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ESTIMATION FOR BOUNDED SOLUTIONS OF INTEGRAL INEQUALITIES INVOLVING INFINITE INTEGRATION LIMITS

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Abstract

Some integral inequalities with infinite integration limits are established as generalizations of a known result due to B.G. Pachpatte.

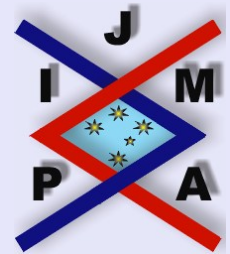
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1. Introduction

As well known, various differential and integral inequalities have played a dominant role in the development of the theories of differential, functional-differential as well as integral equations. The most powerful integral inequalities applied frequently in the literature are the famous Gronwall-Bellman inequality [1] and its first nonlinear generalization due to Bihari (cf., [2]). A large number of generalizations and their applications of the Gronwall-Bellman inequality have been obtained by many authors (cf., [4] – [7], [3], [5]). Pachpatte [6, p. 28] proved the following interesting variant of the Gronwall-Bellman inequality which contains an infinite integration limit:

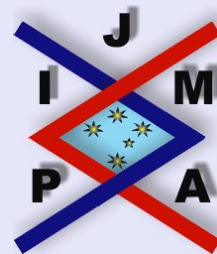
Theorem A. *Let f be a nonnegative continuous function defined for $t \in \mathbb{R}_+ = [0, \infty)$ such that $\int_0^\infty f(s)ds < \infty$ and $c(t) > 0$ be a continuous and decreasing function defined for $t \in \mathbb{R}_+$. If $u(t) \geq 0$ is a bounded continuous function defined for $t \in \mathbb{R}_+$ and satisfies*

$$u(t) \leq c(t) + \int_t^\infty f(s)u(s)ds, \quad t \in \mathbb{R}_+,$$

then

$$u(t) \leq c(t) \exp\left(\int_t^\infty f(s)ds\right), \quad t \in \mathbb{R}_+.$$

We note that, the condition above on $c(t)$ can be relaxed to only require that, it is nonnegative, continuous and nonincreasing on \mathbb{R}_+ . The importance of the last result was indicated in [6] by the fact that, it can be used to derive the Rodrigues' inequality [8] that played a crucial role in the study of many perturbed linear delay differential equations.



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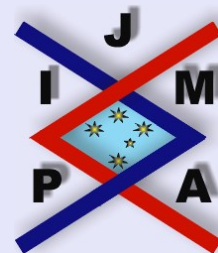
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The aim of the present paper is to establish some new linear and nonlinear generalizations of Theorem A. In the sequel, we denote by $C(S, M)$ the class of continuous functions defined on set S with range contained in set M .



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2. Linear Generalizations

Firstly we show that an inversed version of Theorem A is valid:

Theorem 2.1. *Let $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy the condition $\int_0^\infty f(s)ds < \infty$ and $m \in C(\mathbb{R}_+, (0, \infty))$ be nondecreasing. If $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ is bounded and satisfies the inequality*

$$(2.1) \quad x(t) \geq m(t) + \int_t^\infty f(s)x(s)ds, \quad t \in \mathbb{R}_+,$$

then

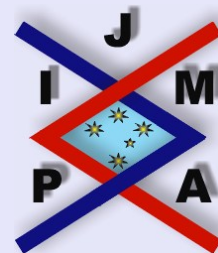
$$(2.2) \quad x(t) \geq m(t) \exp \int_t^\infty f(s)ds, \quad t \in \mathbb{R}_+.$$

Proof. From (2.1) we derive

$$(2.3) \quad \begin{aligned} \frac{x(t)}{m(t) - \varepsilon} &> 1 + \frac{1}{m(t) - \varepsilon} \int_t^\infty f(s)x(s)ds \\ &\geq 1 + \int_t^\infty f(s) \frac{x(s)}{m(s) - \varepsilon} ds, \quad t \in \mathbb{R}_+, \end{aligned}$$

where $\varepsilon > 0$ is an arbitrary number satisfying $m(0) - \varepsilon > 0$. Define a positive and nonincreasing function $V \in C(\mathbb{R}_+, \mathbb{R}_+)$ by the right member of (2.3). Then we have $V(\infty) = 1$ and

$$(2.4) \quad x(t) > [m(t) - \varepsilon]V(t), \quad t \in \mathbb{R}_+.$$



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By differentiation we obtain

$$\frac{dV(t)}{dt} = -f(t) \frac{x(t)}{m(t) - \varepsilon} < -f(t)V(t), \quad t \in \mathbb{R}_+.$$

Rewrite the last relation in the form

$$\frac{dV(t)}{V(t)dt} < -f(t), \quad t \in \mathbb{R}_+,$$

and integrating its both sides from t to ∞ , then we have

$$\ln V(\infty) - \ln V(t) < - \int_t^\infty f(s)ds, \quad t \in \mathbb{R}_+,$$

i.e.,

$$V(t) > \exp \int_t^\infty f(s)ds, \quad t \in \mathbb{R}_+.$$

Substituting the last relation into (2.4), and letting $\varepsilon \rightarrow 0$, the desired inequality (2.2) follows. \square

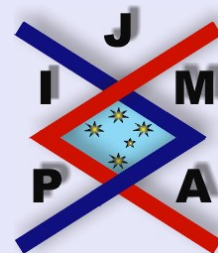
From Theorem A and Theorem 2.1, we obtain the following

Corollary 2.2. *Let $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy $\int_0^\infty f(s)ds < \infty$. Let $c \geq 0$ be a constant. Then the linear integral equation*

$$(2.5) \quad x(t) = c + \int_t^\infty f(s)x(s)ds, \quad t \in \mathbb{R}_+,$$

has an unique bounded continuous solution represented by

$$(2.6) \quad x(t) = c \exp \int_t^\infty f(s)ds, \quad t \in \mathbb{R}_+.$$



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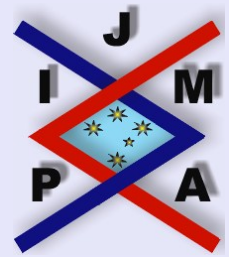


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Proof. If $c > 0$, by letting $c(t) \equiv c$ and $m(t) \equiv c$ respectively in Theorem A and Theorem 2.1, we have $x(t) \leq c \exp \int_t^\infty f(s) ds$ and $x(t) \geq c \exp \int_t^\infty f(s) ds$.

Hence (2.6) is the unique bounded continuous solution of the equation (2.5). By the continuous dependence on c of $x(t)$ given by (2.6), the conclusion holds also when $c = 0$. □

The next result is a new generalization of Pachpatte's inequality in the case when an iterated integral functional is involved.

Theorem 2.3. *Let $n \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nonincreasing. Let $f, h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $g'(t) \geq 0$ and $\int_0^\infty [f(s) + g(s)h(s)] ds < \infty$. If $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ is bounded and satisfies the inequality*

$$(2.7) \quad x(t) \leq n(t) + \int_t^\infty f(s) \left(x(s) + g(s) \int_s^\infty h(k)x(k) dk \right) ds, \quad t \in \mathbb{R}_+,$$

then

$$(2.8) \quad x(t) \leq n(t) \left\{ 1 + \int_t^\infty f(s) \exp \left(\int_s^\infty [f(k) + g(k)h(k)] dk \right) ds \right\}, \quad t \in \mathbb{R}_+.$$

Proof. From (2.7) we have

$$(2.9) \quad \begin{aligned} & \frac{x(t)}{n(t) + \varepsilon} \\ & < 1 + \frac{1}{n(t) + \varepsilon} \int_t^\infty f(s) \left(x(s) + g(s) \int_s^\infty h(k)x(k) dk \right) ds \\ & < 1 + \int_t^\infty f(s) \left(\frac{x(s)}{n(s) + \varepsilon} + g(s) \int_s^\infty h(k) \frac{x(k)}{n(k) + \varepsilon} dk \right) ds, \quad t \in \mathbb{R}_+, \end{aligned}$$

where $\varepsilon > 0$ is an arbitrary positive number. Define a function $V \in C(\mathbb{R}_+, \mathbb{R}_+)$ by the right member of inequality (2.9). Then $V(t)$ is positive and nonincreasing with $V(\infty) = 1$, and by (2.9) we have

$$(2.10) \quad x(t) < [n(t) + \varepsilon] V(t), \quad t \in \mathbb{R}_+.$$

By differentiation we obtain

$$\begin{aligned} \frac{dV(t)}{dt} &= -f(t) \left(\frac{x(t)}{n(t) + \varepsilon} + g(t) \int_t^\infty h(k) \frac{x(k)}{n(k) + \varepsilon} dk \right) \\ &\geq -f(t) \left(V(t) + g(t) \int_t^\infty h(k) V(k) dk \right), \quad t \in \mathbb{R}_+. \end{aligned}$$

Now we define

$$W(t) = V(t) + g(t) \int_t^\infty h(k) V(k) dk.$$

Then $W(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is positive, $W(\infty) = V(\infty) = 1$, and we have

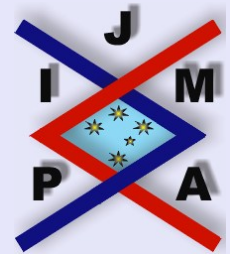
$$(2.11) \quad W(t) \geq V(t), \quad t \in \mathbb{R}_+,$$

and

$$(2.12) \quad \frac{dV(t)}{dt} \geq -f(t) W(t), \quad t \in \mathbb{R}_+.$$

By differentiation we derive

$$(2.13) \quad \begin{aligned} \frac{dW(t)}{dt} &= \frac{dV(t)}{dt} + g'(t) \int_t^\infty h(k) V(k) dk - g(t) h(t) V(t) \\ &\geq -[f(t) + g(t) h(t)] W(t), \quad t \in \mathbb{R}_+, \end{aligned}$$



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here (2.11) and (2.12) are used. Rewrite the last relation in the form

$$\frac{dW(t)}{W(t)dt} \geq -[f(t) + g(t)h(t)], \quad t \in \mathbb{R}_+,$$

and then integrating both sides from t to ∞ , we obtain

$$\ln W(\infty) - \ln W(t) \geq - \int_t^\infty [f(k) + g(k)h(k)]dk,$$

or

$$W(t) \leq \exp \left(\int_t^\infty [f(k) + g(k)h(k)]dk \right), \quad t \in \mathbb{R}_+.$$

Substituting the last inequality into (2.12) and then integrating both sides from t to ∞ , we have

$$V(\infty) - V(t) \geq - \int_t^\infty f(s) \exp \left(\int_s^\infty [f(k) + g(k)h(k)]dk \right) ds,$$

i.e.,

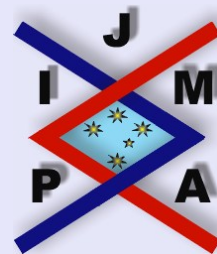
$$V(t) \leq 1 + \int_t^\infty f(s) \exp \left(\int_s^\infty [f(k) + g(k)h(k)]dk \right) ds, \quad t \in \mathbb{R}_+.$$

From inequality (2.10) we obtain

$$x(t) < [n(t) + \varepsilon] \left\{ 1 + \int_t^\infty f(s) \exp \left(\int_s^\infty [f(k) + g(k)h(k)]dk \right) ds \right\}, \quad t \in \mathbb{R}_+.$$

Hence, by letting $\varepsilon \rightarrow 0$ the desired inequality (2.8) follows from the last relation directly. \square

Note that, if $g(t) \equiv 0$ or $h(t) \equiv 0$, then from Theorem 2.3 we derive Theorem A.



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3. Nonlinear Extensions

Theorem 3.1. Let $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy the condition $\int_0^\infty f(s)ds < \infty$ and c is a nonnegative number. Let $\varphi, \psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be strictly increasing and φ^{-1} denote the inverse of φ . If $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ is bounded and satisfies the inequality

$$(3.1) \quad \varphi[x(t)] \leq c + \int_t^\infty f(s)\psi[x(s)]ds, \quad t \in \mathbb{R}_+,$$

then for $t \in (T, \infty)$ we have

$$(3.2) \quad x(t) \leq \varphi^{-1} \circ G_c^{-1} \left(\int_t^\infty f(s)ds \right),$$

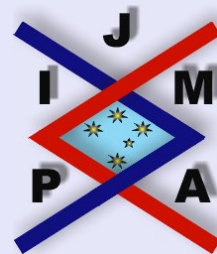
where G_c^{-1} is the inverse of G_c and

$$(3.3) \quad G_c(z) := \int_c^z \frac{ds}{\psi \circ \varphi^{-1}(s)}, \quad z \geq c,$$

and $T > 0$ is the smallest number satisfying the condition

$$(3.4) \quad \int_t^\infty f(s)ds \in \text{Dom}(G^{-1}), \text{ as long as } t \in (T, \infty).$$

Proof. Without loss of generality we may assume $c > 0$. Otherwise we may replace it by an arbitrary positive number ε and then let $\varepsilon \rightarrow 0$ in (3.1) and (3.2) to complete the proof.



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Define a nonincreasing and differentiable function $H \in C(\mathbb{R}_+, [c, \infty))$ by the right member of (3.1), then we have

$$(3.5) \quad x(t) \leq \varphi^{-1}[H(t)], \quad t \in \mathbb{R}_+,$$

and $H(\infty) = c$ holds. By differentiation we obtain

$$\frac{dH(t)}{dt} = -f(t)\psi[x(t)] \geq -f(t)\psi \circ \varphi^{-1}[H(t)], \quad t \in \mathbb{R}_+,$$

where we used inequality (3.5). Rewrite this relation as

$$\frac{dH(t)}{\psi \circ \varphi^{-1}[H(t)]dt} \geq -f(t), \quad t \in \mathbb{R}_+.$$

Integrating both sides from t to ∞ , we derive

$$G_c(H(\infty)) - G_c(H(t)) \geq - \int_t^\infty f(s)ds, \quad t \in \mathbb{R}_+,$$

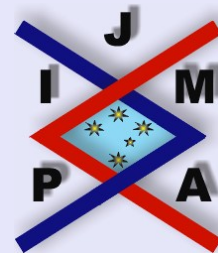
i.e.,

$$G_c(H(t)) \leq G_c(c) + \int_t^\infty f(s)ds, \quad t \in \mathbb{R}_+,$$

where the function G_c is defined by (3.3). Since $G_c(c) = 0$, in view of the choice of T in (3.4), the last relation implies

$$H(t) \leq G_c^{-1} \left(\int_t^\infty f(s)ds \right), \quad t \in (T, \infty).$$

Finally, substituting the last inequality into (3.5), the desired inequality (3.2) follows immediately. \square



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Remark 1. In the case when $c = 0$ and $\varphi(0) = \psi(0) = 0$ hold, to ensure the correct definition of the function $G(z)$, an additional condition is needed, namely,

$$\lim_{\delta \rightarrow 0} \int_{\delta}^1 \frac{ds}{\psi \circ \varphi^{-1}(s)} = M < \infty.$$

Theorem 3.2. Let p, q be positive numbers and $c \in C(\mathbb{R}_+, \mathbb{R}_+)$ be positive and nonincreasing. Let $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy the condition $\int_0^{\infty} f(s)ds < \infty$. If $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ is bounded and satisfies the inequality

$$(3.6) \quad [x(t)]^p \leq c(t) + \int_t^{\infty} f(s)[x(s)]^q ds, \quad t \in \mathbb{R}_+,$$

the following conclusions are true:

(I) If $p > q$,

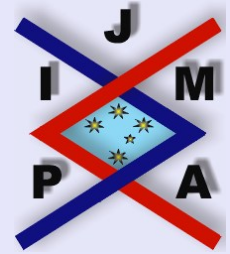
$$(3.7) \quad x(t) \leq c^{1/p}(t) \left[1 + \frac{p-q}{q} \int_t^{\infty} c(s)^{(q-p)/p} f(s) ds \right]^{\frac{1}{p-q}}, \quad t \in \mathbb{R}_+;$$

(II) If $p = q$,

$$(3.8) \quad x(t) \leq c^{1/p}(t) \exp \left[\frac{1}{p} \int_t^{\infty} f(s) ds \right], \quad t \in \mathbb{R}_+;$$

(III) If $p < q$,

$$(3.9) \quad x(t) \leq c^{1/p}(t) \left[1 + \frac{p-q}{p} \int_t^{\infty} c(s)^{(q-p)/p} f(s) ds \right]^{\frac{1}{p-q}}, \quad t \in (T, \infty),$$



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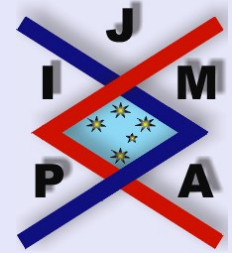


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where T is the smallest non-negative number that satisfies

$$\int_T^\infty c(s)^{(q-p)/p} f(s) ds \leq \frac{p}{q-p}.$$

Proof. (I) If $p > q$ holds, from inequality (3.6) we obtain

$$y^p(t) \leq 1 + \int_t^\infty c(s)^{(q-p)/p} f(s) y^q(s) ds, \quad t \in \mathbb{R}_+,$$

where $y(t) := \frac{x(t)}{c^{1/p}(t)}$. The last integral inequality is a special case of (3.1) when $\varphi(\xi) = \xi^p$, $\psi(\eta) = \eta^q$. By (3.3) we derive

$$G_1(z) = \int_1^z s^{-q/p} ds = \frac{p}{p-q} (z^{(p-q)/p} - 1),$$

and hence,

$$G_1^{-1}(v) = \left[\frac{p-q}{p} v + 1 \right]^{\frac{p}{p-q}}.$$

Since $G_1^{-1}(v) \supset [0, \infty)$ holds, from (3.2) we derive that

$$\begin{aligned} \frac{x(t)}{c^{1/p}(t)} &\leq \varphi^{-1} \circ G_1^{-1} \left[\int_t^\infty c(s)^{(q-p)/p} f(s) ds \right] \\ &= \left\{ G_1^{-1} \left[\int_t^\infty c(s)^{(q-p)/p} f(s) ds \right] \right\}^{\frac{1}{p}} \\ &= \left[1 + \frac{p-q}{p} \int_t^\infty c(s)^{(q-p)/p} f(s) ds \right]^{\frac{1}{p-q}}, \quad t \in \mathbb{R}_+. \end{aligned}$$

The desired inequality (3.7) follows from the last relation directly.

(II) If $p = q$ holds, letting $z(t) = \left[\frac{x(t)}{c^{1/p}(t)} \right]^p$, from (3.6) we derive

$$(3.10) \quad z(t) \leq 1 + \int_t^\infty f(s)z(s)ds, \quad t \in \mathbb{R}_+.$$

Define a positive, nonincreasing and differentiable function $V(t)$ by the right member of (3.10), then $z(t) \leq V(t)$ and $V(\infty) = 1$ hold. Since $c(t), f(t), z(t)$ are nonnegative, by differentiation we obtain from (3.9)

$$V'(t) = -f(t)z(t) \geq -f(t)V(t), \quad t \in \mathbb{R}_+,$$

i.e.,

$$\frac{V'(t)}{V(t)} \geq -f(t), \quad t \in \mathbb{R}_+.$$

Integrating both sides of the last relation from t to ∞ , then we have

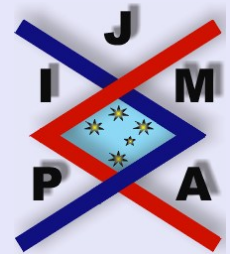
$$\ln V(\infty) - \ln V(t) \geq - \int_t^\infty f(s)ds, \quad t \in \mathbb{R}_+,$$

or

$$\ln V(t) \leq \ln V(\infty) + \int_t^\infty f(s)ds, \quad t \in \mathbb{R}_+.$$

Hence we obtain

$$\left[\frac{x(t)}{c^{1/p}(t)} \right]^p = z(t) \leq V(t) \leq \exp \left(\int_t^\infty f(s)ds \right), \quad t \in \mathbb{R}_+.$$



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This relation implies the desired inequality (3.8) immediately.

(III) If $p < q$ holds, similar to the process of (I), we can get

$$G_1(z) = \frac{p}{p-q}(z^{(p-q)/p} - 1), \quad G_1^{-1}(v) = \left[\frac{p-q}{p}v + 1 \right]^{\frac{p}{p-q}}.$$

Since

$$\int_T^\infty c(s)^{(q-p)/p} f(s) ds = \frac{p}{q-p},$$

we can derive

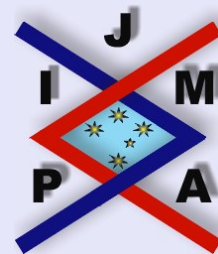
$$(3.11) \quad 1 + \frac{p-q}{p} \int_t^\infty c(s)^{(q-p)/p} f(s) ds > 0, \quad \text{for } t \in (T, \infty).$$

Inequality (3.11) ensures that $G_1^{-1} \left(\int_t^\infty c(s)^{(q-p)/p} f(s) ds \right)$ exists for $t \in (T, \infty)$. Then we get the desired inequality (3.9). \square

Note that, Theorem A is a special case of Theorem 3.2 (II), when $p = q = 1$. Some similar integral inequalities without infinite integration limits had been established by Yang [8, 9].

Corollary 3.3. *Let p, q be positive numbers with $p \leq q$. Let $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy the condition $\int_0^\infty f(s) ds < \infty$. Then $x(t) \equiv 0$ ($t \in \mathbb{R}_+$) is the unique bounded continuous and nonnegative solution of inequality*

$$(3.12) \quad [x(t)]^p \leq \int_t^\infty f(s)[x(s)]^q ds, \quad t \in \mathbb{R}_+.$$



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Proof. Let $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ be any bounded function satisfying (3.12). We obtain

$$(3.13) \quad [x(t)]^p \leq \varepsilon + \int_t^\infty f(s)[x(s)]^q ds, \quad t \in \mathbb{R}_+,$$

where ε is an arbitrary positive number.

When $p < q$ and ε is small enough, the inequality

$$\int_t^\infty \varepsilon^{(q-p)/p} f(s) ds < \frac{p}{q-p}$$

holds for all $t \in \mathbb{R}_+$.

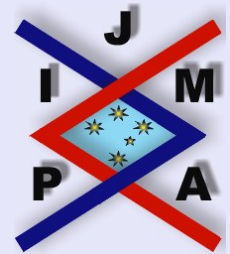
A suitable application of Theorem 3.2 to (3.13) yields that, for $t \in \mathbb{R}_+$

$$x(t) \leq \begin{cases} \varepsilon^{1/p} \left[1 + \frac{p-q}{p} \int_t^\infty \varepsilon^{(q-p)/p} f(s) ds \right]^{\frac{1}{p-q}}, & p < q; \\ \varepsilon^{1/p} \exp \left[\frac{1}{p} \int_t^\infty f(s) ds \right], & p = q. \end{cases}$$

Finally, letting $\varepsilon \rightarrow 0$, from the last relation we obtain $x(t) \equiv 0, t \in \mathbb{R}_+$. \square

If the condition $p \leq q$ is replaced by $p > q$, the result $x(t) \equiv 0$ cannot be derived directly from Theorem 3.2. In fact, if $p > q$ and $M(t) := \int_t^\infty f(s) ds$, then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{1/p} \left[1 + \frac{p-q}{p} \int_t^\infty \varepsilon^{(q-p)/p} f(s) ds \right]^{\frac{1}{p-q}} = \left[\frac{p-q}{p} M(t) \right]^{\frac{1}{p-q}}.$$



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4. Examples

Example 4.1. Let $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ be bounded and satisfy the integral inequality

$$x(t) \geq 1 + \int_t^\infty s e^{-3s} x(s) ds, \quad t \in \mathbb{R}_+.$$

Then by Theorem 2.1, we have

$$x(t) \geq \exp \int_t^\infty s e^{-3s} ds = \exp \left[\frac{3t+1}{9} e^{-3t} \right], \quad t \in \mathbb{R}_+.$$

Example 4.2. Let $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a bounded function satisfying the inequality

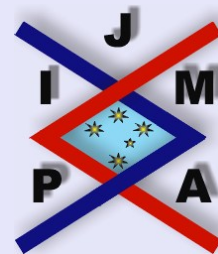
$$x(t) \leq 1 + \int_t^\infty e^{-s} x(s) ds + \int_t^\infty e^{-s} \int_s^\infty (e^{-k} x(k) dk) ds, \quad t \in \mathbb{R}_+.$$

Then by Theorem 2.3, we easily establish

$$\begin{aligned} x(t) &\leq 1 + \int_t^\infty e^{-s} \exp \left[\int_s^\infty 2e^{-k} dk \right] ds \\ &= \frac{1}{2} [1 + \exp(2e^{-t})], \quad t \in \mathbb{R}_+. \end{aligned}$$

Example 4.3. Let $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a bounded function satisfying the inequality

$$x^{1/2}(t) \leq 1 + \int_t^\infty e^{-3s} x(s) ds, \quad t \in \mathbb{R}_+.$$



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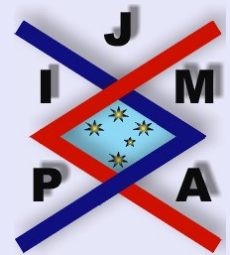
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Since

$$\text{dom} \left(G_1^{-1} \left(\int_t^\infty e^{-3s} ds \right) \right) = \text{dom} \left(\frac{3}{3 - e^{-3t}} \right) \supset \mathbb{R}_+$$

holds, referring to the proof of Theorem 3.2, we obtain

$$x(t) \leq \left[\frac{3}{3 - e^{-3t}} \right]^2, \quad t \in \mathbb{R}_+.$$



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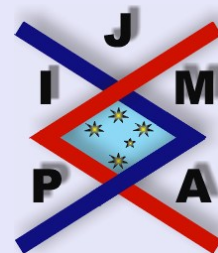
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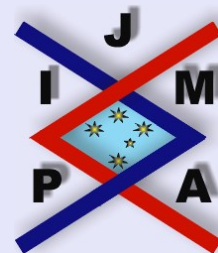
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