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ESTIMATION FOR BOUNDED SOLUTIONS OF INTEGRAL INEQUALITIES INVOLVING INFINITE INTEGRATION LIMITS

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Abstract

Some integral inequalities with infinite integration limits are established as generalizations of a known result due to B.G. Pachpatte.

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Contents

1. Introduction

As well known, various differential and integral inequalities have played a dominant role in the development of the theories of differential, functional-differential as well as integral equations. The most powerful integral inequalities applied frequently in the literature are the famous Gronwall-Bellman inequality [1] and its first nonlinear generalization due to Bihari (cf., [2]). A large number of generalizations and their applications of the Gronwall-Bellman inequality have been obtained by many authors (cf., [4] – [7], [3], [5]). Pachpatte [6, p. 28] proved the following interesting variant of the Gronwall-Bellman inequality which contains an infinite integration limit:

Theorem A. *Let f be a nonnegative continuous function defined for $t \in \mathbb{R}_+ = [0,\infty)$ such that $\int_0^\infty f(s)ds < \infty$ and $c(t) > 0$ be a continuous and decreasing function defined for $t \in \mathbb{R}_+$. If $u(t) \geq 0$ is a bounded continuous function defined for $t \in \mathbb{R}_+$ and satisfies*

$$u(t) \leq c(t) + \int_t^\infty f(s)u(s)ds, \qquad t \in \mathbb{R}_+,$$

then

$$u(t) \leq c(t) \exp\left(\int_t^\infty f(s)ds\right), \qquad t \in \mathbb{R}_+ .$$

We note that, the condition above on $c(t)$ can be relaxed to only require that, it is nonnegative, continuous and nonincreasing on $\mathbb{R}_+$. The importance of the last result was indicated in [6] by the fact that, it can be used to derive the Rodrigues' inequality [8] that played a crucial role in the study of many perturbed linear delay differential equations.

The aim of the present paper is to establish some new linear and nonlinear generalizations of Theorem A. In the sequel, we denote by $C(S, M)$ the class of continuous functions defined on set S with range contained in set M.

2. Linear Generalizations

Firstly we show that an inversed version of Theorem A is valid:

Theorem 2.1. *Let $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy the condition $\int_0^\infty f(s)ds < \infty$ and $m \in C(\mathbb{R}_+, (0, \infty))$ be nondecreasing. If $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ is bounded and satisfies the inequality*

$$x(t) \geq m(t) + \int_t^\infty f(s)x(s)ds, \quad t \in \mathbb{R}_+, \tag{2.1}$$

then

$$x(t) \geq m(t) \exp \int_t^\infty f(s)ds, \quad t \in \mathbb{R}_+. \tag{2.2}$$

Proof. From (2.1) we derive

$$\begin{aligned} \frac{x(t)}{m(t)-\varepsilon} &> 1 + \frac{1}{m(t)-\varepsilon}\int_t^\infty f(s)x(s)ds \\ &\geq 1 + \int_t^\infty f(s)\frac{x(s)}{m(s)-\varepsilon}ds, \qquad t \in \mathbb{R}_+, \end{aligned} \tag{2.3}$$

where $\varepsilon > 0$ is an arbitrary number satisfying $m(0) - \varepsilon > 0$. Define a positive and nonincreasing function $V \in C(\mathbb{R}_+, \mathbb{R}_+)$ by the right member of (2.3). Then we have $V(\infty) = 1$ and

$$x(t) > [m(t) - \varepsilon]V(t), \qquad t \in \mathbb{R}_+. \tag{2.4}$$

By differentiation we obtain

$$\frac{dV(t)}{dt} = -f(t)\frac{x(t)}{m(t)-\varepsilon} < -f(t)V(t), \qquad t \in \mathbb{R}_+.$$

Rewrite the last relation in the form

$$\frac{dV(t)}{V(t)dt} < -f(t), \qquad t \in \mathbb{R}_+,$$

and integrating its both sides from t to ∞,then we have

$$\ln V(\infty) - \ln V(t) < -\int_t^\infty f(s)ds, \qquad t \in \mathbb{R}_+ ,$$

i.e.,

$$V(t) > \exp \int_t^\infty f(s)ds\,, \qquad t \in \mathbb{R}_+ .$$

Substituting the last relation into (2.4), and letting $\varepsilon \to 0$, the desired inequality (2.2) follows. □

From Theorem A and Theorem 2.1, we obtain the following

Corollary 2.2. *Let $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy $\int_0^\infty f(s)ds < \infty$. Let $c \geq 0$ be a constant. Then the linear integral equation*

$$x(t) = c + \int_t^\infty f(s)x(s)ds, \qquad t \in \mathbb{R}_+, \tag{2.5}$$

has an unique bounded continuous solution represented by

$$x(t) = c\exp\int_t^\infty f(s)ds, \qquad t \in \mathbb{R}_+ . \tag{2.6}$$

Proof. If $c > 0$, by letting $c(t) \equiv c$ and $m(t) \equiv c$ respectively in Theorem A and Theorem 2.1, we have $x(t) \leq c \exp \int_t^\infty f(s)ds$ and $x(t) \geq c \exp \int_t^\infty f(s)ds$.

Hence (2.6) is the unique bounded continuous solution of the equation (2.5). By the continuous dependence on c of $x(t)$ given by (2.6), the conclusion holds also when $c = 0$. □

The next result is a new generalization of Pachpatte's inequality in the case when an iterated integral functional is involved.

Theorem 2.3. *Let $n \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nonincreasing. Let $f, h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $g'(t) \geq 0$ and $\int_0^\infty [f(s) + g(s)h(s)]ds < \infty$. If $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ is bounded and satisfies the inequality*

$$x(t) \leq n(t) + \int_t^\infty f(s) \left(x(s) + g(s) \int_s^\infty h(k)x(k)dk \right) ds, \quad t \in \mathbb{R}_+, \tag{2.7}$$

then

$$x(t) \leq n(t) \left\{ 1 + \int_t^\infty f(s) \exp\left(\int_s^\infty [f(k) + g(k)h(k)]dk \right) ds \right\}, \ t \in \mathbb{R}_+. \tag{2.8}$$

Proof. From (2.7) we have

$$\begin{aligned}
&\frac{x(t)}{n(t) + \varepsilon} \\
&< 1 + \frac{1}{n(t) + \varepsilon} \int_t^\infty f(s) \left(x(s) + g(s) \int_s^\infty h(k)x(k)dk \right) ds \\
&< 1 + \int_t^\infty f(s) \left(\frac{x(s)}{n(s) + \varepsilon} + g(s) \int_s^\infty h(k) \frac{x(k)}{n(k) + \varepsilon} dk \right) ds, \ \ t \in \mathbb{R}_+,
\end{aligned} \tag{2.9}$$

where $\varepsilon > 0$ is an arbitrary positive number. Define a function $V \in C(\mathbb{R}_+, \mathbb{R}_+)$ by the right member of inequality (2.9). Then $V(t)$ is positive and nonincreasing with $V(\infty) = 1$, and by (2.9) we have

$$x(t) < [n(t) + \varepsilon]\, V(t), \qquad t \in \mathbb{R}_+. \tag{2.10}$$

By differentiation we obtain

$$\begin{aligned} \frac{dV(t)}{dt} &= -f(t)\left(\frac{x(t)}{n(t)+\varepsilon} + g(t)\int_t^{\infty} h(k)\frac{x(k)}{n(k)+\varepsilon}dk\right) \\ &\geq -f(t)\left(V(t) + g(t)\int_t^{\infty} h(k)V(k)dk\right), \qquad t \in \mathbb{R}_+. \end{aligned}$$

Now we define

$$W(t) = V(t) + g(t)\int_t^{\infty} h(k)V(k)dk.$$

Then $W(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is positive, $W(\infty) = V(\infty) = 1$, and we have

$$W(t) \geq V(t), \qquad t \in \mathbb{R}_+\,, \tag{2.11}$$

and

$$\frac{dV(t)}{dt} \geq -f(t)\, W(t), \qquad t \in \mathbb{R}_+\,. \tag{2.12}$$

By differentiation we derive

$$\begin{aligned} \frac{dW(t)}{dt} &= \frac{dV(t)}{dt} + g'(t)\int_t^{\infty} h(k)V(k)dk - g(t)h(t)V(t) \\ &\geq -[f(t) + g(t)h(t)]\, W(t), \quad t \in \mathbb{R}_+, \end{aligned} \tag{2.13}$$

here (2.11) and (2.12) are used. Rewrite the last relation in the form

$$\frac{dW(t)}{W(t)dt} \geq -[f(t) + g(t)h(t)], \qquad t \in \mathbb{R}_+,$$

and then integrating both sides from t to ∞, we obtain

$$\ln W(\infty) - \ln W(t) \geq -\int_t^\infty [f(k) + g(k)h(k)]dk,$$

or

$$W(t) \leq \exp\left(\int_t^\infty [f(k) + g(k)h(k)]dk\right), \qquad t \in \mathbb{R}_+.$$

Substituting the last inequality into (2.12) and then integrating both sides from t to ∞, we have

$$V(\infty) - V(t) \geq -\int_t^\infty f(s)\exp\left(\int_s^\infty [f(k) + g(k)h(k)]dk\right) ds,$$

i.e.,

$$V(t) \leq 1 + \int_t^\infty f(s)\exp\left(\int_s^\infty [f(k) + g(k)h(k)]dk\right) ds, \qquad t \in \mathbb{R}_+.$$

From inequality (2.10) we obtain

$$x(t) < [n(t)+\varepsilon]\left\{1 + \int_t^\infty f(s)\exp\left(\int_s^\infty [f(k) + g(k)h(k)]dk\right) ds\right\}, \ t \in \mathbb{R}_+.$$

Hence, by letting $\varepsilon \to 0$ the desired inequality (2.8) follows from the last relation directly. □

Note that, if $g(t) \equiv 0$ or $h(t) \equiv 0$, then from Theorem 2.3 we derive Theorem A.

3. Nonlinear Extensions

Theorem 3.1. *Let* $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ *satisfy the condition* $\int_0^\infty f(s)ds < \infty$ *and* c *is a nonnegative number. Let* $\varphi, \psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be strictly increasing and* φ^{-1} *denote the inverse of* φ*. If* $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ *is bounded and satisfies the inequality*

$$\varphi[x(t)] \leq c + \int_t^\infty f(s)\psi[x(s)]ds, \qquad t \in \mathbb{R}_+, \tag{3.1}$$

then for $t \in (T, \infty)$ *we have*

$$x(t) \leq \varphi^{-1} \circ G_c^{-1}\left(\int_t^\infty f(s)ds\right), \tag{3.2}$$

where G_c^{-1} *is the inverse of* G_c *and*

$$G_c(z) := \int_c^z \frac{ds}{\psi \circ \varphi^{-1}(s)}, \qquad z \geq c, \tag{3.3}$$

and $T > 0$ *is the smallest number satisfying the condition*

$$\int_t^\infty f(s)ds \in Dom(G^{-1}), \text{ as long as } t \in (T, \infty). \tag{3.4}$$

Proof. Without loss of generality we may assume $c > 0$. Otherwise we may replace it by an arbitrary positive number ε and then let $\varepsilon \to 0$ in (3.1) and (3.2) to complete the proof.

Define a nonincreasing and differentiable function $H \in C(\mathbb{R}_+, [c, \infty))$ by the right member of (3.1), then we have

$$x(t) \le \varphi^{-1}[H(t)], \qquad t \in \mathbb{R}_+, \tag{3.5}$$

and $H(\infty) = c$ holds. By differentiation we obtain

$$\frac{dH(t)}{dt} = -f(t)\psi[x(t)] \ge -f(t)\psi \circ \varphi^{-1}[H(t)], \qquad t \in \mathbb{R}_+,$$

where we used inequality (3.5). Rewrite this relation as

$$\frac{dH(t)}{\psi \circ \varphi^{-1}[H(t)]dt} \ge -f(t), \qquad t \in \mathbb{R}_+.$$

Integrating both sides from t to ∞, we derive

$$G_c(H(\infty)) - G_c(H(t)) \ge -\int_t^\infty f(s)ds, \qquad t \in \mathbb{R}_+,$$

i.e.,

$$G_c(H(t)) \le G_c(c) + \int_t^\infty f(s)ds, \qquad t \in \mathbb{R}_+,$$

where the function G_c is defined by (3.3). Since $G_c(c) = 0$, in view of the choice of T in (3.4), the last relation implies

$$H(t) \le G_c^{-1}\left(\int_t^\infty f(s)ds\right), \qquad t \in (T, \infty).$$

Finally, substituting the last inequality into (3.5), the desired inequality (3.2) follows immediately. □

Remark 1. *In the case when* $c = 0$ *and* $\varphi(0) = \psi(0) = 0$ *hold, to ensure the correct definition of the function* $G(z)$*, an additional condition is needed, namely,*

$$\lim_{\delta \to 0} \int_{\delta}^{1} \frac{ds}{\psi \circ \varphi^{-1}(s)} = M < \infty.$$

Theorem 3.2. *Let* p, q *be positive numbers and* $c \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be positive and nonincreasing. Let* $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ *satisfy the condition* $\int_0^\infty f(s)ds < \infty$*. If* $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ *is bounded and satisfies the inequality*

$$[x(t)]^p \leq c(t) + \int_t^\infty f(s)[x(s)]^q ds, \qquad t \in \mathbb{R}_+, \tag{3.6}$$

the following conclusions are true:

(I) If $p > q$,

$$x(t) \leq c^{1/p}(t) \left[1 + \frac{p-q}{q} \int_t^\infty c(s)^{(q-p)/p} f(s)ds\right]^{\frac{1}{p-q}}, \quad t \in \mathbb{R}_+; \tag{3.7}$$

(II) If $p = q$,

$$x(t) \leq c^{1/p}(t) \exp\left[\frac{1}{p} \int_t^\infty f(s)ds\right], \quad t \in \mathbb{R}_+; \tag{3.8}$$

(III) If $p < q$,

$$x(t) \leq c^{1/p}(t) \left[1 + \frac{p-q}{p} \int_t^\infty c(s)^{(q-p)/p} f(s)ds\right]^{\frac{1}{p-q}}, \quad t \in (T, \infty), \tag{3.9}$$

where T is the smallest non-negative number that satisfies

$$\int_T^\infty c(s)^{(q-p)/p} f(s)ds \le \frac{p}{q-p}.$$

Proof. (I) If $p > q$ holds, from inequality (3.6) we obtain

$$y^p(t) \le 1 + \int_t^\infty c(s)^{(q-p)/p} f(s) y^q(s) ds, \qquad t \in \mathbb{R}_+,$$

where $y(t) := \frac{x(t)}{c^{1/p}(t)}$. The last integral inequality is a special case of (3.1) when $\varphi(\xi) = \xi^p$, $\psi(\eta) = \eta^q$. By (3.3) we derive

$$G_1(z) = \int_1^z s^{-q/p} ds = \frac{p}{p-q}(z^{(p-q)/p} - 1),$$

and hence,

$$G_1^{-1}(v) = \left[\frac{p-q}{p} v + 1\right]^{\frac{p}{p-q}}.$$

Since $G_1^{-1}(v) \supset [0, \infty)$ holds, from (3.2) we derive that

$$\begin{aligned} \frac{x(t)}{c^{1/p}(t)} &\le \varphi^{-1} \circ G_1^{-1} \left[\int_t^\infty c(s)^{(q-p)/p} f(s) ds\right] \\ &= \left\{G_1^{-1}\left[\int_t^\infty c(s)^{(q-p)/p} f(s) ds\right]\right\}^{\frac{1}{p}} \\ &= \left[1 + \frac{p-q}{p}\int_t^\infty c(s)^{(q-p)/p} f(s) ds\right]^{\frac{1}{p-q}}, \qquad t \in \mathbb{R}_+. \end{aligned}$$

The desired inequality (3.7) follows from the last relation directly.

(II) If $p = q$ holds, letting $z(t) = \left[\frac{x(t)}{c^{1/p}(t)}\right]^p$,from (3.6) we derive

$$z(t) \le 1 + \int_t^\infty f(s)z(s)ds, \qquad t \in R_+. \tag{3.10}$$

Define a positive, nonincreasing and differentiable function $V(t)$ by the right member of (3.10), then $z(t) \le V(t)$ and $V(\infty) = 1$ hold. Since $c(t), f(t), z(t)$ are nonnegative, by differentiation we obtain from (3.9)

$$V'(t) = -f(t)z(t) \ge -f(t)V(t), \quad t \in \mathbb{R}_+,$$

i.e.,

$$\frac{V'(t)}{V(t)} \ge -f(t), \qquad t \in \mathbb{R}_+.$$

Integrating both sides of the last relation from t to ∞, then we have

$$\ln V(\infty) - \ln V(t) \ge -\int_t^\infty f(s)ds, \qquad t \in \mathbb{R}_+,$$

or

$$\ln V(t) \le \ln V(\infty) + \int_t^\infty f(s)ds, \qquad t \in \mathbb{R}_+.$$

Hence we obtain

$$\left[\frac{x(t)}{c^{1/p}(t)}\right]^p = z(t) \le V(t) \le \exp\left(\int_t^\infty f(s)ds\right), \qquad t \in \mathbb{R}_+.$$

This relation implies the desired inequality (3.8) immediately.

(III) If $p < q$ holds, similar to the process of (I), we can get

$$G_1(z) = \frac{p}{p-q}\left(z^{(p-q)/p} - 1\right), \quad G_1^{-1}(v) = \left[\frac{p-q}{p}v + 1\right]^{\frac{p}{p-q}}.$$

Since

$$\int_T^{\infty} c(s)^{(q-p)/p} f(s)ds = \frac{p}{q-p},$$

we can derive

$$1 + \frac{p-q}{p}\int_t^{\infty} c(s)^{(q-p)/p} f(s)ds > 0, \quad \text{for } t \in (T, \infty). \tag{3.11}$$

Inequality (3.11) ensures that $G_1^{-1}\left(\int_t^{\infty} c(s)^{(q-p)/p} f(s)ds\right)$ exists for $t \in (T, \infty)$. Then we get the desired inequality (3.9). □

Note that, Theorem A is a special case of Theorem 3.2 (II), when $p = q = 1$. Some similar integral inequalities without infinite integration limits had been established by Yang [8, 9].

Corollary 3.3. *Let p, q be positive numbers with $p \leq q$. Let $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy the condition $\int_0^{\infty} f(s)ds < \infty$. Then $x(t) \equiv 0\,(t \in \mathbb{R}_+)$ is the unique bounded continuous and nonnegative solution of inequality*

$$[x(t)]^p \leq \int_t^{\infty} f(s)[x(s)]^q ds, \quad t \in \mathbb{R}_+. \tag{3.12}$$

Proof. Let $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ be any bounded function satisfying (3.12). We obtain

$$[x(t)]^p \leq \varepsilon + \int_t^\infty f(s)[x(s)]^q ds, \quad t \in \mathbb{R}_+, \tag{3.13}$$

where ε is an arbitrary positive number.

When $p < q$ and ε is small enough, the inequality

$$\int_t^\infty \varepsilon^{(q-p)/p} f(s)ds < \frac{p}{q-p}$$

holds for all $t \in \mathbb{R}_+$.

A suitable application of Theorem 3.2 to (3.13) yields that, for $t \in \mathbb{R}_+$

$$x(t) \leq \begin{cases} \varepsilon^{1/p}\left[1 + \frac{p-q}{p}\int_t^\infty \varepsilon^{(q-p)/p} f(s)ds\right]^{\frac{1}{p-q}}, & p < q; \\ \\ \varepsilon^{1/p} \exp\left[\frac{1}{p}\int_t^\infty f(s)ds\right], & p = q. \end{cases}$$

Finally, letting $\varepsilon \to 0$, from the last relation we obtain $x(t) \equiv 0$, $t \in \mathbb{R}_+$. □

If the condition $p \leq q$ is replaced by $p > q$, the result $x(t) \equiv 0$ cannot be derived directly from Theorem 3.2. In fact, if $p > q$ and $M(t) := \int_t^\infty f(s)ds$, then

$$\lim_{\varepsilon \to 0} \varepsilon^{1/p}\left[1 + \frac{p-q}{p}\int_t^\infty \varepsilon^{(q-p)/p} f(s)ds\right]^{\frac{1}{p-q}} = \left[\frac{p-q}{p}M(t)\right]^{\frac{1}{p-q}}.$$

4. Examples

Example 4.1. *Let $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ be bounded and satisfy the integral inequality*

$$x(t) \geq 1 + \int_t^\infty s\, e^{-3s} x(s) ds, \quad t \in \mathbb{R}_+.$$

Then by Theorem 2.1, we have

$$x(t) \geq \exp \int_t^\infty s e^{-3s} ds = \exp\left[\frac{3t+1}{9} e^{-3t}\right], \quad t \in \mathbb{R}_+.$$

Example 4.2. *Let $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a bounded function satisfying the inequality*

$$x(t) \leq 1 + \int_t^\infty e^{-s} x(s) ds + \int_t^\infty e^{-s} \int_s^\infty \left(e^{-k} x(k) dk\right) ds, \quad t \in \mathbb{R}_+.$$

Then by Theorem 2.3 ,we easily establish

$$\begin{aligned} x(t) &\leq 1 + \int_t^\infty e^{-s} \exp\left[\int_s^\infty 2e^{-k} dk\right] ds \\ &= \frac{1}{2}\left[1 + \exp\left(2e^{-t}\right)\right], \qquad t \in \mathbb{R}_+. \end{aligned}$$

Example 4.3. *Let $x \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a bounded function satisfying the inequality*

$$x^{1/2}(t) \leq 1 + \int_t^\infty e^{-3s} x(s) ds, \quad t \in \mathbb{R}_+.$$

Since

$$\operatorname{dom}\left(G_1^{-1}\left(\int_t^{\infty} e^{-3s} ds\right)\right) = \operatorname{dom}\left(\frac{3}{3-e^{-3t}}\right) \supset \mathbb{R}_+$$

holds, referring to the proof of Theorem 3.2, we obtain

$$x(t) \le \left[\frac{3}{3-e^{-3t}}\right]^2, \quad t \in \mathbb{R}_+.$$

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