^2$.

Proof. From (2.1) it is clear that

$$(2.5) \qquad \sum_{i=1}^{n} \mathrm{Re}(\mu_i) = \sum_{i=1}^{n} d_i.$$

By Schur's unitary triangularization theorem, there is a unitary matrix U such that $U^* L(G) U = T = [t_{ij}]$, where T is an upper triangular matrix with diagonal entries $t_{ii} = \mu_i$, $i = 1, 2, \ldots, n$, i.e. $L(G) = [s_{ij}]$ and $T = [t_{ij}]$ are unitarily equivalent. That is,

$$\sum_{i,j=1}^{n} |s_{ij}|^2 = \sum_{i,j=1}^{n} |t_{ij}|^2 \ge \sum_{i=1}^{n} |t_{ii}|^2 = \sum_{i=1}^{n} |\mu_i|^2.$$

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Thus

$$(2.6) \qquad \sum_{i=1}^{n} d_i^2 + 2m \geq \sum_{i=1}^{n} |\mu_i|^2.$$

Let $\gamma_i = \mu_i - \frac{2m}{n}$, $i = 1, 2, \ldots, n$. By the Cauchy-Schwarz inequality applied to the Euclidean vectors $(|\gamma_1|, |\gamma_2|, \ldots, |\gamma_n|)$ and $(1, 1, \ldots, 1)$, we have

$$(2.7) \qquad SLE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| = \sum_{i=1}^{n} |\gamma_i| \leq \sqrt{\sum_{i=1}^{n} |\gamma_i|^2} \sqrt{n}.$$

Now by (2.5) and (2.6),

$$\sum_{i=1}^{n} |\gamma_i|^2 = \sum_{i=1}^{n} \left(\left| \mu_i - \frac{2m}{n} \right| \right) \left(\left| \overline{\mu_i} - \frac{2m}{n} \right| \right)$$

$$= \sum_{i=1}^{n} |\mu_i|^2 - \frac{2m}{n} \sum_{i=1}^{n} 2 \operatorname{Re} \mu_i + \frac{4m^2}{n}$$

$$\leq 2m + \sum_{i=1}^{n} d_i^2 - \frac{4m}{n} \sum_{i=1}^{n} d_i + \frac{4m^2}{n}$$

$$(2.8) \qquad = 2M_1.$$

Using (2.8) in (2.7), we conclude that

$$SLE(G) \leq \sqrt{2M_1 n}.$$

$\square$

Second Proof. Consider the sum

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} (|\gamma_i| - |\gamma_j|)^2.$$

By direct calculation

$$S = 2n \sum_{i=1}^{n} |\gamma_i|^2 - 2 \left(\sum_{i=1}^{n} |\gamma_i| \sum_{j=1}^{n} |\gamma_j| \right).$$

It follows from (2.8) and the definition of $SLE(G)$ that

$$S \le 4nM_1 - 2SLE(G)^2.$$

Since $S \ge 0$, we have $SLE(G) \le \sqrt{2M_1 n}$. □

If $E(G_1)$ is the ordinary energy of a simple graph G_1 it is well-known [7] that

$$(2.9) \qquad E(G_1) \le \frac{2m}{n} + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]}.$$

We prove an inequality similar to (2.9) involving the skew Laplacian energy of a digraph. Let G be an (n, m)-digraph. Suppose $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the Laplacian matrix $L(G)$ with $|\gamma_1| \le |\gamma_2| \le \cdots \le |\gamma_n| = k$, where $\gamma_i = \mu_i - \frac{2m}{n}$, $i = 1, 2, \ldots, n$. Let $X = (|\gamma_1| \le |\gamma_2| \le \cdots \le |\gamma_{n-1}|)$ and $Y = (1, 1, \ldots, 1)$. By the Cauchy-Schwarz inequality we have

$$\left(\sum_{i=1}^{n-1} |\gamma_i| \right)^2 \le (n-1) \sum_{i=1}^{n-1} |\gamma_i|^2.$$

That is,

$$(SLE(G) - |\gamma_n|)^2 \leq (n-1)\left(\sum_{i=1}^{n}|\gamma_i|^2 - |\gamma_n|^2\right).$$

Using (2.8) in the above inequality we obtain

$$SLE(G) \leq k + \sqrt{(n-1)(2M_1 - k^2)},$$

where $k = |\gamma_n|$ and M_1 is as in Theorem 2.2.

Theorem 2.3. *We have*

$$2\sqrt{|M|} \leq SLE(G) \leq 2M_1.$$

Proof. Since $\sum_{i=1}^{n}\gamma_i = 0$, we have

$$\sum_{i=1}^{n}\gamma_i^2 + 2\sum_{i<j}^{n}\gamma_i\gamma_j = 0.$$

Now, using (2.4) in the above equation we have

$$2M = -2\sum_{i<j}^{n}\gamma_i\gamma_j.$$

This implies

$$(2.10) \qquad 2|M| = 2\left|\sum_{i<j}^{n}\gamma_i\gamma_j\right| \leq 2\sum_{i<j}^{n}|\gamma_i||\gamma_j|.$$

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Now by (2.4),

$$SLE(G)^2 = \left(\sum_{i=1}^{n} |\gamma_i|\right)^2 = \sum_{i=1}^{n} |\gamma_i|^2 + 2\sum_{i<j}^{n} |\gamma_i||\gamma_j|$$

$$\geq 2|M| + 2\sum_{i<j}^{n} |\gamma_i||\gamma_j|,$$
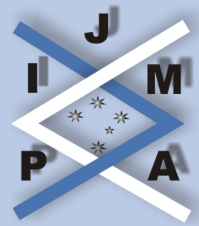
which combined with (2.10) yields $SLE(G)^2 \geq 4|M|$. Thus

$$2\sqrt{|M|} \leq SLE(G).$$

To prove the right-hand inequality, note that for a graph with m edges and no isolated vertex, $n \leq 2m$. By Theorem 2.2, we have

$$SLE(G) \leq \sqrt{2M_1 n} \leq \sqrt{2M_1(2m)} = 2\sqrt{M_1 m}.$$

Since $M_1 \geq m$, we obtain $SLE(G) \leq 2M_1$. $\qquad\square$

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