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MONOTONE TRAJECTORIES OF DYNAMICAL SYSTEMS AND CLARKE'S GENERALIZED JACOBIAN

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ABSTRACT. We generalize some results due to Pappalardo and Passacantando [10]. We prove necessary and sufficient conditions for the monotonicity of a trajectory of an autonomous dynamical system with locally Lipschitz data, by means of Clarke's generalized Jacobian. Some of the results are developed in the framework of variational inequalities.

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1. INTRODUCTION

Existence of solutions to a dynamical system has been variously investigated (see e.g. [8]). Recently, in [10] the authors prove, in the framework of variational inequalities, necessary and sufficient conditions for the existence of monotone trajectories of the autonomous dynamical system

$$x'(t) = -F\left(x\left(t\right)\right)$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1. However, existence and uniqueness of solutions of the latter problem are known to hold under weaker assumptions on F. Namely, in [8] one can find local Lipschitzianity of F is sufficient.

Here we propose a generalization of Theorems 2.2. and 2.5 in [10] to the case where F is locally Lipschitz. We develop necessary and sufficient conditions to have monotone trajectories

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of the autonomous (projected) dynamical system, expressed in terms of Clarke's generalized Jacobian [3]. The main results are proved in Section 3, while Section 2 is devoted to preliminary results and definitions.

2. PRELIMINARIES

Throughout the paper we make use of some relations between differential inclusions and variational inequalities. For the sake of completeness, we recall some of them together with the standard notation. We shall consider a convex and closed feasible region $K \subset \mathbb{R}^n$ and an upper semi-continuous (u.s.c.) map F from $\mathbb{R}^n$ to $2^{\mathbb{R}^n}$, with nonempty convex and compact values.

2.1. Differential Inclusions.

We start by recalling from [1] the following result about projection:

Theorem 2.1. *We can associate to every $x \in \mathbb{R}^n$ a unique element $\pi_K(x) \in K$, satisfying:*

$$\|x - \pi_K(x)\| = \min_{y \in K} \|x - y\|.$$

It is characterized by the following inequality:

$$\langle \pi_K(x) - x, \pi_K(x) - y \rangle \leq 0, \quad \forall y \in K.$$

Furthermore the map $\pi_K(\cdot)$ is non expansive, i.e.:

$$\|\pi_K(x) - \pi_K(y)\| \leq \|x - y\|.$$

The map π_K is said to be the projector (of best approximation) onto K. When K is a linear subspace, then π_K is linear (see [1]). For our aims, we set also:

$$\pi_K(A) = \bigcup_{x \in A} \pi_K(x).$$

The following notation should be common:

$$C^- = \{v \in \mathbb{R}^n : \langle v, a \rangle \leq 0, \forall a \in C\}$$

is the (negative) polar cone of the set $C \subseteq \mathbb{R}^n$, while:

$$T(C, x) = \{v \in \mathbb{R}^n : \exists v_n \to v,\ \alpha_n > 0,\ \alpha_n \to 0,\ x + \alpha_n v_n \in C\}$$

is the Bouligand tangent cone to the set C at $x \in \mathrm{cl}C$ and $N(C, x) = [T(C, x)]^-$ stands for the normal cone to C at $x \in \mathrm{cl}C$.

It is known that $T(C, x)$ and $N(C, x)$ are closed sets and $N(C, x)$ is convex. Furthermore, when we consider a closed convex set $K \subseteq \mathbb{R}^n$, then $T(K, x) = \mathrm{cl}\,\mathrm{cone}\,(K - x)$ ($\mathrm{cone}\,A$ denotes the cone generated by the set A), so that also the tangent cone is convex.

Given a map $G : K \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$, a differential inclusion is the problem of finding an absolutely continuous function $x(\cdot)$, defined on an interval $[0, T]$, such that:

$$\begin{cases} \forall t \in [0, T], & x(t) \in K, \\ \text{for a.a. } t \in [0, T], & x'(t) \in G(x(t)). \end{cases}$$

The solutions of the previous problem are also called *trajectories* of the differential inclusion.

We are concerned with the following problem, which is a special case of differential inclusion.

Problem 2.1. Find an absolutely continuous function $x(\cdot)$ from $[0, T]$ into $\mathbb{R}^n$, satisfying:

$$(PDI(F, K)) \qquad \begin{cases} \forall t \in [0, T], & x(t) \in K, \\ \text{for a.a. } t \in [0, T], & x'(t) \in \pi_{T(K, x(t))}\left(-F(x(t)\right), \end{cases}$$

The previous problem is usually named “projected differential inclusion” (for short, PDI).

Theorem 2.2. *The solutions of Problem 2.1 are the solutions of the “differential variational inequality” (DVI):*

$$(DVI(F,K))\qquad \begin{cases} \forall t \in [0,T], & x(t) \in K, \\ \textit{for a.a. } t \in [0,T], & x'(t) \in -F(x(t)) - N(K, x(t)) \end{cases}$$

and conversely.

Remark 2.3. We recall that when F is a single-valued operator, then the corresponding “projected differential equation” and its applications have been studied for instance in [5, 9, 10].

Definition 2.1. A point $x^* \in K$ is an equilibrium point for $PDI(F,K)$, when:

$$0 \in -F(x^*) - N(K, x^*).$$

In our main results we make use of the monotonicity of a trajectory of $PDI(F,K)$, as stated in [1].

Definition 2.2. Let V be a function from K to $\mathbb{R}^+$. A trajectory $x(t)$ of $PDI(F,K)$ is monotone (with respect to V) when:

$$\forall t \geq s, \quad V(x(t)) - V(x(s)) \leq 0.$$

If the previous inequality holds strictly $\forall t > s$, then we say that $x(t)$ is strictly monotone w.r.t. V.

We apply the previous definition to the function:

$$\tilde{V}_{x^*}(x) = \frac{\|x - x^*\|^2}{2},$$

where x^* is an equilibrium point of $PDI(F,K)$.

2.2. **Variational Inequalities.**

Definition 2.3. A point $x^* \in K$ is a solution of a Strong Minty Variational Inequality (for short, $SMVI$), when:

$$(SMVI(F,K))\qquad \langle \xi, y - x^* \rangle \geq 0, \quad \forall y \in K,\ \forall \xi \in F(y).$$

Definition 2.4. A point $x^* \in K$ is a solution of a Weak Minty Variational Inequality (for short, $WMVI$), when $\forall y \in K$, $\exists \xi \in F(y)$ such that:

$$(WMVI(F,K))\qquad \langle \xi, y - x^* \rangle \geq 0.$$

Definition 2.5. If in Definition 2.3 (resp. 2.4), strict inequality holds $\forall y \in K$, $y \neq x^*$, then we say that x^* is a “strict” solution of $SMVI(F,K)$ (resp. of $WMVI(F,K)$).

Remark 2.4. When F is single valued, Definitions 2.3 and 2.4 reduce to the classical notion of (MVI).

The following results relate the monotonicity of trajectories of $PDI(F,K)$ w.r.t. $\tilde{V}_{x^*}$ to solutions of Minty Variational Inequalities.

Definition 2.6. A set valued map $F : \mathbb{R}^n \rightrightarrows 2^{\mathbb{R}^n}$ is said to be upper semicontinuous (u.s.c.) at $x_0 \in \mathbb{R}^n$, when for every open set N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M) \subseteq N$.
F is said to be u.s.c. when it is so at every $x_0 \in \mathbb{R}^n$.

Theorem 2.5 ([4]). *If $x^* \in K$ is a solution of $SMVI(F,K)$, where F is u.s.c. with nonempty convex compact values, then every trajectory $x(t)$ of $PDI(F,K)$ is monotone w.r.t. function $\tilde{V}_{x^*}$.*

Theorem 2.6 ([4]). *Let x^* be an equilibrium point of $PDI(F,K)$. If for any point $x \in K$ there exists a trajectory of $PDI(F,K)$ starting at x and monotone w.r.t. function $\tilde{V}_{x^*}$, then x^* solves $WMVI(F,K)$.*

Proposition 2.7 ([4]). *Let x^* be a strict solution of $SMVI(F,K)$, then:*

i) *x^* is the unique equilibrium point of $PDI(F,K)$;*
ii) *every trajectory of $PDI(F,K)$, starting at a point $x_0 \in K$ and defined on $[0,+\infty)$ is strictly monotone w.r.t. $\tilde{V}_{x^*}$ and converges to x^*.*

Example 2.1. Let $K = \mathbb{R}^2$ and consider the system of autonomous differential equations:

$$x'(t) = -F(x(t)),$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a single-valued map defined as:

$$F(x,y) = \begin{bmatrix} -y + x|1-x^2-y^2| \\ x + y|1-x^2-y^2| \end{bmatrix}.$$

Clearly $(x^*, y^*) = (0,0)$ is an equilibrium point and one has $\langle F(x,y),(x,y)\rangle \geq 0 \,\forall (x,y) \in \mathbb{R}^2$, so that $(0,0)$ is a solution of $GMVI(F,K)$ and hence, according to Theorem 2.5, every solution $x(t)$ of the considered system of differential equations is monotone w.r.t. $\tilde{V}_{x^*}$. Anyway, not all the solutions of the system converge to $(0,0)$. In fact, passing to polar coordinates, the system can be written as:

$$\begin{cases} \rho'(t) = -\rho(t)|1-\rho^2(t)| \\ \theta'(t) = -1 \end{cases}$$

and solving the system, one can easily see that the solutions that start at a point (ρ,θ), with $\rho \geq 1$ do not converge to $(0,0)$, while the solutions that start at a point (ρ,θ) with $\rho < 1$ converge to $(0,0)$. This last fact can be checked on observing that for every $c < 1$, $(0,0)$ is a strict solution of $SMVI(F,K_c)$ where:

$$K_c := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < c\}.$$

Proposition 2.7 is useful in the proof of necessary and sufficient conditions for the existence of monotone trajectories of $DS(F)$, expressed by means of Clarke's generalized Jacobian [3].

Definition 2.7. Let G be a locally Lipschitz function from K to $\mathbb{R}^m$. Clarke's generalized Jacobian of G at x is the subset of the space $\mathbb{R}^{n\times m}$ of $n \times m$ matrices, defined as:

$$J_C G(x) = \operatorname{conv}\{\lim JG(x_k) : x_k \to x,\ G \text{ is differentiable at } x_k\}$$

(here JG denotes the Jacobian of G and $\operatorname{conv} A$ stands for the convex hull of the set $A \subseteq \mathbb{R}^n$).

The following proposition summarizes the main properties of the generalized Jacobian.

Proposition 2.8.

i) *$J_C F(x)$ is a nonempty, convex and compact subset of $\mathbb{R}^{n\times m}$;*
ii) *the map $x \to J_C F(x)$ is u.s.c.;*
iii) *(Mean value Theorem) For all $x,y \in K$ we have $F(y) - F(x) \in \operatorname{conv}\{J_C F(x+\delta(y-x))(y-x),\ \delta \in [0,1]\}$.*

Definition 2.8. Let $G(\cdot)$ be a map from $\mathbb{R}^n$ into the subsets of the space $\mathbb{R}^{n\times n}$ of $n \times n$ matrices. We say that $G(\cdot)$ is positively defined at x (respectively weakly positively defined) on K when:

$$\inf_{G\in G(x)} u^\top G u \geq 0, \quad \forall u \in T(K,x)$$

$$\left(\sup_{G\in G(x)} u^\top G u \geq 0, \quad \forall u \in T(K,x)\right)$$

If the inequality is strict (for $u \neq 0$), we say that $G(x)$ is strictly positively defined (resp. strictly weakly positively defined).

3. MAIN RESULTS

Theorem 3.1. *Let $F : K \to \mathbb{R}^n$ be locally Lipschitz and let x^* be an equilibrium point of $PDI(F,K)$. If there exists a positive number δ such that for any $x_0 \in K$ with $\|x_0 - x^*\| < \delta$, there exists a trajectory $x(t)$ of $PDI(F,K)$ starting at x_0 and monotone w.r.t. $\tilde{V}_{x^*}$, then Clarke's generalized Jacobian of F at x^* is weakly positively defined on K.*

Proof. Let $B(x^*,\delta)$ be the open ball with center in x^* and radius δ. Fix $z \in B(x^*,\delta)\cap K$ and let $y(\alpha) = x^* + \alpha(z - x^*)$, for $\alpha \in [0,1]$ (clearly $y(\alpha) \in B(x^*,\delta)\cap K$). Let $x(t)$ be a trajectory of $PDI(F,K)$ starting at $y(\alpha)$; for $v(t) = \tilde{V}_{x^*}(x(t))$, we have:

$$0 \geq v'(0) = \langle x'(0), y(\alpha) - x^*\rangle,$$

and:

$$x'(0) = -F(y(\alpha)) - n, \quad n \in N(K, y(\alpha))$$

so that:

$$\langle F(y(\alpha)), y(\alpha) - x^*\rangle \geq -\langle n, y(\alpha) - x^*\rangle \geq 0.$$

Now, applying the mean value theorem, since x^* is an equilibrium of $PDI(F,K)$, we get, for some $n^*(\alpha) \in N(K,x^*)$:

$$\begin{aligned} F(y(\alpha)) + n^*(\alpha) &= F(y(\alpha)) - F(x^*) \\ &\in \operatorname{conv}\big\{\alpha J_C F(x^* + \rho(z - x^*))(z - x^*),\ \delta \in [0,\alpha]\big\} = A(\alpha). \end{aligned}$$

Since $J_C F(\cdot)$ is u.s.c., $\forall\varepsilon > 0$ and for ρ "small enough", say $\rho \in [0,\beta(\epsilon)]$ we have:

$$J_C F(x^* + \rho(z - x^*)) \subseteq J_C F(x^*) + \varepsilon B := J_\varepsilon F(x^*)$$

(here B denotes the open unit ball in $\mathbb{R}^{n\times n}$). So, it follows, for $\alpha = \beta(\varepsilon)$:

$$A(\beta(\varepsilon)) \subseteq \beta(\varepsilon) J_\varepsilon F(x^*)(z - x^*),$$

and hence, for any $\varepsilon > 0$, $F(y(\beta(\varepsilon))) \in \beta(\varepsilon) J_\varepsilon F(x^*)(z - x^*)$.

Now, let $\varepsilon_n = 1/n$ and $\alpha_n = \beta(\varepsilon_n)$. We have $\langle F(y(\alpha_n)) + n^*(\alpha_n), y(\alpha_n) - x^*\rangle \geq 0$, that is:

$$\alpha_n^2 (z - x^*)^\top (d(\alpha_n) + \gamma(\alpha_n))(z - x^*) \geq 0,$$

with $\gamma(\alpha_n) \in \frac{1}{n}B$ and $d(\alpha_n) \in J_C F(x^*)$. So we obtain:

$$(z - x^*)^\top d(\alpha_n)(z - x^*) \geq -(z - x^*)^\top \gamma(\alpha_n)(z - x^*) = -\frac{1}{n}(z - x^*) b_n (z - x^*),$$

with $b_n \in B$. Sending n to $+\infty$ we can can assume $d(\alpha_n) \to d \in J_C F(x^*)$ while the right side converges to 0 and we get:

$$(z - x^*)^\top d\,(z - x^*) \geq 0.$$

Since z is arbitrary in $B(x^*,\delta)\cap K$.
Hence

$$\sup_{A\in J_C F(x^*)} (z-x^*)^\top A(z-x^*) \geq 0 \quad \forall z \in B(x^*,\delta)\cap K.$$

Now let $y=\lim \lambda_n(z_n-x^*)$, $z_n\in B(x^*,\delta)\cap K$ be some element in $T(K,x^*)$. We have

$$\sup_{A\in J_C F(x^*)} y^\top Ay \geq 0$$

and

$$\sup_{A\in J_C F(x^*)} y^\top Ay \geq 0 \quad \forall y\in T(K,x^*).$$

that is, $J_C F(x^*)$ is weakly positive defined on K. □

Example 3.1. The condition of the previous theorem is necessary but not sufficient for the existence of monotone trajectories (w.r.t. $\tilde{V}$). Consider the locally Lipschitz function $F:\mathbb{R}\to\mathbb{R}$ defined as:

$$F(x)=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

and the autonomous differential equation $x'(t)=-F(x(t))$. Clearly $x^*=0$ is an equilibrium point and it is known that $J_C F(0)=[-1,1]$. Hence the necessary condition of Theorem 3.1 is satisfied, but it is easily seen that any trajectory $x(t)$ of the considered differential equation (apart from the trivial solution $x(t)\equiv 0$) is not monotone w.r.t. $\tilde{V}_{x^*}$.

Theorem 3.2. *Assume that $J_C F(x^*)$ is strictly positively defined. Then, every trajectory $x(t)$ of $PDI(F)$ starting "sufficiently near" x^* and defined on $[0,+\infty)$ is strictly monotone w.r.t. $\tilde{V}_{x^*}$ and converges to x^*.*

Proof. By assumption:

$$\inf_{A\in J_C F(x^*)} u^\top Au > 0, \quad \forall u\in T(K,x^*)\setminus\{0\},$$

and this condition is equivalent to the existence of a positive number m such that $\inf_{A\in J_C F(x^*)} v^\top Av > m,\ \ \forall v\in S^1\cap\left(T(K,x^*)\setminus\{0\}\right)$ (where S^1 is the unit sphere in $\mathbb{R}^n$). Indeed, if this is not the case, there would exist some sequence $\{v_n\}\in S^1$, converging to some $v\in S^1$, such that:

$$\inf_{A\in J_C F(x^*)} v_n^\top Av_n \leq \frac{1}{n}$$

by compactness of $J_C F(x^*)$, we would have, for every n some $A_n\in J_C F(x^*)$ such that:

$$\inf_{A\in J_C F(x^*)} v_n^\top Av_n = v_n^\top A_n v_n$$

and $A_n\to\bar{A}\in J_C F(x^*)$. Therefore we have $v_n^\top A_n v_n\to v^\top\bar{A}v\leq 0$ for $n\to+\infty$ and the contradiction

$$\inf_{A\in J_C F(x^*)} u^\top Au \leq 0.$$

Let $\varepsilon>0$ and consider the set:

$$J_\varepsilon F(x^*) := J_C F(x^*) + \varepsilon B.$$

We claim:

$$\inf_{A\in J_\varepsilon F(x^*)} u^\top Au > 0, \quad \forall u\in T(K,x^*)\setminus\{0\},$$

for ε "small enough". Indeed, $A \in J_\varepsilon F(x^*)$ if and only if $A = A' + A''$, with $A' \in J_C F(x^*)$ and $A'' \in \varepsilon B$ and hence, for $u \in \mathbb{R}^n \backslash \{0\}$:

$$\inf_{A \in J_\varepsilon F(x^*)} u^\top A u \geq \inf_{A' \in J_C F(x^*)} u^\top A' u + \inf_{A'' \in \varepsilon B} u^\top A'' u.$$

Since $A'' \in \varepsilon B$, we have $|u^\top A'' u| \leq \|A''\| \|u\|^2 \leq \varepsilon \|u\|^2$ and we get:

$$\inf_{A' \in J_C F(x^*)} u^\top A' u + \inf_{A'' \in \varepsilon B} u^\top A'' u \geq \inf_{A' \in J_C F(x^*)} u^\top A' u - \varepsilon \|u\|^2.$$

Therefore:

$$\inf_{A \in J_\varepsilon F(x^*)} \frac{u^\top A u}{\|u\|^2} \geq \inf_{A' \in J_C F(x^*)} \frac{u^\top A' u}{\|u\|^2} - \varepsilon$$

and for $\varepsilon < m$, the right-hand side is positive.

If we fix ε in $(0, m)$, for a suitable $\delta > 0$ we have, for all $x \in B(x^*, \delta) \cap K$:

$$J_C F(x^* + \alpha(x - x^*)) \subseteq J_\varepsilon F(x^*), \quad \forall \alpha \in (0, 1)$$

and from the mean value theorem and the convexity of the generalized Jacobian, we obtain, for some $n^* \in N(K, x^*)$:

$$\begin{aligned} F(x) + n^* &= F(x) - F(x^*) \\ &\in \operatorname{conv}\left\{ J_C F(x^* + \delta(x - x^*))(x - x^*),\ \delta \in [0, 1] \right\} \\ &\subseteq J_\varepsilon F(x^*)(x - x^*). \end{aligned}$$

Hence we conclude:

$$\langle F(x), x - x^* \rangle > 0, \quad \forall x \in (K \cap B(x^*, \delta)) \backslash \backslash \left\{ x^* \right\}$$

and so x^* is a strict solution of $SMVI(F, \mathbb{R}^n \cap \bar{B}(x^*, \delta))$. The proof now follows from Proposition 2.7. $\square$

Example 3.2. The condition of the previous theorem is sufficient but not necessary for the monotonicity of trajectories. Consider the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ defined as:

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x} + ax, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where $0 < a < 1$, and the autonomous differential equation $x'(t) = -F(x(t))$, for which $x^* = 0$ is an equilibrium point. In a suitable neighborhood U of 0 we have $F(x) > 0$ if $x > 0$, while $F(x) < 0$, if $x < 0$ and hence it is easily seen that every solution of the considered differential equation, starting "near" 0, is strictly monotone w.r.t. $\tilde{V}_{x^*}$ and converges to 0. If we calculate the generalized Jacobian of F at 0 we get $J_C F(0) = [-1 + a, 1 + a]$ and the sufficient condition of the previous theorem is not satisfied.

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