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INEQUALITIES DEFINING CERTAIN SUBCLASSES OF ANALYTIC AND MULTIVALENT FUNCTIONS INVOLVING FRACTIONAL CALCULUS OPERATORS

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ABSTRACT. Making use of a certain fractional calculus operator, we introduce two new subclasses $M_\delta(p;\lambda,\mu,\eta)$ and $T_\delta(p;\lambda,\mu,\eta)$ of analytic and $p-$valent functions in the open unit disk. The results investigated exhibit the sufficiency conditions for a function to belong to each of these classes. Several geometric properties of such multivalent functions follow, and these consequences are also mentioned.

Key words and phrases: Analytic functions, Multivalent functions, Starlike functions, Convex functions, Fractional calculus operators.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic and $p-$valent in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

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A function $f(z) \in \mathcal{A}_p$ is said to be $p-$valently starlike in $\mathcal{U}$, if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in \mathcal{U}), \tag{1.2}$$

and the function $f(z) \in \mathcal{A}_p$ is said to be $p-$valently convex in $\mathcal{U}$, if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.3}$$

Further, a function $f(z) \in \mathcal{A}_p$ is said to be $p-$valently close-to-convex in $\mathcal{U}$, if

$$\Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.4}$$

One may refer to [1], [2] and [9] for above definitions and other related details.

The operator $J_{0,z}^{\lambda,\mu,\eta}$ occurring in this paper is the Saigo type fractional calculus operator defined as follows ([8]):

Definition 1.1. Let $0 \leq \lambda < 1$ and $\mu,\eta \in \mathbb{R}$, then

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)}\int_0^z (z-t)^{-\lambda} F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{t}{z}\right) f(t)dt\right), \tag{1.5}$$

where the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, with the order

$$f(z) = O\left(|z|^{\varepsilon}\right)(z \to 0), \text{ where } \varepsilon > \max\{0, \mu-\eta\} - 1.$$

It being understood that $(z-t)^{-\lambda}$ denotes the principal value for $0 \leq \arg(z-t) < 2\pi$. The ${}_2F_1$ function occurring in the right-hand side of (1.5) is the familiar Gaussian hypergeometric function (see [9] for its definition).

Definition 1.2. Under the hypotheses of Definition 1.1, a fractional calculus operator $J_{0,z}^{\lambda+m,\mu+m,\eta+m}$ is defined by ([7])

$$J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z) \quad (z \in \mathcal{U}; m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}). \tag{1.6}$$

We observe that

$$D_z^{\lambda} f(z) = J_{0,z}^{\lambda,\lambda,\eta} f(z) \qquad (0 \leq \lambda < 1), \tag{1.7}$$

and

$$D_z^{\lambda+m} f(z) = J_{0,z}^{\lambda+m,\lambda+m,\eta+m} f(z) \qquad (0 \leq \lambda < 1; m \in \mathbb{N}_0), \tag{1.8}$$

where $D_z^{\lambda+m}$ is the well known fractional derivative operator ([6], [9]).

We introduce here two subclasses of functions $\mathcal{M}_\delta(p;\lambda,\mu,\eta)$ and $\mathcal{T}_\delta(p;\lambda,\mu,\eta)$ which are defined as follows.

Definition 1.3. Let $\delta \in \mathbb{R} \setminus \{0\}$, $p \in \mathbb{N}$, $0 \leq \lambda < 1$, $\mu < 1$, and $\eta > \max(\lambda,\mu) - p - 1$. Then the function $f(z) \in \mathcal{A}_p$ is said to belong to $\mathcal{M}_\delta(p;\lambda,\mu,\eta)$ if it satisfies the inequality

$$\left|\left(\frac{zJ_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{J_{0,z}^{\lambda,\mu,\eta} f(z)}\right)^{\delta} - (p-\mu)^{\delta}\right| < (p-\mu)^{\delta} \qquad (z \in \mathcal{U}), \tag{1.9}$$

where the value of $\left(z\, J_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z) / J_{0,z}^{\lambda,\mu,\eta} f(z)\right)^{\delta}$ is taken as its principal value.

Definition 1.4. Under the hypotheses of Definition 1.3, the function $f(z) \in \mathcal{A}_p$ is said to belong to $\mathcal{T}_\delta(p;\lambda,\mu,\eta)$ if it satisfies the inequality

$$\left|\left(z^{\mu-p}J_{0,z}^{\lambda,\mu,\eta}f(z)\right)^\delta - \left(\frac{\Gamma(p+1)\Gamma(p+\eta-\mu+1)}{\Gamma(p-\mu+1)\Gamma(p+\eta-\lambda+1)}\right)^\delta\right| < \left(\frac{\Gamma(p+1)\Gamma(p+\eta-\mu+1)}{\Gamma(p-\mu+1)\Gamma(p+\eta-\lambda+1)}\right)^\delta \qquad (z \in \mathcal{U}), \tag{1.10}$$

where the value of $\left(z^{\mu-p}J_{0,z}^{\lambda,\mu,\eta}f(z)\right)^\delta$ is considered to be its principal value. For $\lambda = \mu$, we have

$$\mathcal{M}_\delta(p;\mu,\mu,\eta) = \mathcal{M}_\delta(p;\mu), \tag{1.11}$$

and

$$\mathcal{T}_\delta(p;\mu,\mu,\eta) = \mathcal{T}_\delta(p;\mu). \tag{1.12}$$

The classes $\mathcal{M}_\delta(p;\mu)$and $\mathcal{T}_\delta(p;\mu)$ were studied recently in [4]. In view of the operational relation (1.8), it may be noted that the functions in $\mathcal{M}_1(p;0)$ are $p-$valently starlike in $\mathcal{U}$, whereas, the functions in$\mathcal{T}_1(p;1)$ are $p-$valently close-to-convex in $\mathcal{U}$.

In this paper we investigate characterization properties giving sufficiency conditions for functions of the form (1.1) to belong to the classes $\mathcal{M}_\delta(p;\lambda,\mu,\eta)$ and $\mathcal{T}_\delta(p;\lambda,\mu,\eta)$ involving the fractional calculus operator (1.6). Several consequences of the main results and their relevance to known results are also pointed out.

2. RESULTS REQUIRED

We mention the following results which are used in the sequel:

Lemma 2.1. *([8]). If $0 \le \lambda < 1$; μ,$\eta \in \mathbb{R}$ and $k > \max\{0, \mu-\eta\} - 1$, then*

$$J_{0,z}^{\lambda,\mu,\eta}z^k = \frac{\Gamma(1+k)\Gamma(1-\mu+\eta+k)}{\Gamma(1-\mu+k)\Gamma(1-\lambda+\eta+k)}z^{k-\mu}. \tag{2.1}$$

Lemma 2.2. *([5]). Let $w(z)$ be an analytic function in the unit disk $\mathcal{U}$ with $w(0) = 0$, and let $0 < r < 1$. If $|w(z)|$ attains at z_0 its maximum value on the circle $|z| = r$, then*

$$z_0w'(z_0) = kw(z_0) \quad (k \ge 1). \tag{2.2}$$

3. MAIN RESULTS

We begin by proving

Theorem 3.1. *Let $\delta \in \mathbb{R}\setminus\{0\}$, $p \in \mathbb{N}$, $0 \le \lambda < 1$, $\mu < 1$, $\eta > \max(\lambda,\mu) - p - 1$, and $a > 0, b \ge 0$, such that $a + 2b \le 1$. If a function $f(z) \in \mathcal{A}_p$ satisfies the inequality*

$$\Re\left[1 + z\left(\frac{J_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{J_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} - \frac{J_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{J_{0,z}^{\lambda,\mu,\eta}f(z)}\right)\right] \begin{cases} < \dfrac{a+b}{\delta(1+a)(1-b)} & (\delta > 0) \\ > \dfrac{a+b}{\delta(1+a)(1-b)} & (\delta < 0) \end{cases} \qquad (z \in \mathcal{U}), \tag{3.1}$$

then $f(z) \in \mathcal{M}_\delta(p;\lambda,\mu,\eta)$.

Proof. Let $f(z) \in \mathcal{A}_p$, and define a function $w(z)$ by

$$\left(\frac{zJ_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{J_{0,z}^{\lambda,\mu,\eta}f(z)}\right)^{\delta} = (p-\mu)^{\delta}\left(\frac{1+aw(z)}{1-bw(z)}\right) \qquad (z \in \mathcal{U}). \tag{3.2}$$

Then it follows from (2.1) that $w(z)$ is analytic function in $\mathcal{U}$, and $w(0) = 0$. Differentiation of (3.2) gives

$$\left\{1 + z\left(\frac{J_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{J_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} - \frac{J_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{J_{0,z}^{\lambda,\mu,\eta}f(z)}\right)\right\} = \frac{1}{\delta}\left(\frac{(a+b)zw'(z)}{(1+aw(z))(1-bw(z))}\right) = \phi(z) \text{ (say)}. \tag{3.3}$$

Assume that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, applying Lemma 2.2, we can write

$$z_0 w'(z_0) = kw(z_0) \qquad (k \geq 1),$$

and $w(z_0) = e^{i\theta}$ $(\theta \in [0, 2\pi))$, so that from (3.3) we have

$$\begin{aligned}
\Re\{\phi(z_0)\} &= \frac{k(a+b)}{\delta}\Re\left\{\frac{w(z_0)}{(1+aw(z_0))(1-bw(z_0))}\right\} \\
&= \frac{k}{\delta}\Re\left\{\frac{1}{1-bw(z_0)} - \frac{1}{1+aw(z_0)}\right\} \\
&= \frac{k}{\delta}\Re\left\{\frac{1-be^{-i\theta}}{1+b^2-2b\cos\theta} - \frac{1+ae^{-i\theta}}{1+a^2+2a\cos\theta}\right\} \\
&= \frac{k}{\delta}\left\{\frac{1}{2+\frac{b^2-1}{1-b\cos\theta}} - \frac{1}{2+\frac{a^2-1}{1+a\cos\theta}}\right\} = \frac{k\Delta}{\delta},
\end{aligned}$$

where $\theta \neq \cos^{-1}(-1/a)$ and $\theta \neq \cos^{-1}(-1/b)$.

Simple calculations (under the constraints mentioned with the hypotheses for the parameters a and b) yield that $\Delta \geq \frac{(a+b)}{(1+a)(1-b)}$, and since $k \geq 1$, it follows that

$$\Re\{\phi(z_0)\} = \frac{k\Delta}{\delta}\begin{cases} > \frac{(a+b)}{\delta(1+a)(1-b)} & (\delta > 0), \\ < \frac{(a+b)}{\delta(1+a)(1-b)} & (\delta < 0). \end{cases} \tag{3.4}$$

This contradicts our condition (3.1), and we conclude from (3.2) that

$$\begin{aligned}
\left|\left(\frac{zJ_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{J_{0,z}^{\lambda,\mu,\eta}}\right)^{\delta} - (p-\mu)^{\delta}\right| &= (p-\mu)^{\delta}\left|\frac{(a+b)w(z)}{1-bw(z)}\right| \\
&< (p-\mu)^{\delta}\left(\frac{a+b}{1-b}\right) \leq (p-\mu)^{\delta}.
\end{aligned}$$

This completes the proof of Theorem 3.1. □

Next we prove

Theorem 3.2. *Let $\delta \in \mathbb{R} \setminus \{0\}$, $p \in \mathbb{N}$, $0 \leq \lambda < 1$, $\mu < 1$, $\eta > \max(\lambda, \mu) - p - 1$, and $a > 0$, $b \geq 0$ such that $a + 2b \leq 1$. If a function $f(z) \in \mathcal{A}_p$ satisfies the inequality*

$$\Re\left(\frac{zJ_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{J_{0,z}^{\lambda,\mu,\eta}}\right)\begin{cases} < p-\mu+\frac{a+b}{\delta(1+a)(1-b)} & (\delta>0) \\ > p-\mu+\frac{a+b}{\delta(1+a)(1-b)} & (\delta>0) \end{cases} \qquad (z \in \mathcal{U}), \tag{3.5}$$

then $f(z) \in T_\delta(p; \lambda, \mu, \eta)$.

Proof. Consider

$$\left(z^{\mu-p}J_{0,z}^{\lambda,\mu,\eta}f(z)\right)^\delta = \left(\frac{\Gamma(1+p)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}\right)^\delta \left(\frac{1+aw(z)}{1-bw(z)}\right) \quad (z \in \mathcal{U}). \tag{3.6}$$

Using the same method as elucidated in the proof of Theorem 3.1, we arrive at the desired result. □

Remark 3.3. If we set $\lambda = \mu, a = 1, b = 0$, then Theorems 3.1 and 3.2 by appealing to the operational relation (1.8) correspond to the recently established results due to Irmak et al. [4, pp. 271–272].

Theorems 3.1 and 3.2 would also yield various results involving analytic and multivalent functions by suitably choosing the values of a, b, δ, μ and p. Setting $\delta = 1$ in Theorems 3.1 and 3.2, we have

Corollary 3.4. *Let $p \in \mathbb{N}$, $0 \leq \lambda < 1$, $\mu < 1$, $\eta > \max(\lambda, \mu) - p - 1$, and $a > 0$, $b \geq 0$ such that $a + 2b \leq 1$. If a function $f(z) \in \mathcal{A}_p$ satisfies the inequality*

$$\Re\left\{1 + z\left(\frac{J_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{J_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} - \frac{J_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{J_{0,z}^{\lambda,\mu,\eta}f(z)}\right)\right\} < \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}), \tag{3.7}$$

then $f(z) \in \mathcal{M}_1(p; \lambda, \mu, \eta)$.

Corollary 3.5. *Let $p \in \mathbb{N}$, $0 \leq \lambda < 1$, $\mu < 1$, $\eta > \max(\lambda, \mu) - p - 1$, and $a > 0, b \geq 0$ such that $a + 2b \leq 1$. If a function $f(z) \in \mathcal{A}_p$ satisfies the inequality*

$$\Re\left(\frac{zJ_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{J_{0,z}^{\lambda,\mu,\eta}f(z)}\right) < p - \mu + \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}), \tag{3.8}$$

then $f(z) \in \mathcal{T}_1\,(p; \lambda, \mu, \eta)$.

Corollaries 3.4 and 3.5 on putting $\lambda = \mu = 0$, and using (1.8) give the following results:

Corollary 3.6. *Let $p \in \mathbb{N}$, $a > 0$, $b \geq 0$ such that $a + 2b \leq 1$. If a function $f(z) \in \mathcal{A}_p$ satisfies the inequality*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} < \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}), \tag{3.9}$$

then $f(z)$ is p-valently starlike in $\mathcal{U}$.

Corollary 3.7. *Let $p \in \mathbb{N}$, $a > 0$, $b \geq 0$ such that $a + 2b \leq 1$. If a function $f(z) \in \mathcal{A}_p$ satisfies the inequality*

$$\Re\left\{\frac{zf\prime(z)}{f(z)}\right\} < p + \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}), \tag{3.10}$$

then $\Re\left\{\frac{f(z)}{z^p}\right\} > 0, \quad (z \in \mathcal{U}).$

Lastly, Corollaries 3.4 and 3.5 on putting $\lambda = \mu = 1$, and using (1.8) give

Corollary 3.8. *Let $p \in \mathbb{N}$, $a > 0$, $b \geq 0$ such that $a + 2b \leq 1$. If a function $f(z) \in \mathcal{A}_p$ satisfies the inequality*

$$\Re\left\{1 + \frac{zf'''(z)}{f''(z)} - \frac{zf''(z)}{f'(z)}\right\} < \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}), \tag{3.11}$$

then $f(z)$ is $p-$valently convex in $\mathcal{U}$.

Corollary 3.9. *Let $p \in \mathbb{N}$, $a > 0$, $b \geq 0$ such that $a + 2b \leq 1$. If a function $f(z) \in \mathcal{A}_p$ satisfies the inequality*

$$\Re\left\{\frac{zf''(z)}{f'(z)}\right\} < p - 1 + \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}), \tag{3.12}$$

then $f(z)$ is $p-$valently close-to -convex in $\mathcal{U}$.

Remark 3.10. When $a = 1$, $b = 0$, then the Corollaries 3.6 – 3.9 correspond to the known results [3, pp. 457–458] involving inequalities on $p-$valent functions.

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