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INEQUALITIES ASSOCIATING HYPERGEOMETRIC FUNCTIONS WITH PLANER HARMONIC MAPPINGS

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ABSTRACT. Though connections between a well established theory of analytic univalent functions and hypergeometric functions have been investigated by several researchers, yet analogous connections between planer harmonic mappings and hypergeometric functions have not been explored. The purpose of this paper is to uncover some of the inequalities associating hypergeometric functions with planer harmonic mappings.

Key words and phrases: Planer harmonic mappings, Hypergeometric functions, Convolution multipliers, Harmonic starlike, Harmonic convex, Inequalities.

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1. INTRODUCTION

Let H be the class consisting of continuous complex-valued functions which are harmonic in the unit disk $\Delta = \{z : |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in Δ. Clunie and Sheil-Small in [1] developed the basic theory of planer harmonic mappings $f \in H$ which are univalent in Δ and have the normalization $f(0) = 0 = f_z(0) - 1$. Such functions, also known as planer mappings, may be written as $f = h + \overline{g}$, where $h, g \in \mathrm{A}$. A function $f \in H$ is said to be locally univalent and sense-preserving if the Jacobian $J(f) = |h'|^2 - |g'|^2$ is positive in Δ; or equivalently $|g'(z)| < |h'(z)|$ $(z \in \Delta)$. Thus for $f = h + \overline{g} \in H$

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we may write

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad g(z) = \sum_{n=1}^{\infty} B_n z^n, \quad |B_1| < 1. \tag{1.1}$$

Let $S_H$denote the family of functions $h+\overline{g}$ which are harmonic, univalent, and sense-preserving in Δ where $h, g \in \mathrm{A}$ and are of the form (1.1). Imposing the additional normalization condition $f_{\overline{z}}(0) = 0$, Clunie and Sheil-Small [1] distinguished the class S_H^0 from S_H. Both the families S_H and S_H^0 are normal families. But, S_H^0 is the only compact family with respect to the topology of locally uniform convergence [1].

Let S_H^* and K_H be the subclasses of S_H consisting of functions f which map Δ, respectively, onto starlike and convex domains. If $f_j = h_j + \overline{g_j}$, $j = 1, 2$ are in the class S_H (or S_H^0), then we define the convolution $f_1 * f_2$ of f_1 and $f_2$in the natural way $h_1 * h_2 + \overline{g_1 * g_2}$. If ϕ_1 and ϕ_2 are analytic and $f = h + \overline{g}$ is in S_H, we define

$$f \widetilde{*} (\phi_1 + \overline{\phi}_2) = h * \phi_1 + \overline{g * \phi_2}. \tag{1.2}$$

Let a, b, c be complex numbers with $c \neq 0, -1, -2, -3, \ldots$. Then the Gauss hypergeometric function written as ${}_2F_1(a, b; c; z)$ or simply as $F(a, b; c; z)$ is defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{1.3}$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \lambda(\lambda+1)\cdots(\lambda+n-1) \text{ for } n = 1, 2, 3, \ldots \quad \text{and } (\lambda)_0 = 1. \tag{1.4}$$

Since the hypergeometric series in (1.3) converges absolutely in Δ, it follows that $F(a, b; c; z)$ defines a function which is analytic in Δ, provided that c is neither zero nor a negative integer. As a matter of fact, in terms of Gamma functions, we are led to the well-known Gauss's summation theorem: If $\mathrm{Re}(c - a - b) > 0$, then

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \ c \neq 0, -1, -2, \ldots. \tag{1.5}$$

In particular, the incomplete beta function, related to the Gauss hypergeometric function, $\varphi(a, c; z)$, is defined by

$$\varphi(a, c; z) := zF(a, 1; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \ z \in \Delta, \ c \neq 0, \ -1, \ -2, \ldots. \tag{1.6}$$

It has an analytic continuation to the z-plane cut along the positive real axis from 1 to ∞. Note that $\varphi(a, 1; z) = \frac{z}{(1-z)^a}$. Moreover, $\varphi(2, 1; z) = \frac{z}{(1-z)^2}$ is the Koebe function.

The hypergeometric series in (1.3) and (1.6) converge absolutely in Δ and thus $F(a, b; c; z)$ and $\varphi(a, c; z)$ are analytic functions in Δ, provided that c is neither zero nor a negative integer. For further information about hypergeometric functions, one may refer to [2], [6], and [11].

Throughout this paper, let $G(z) := \phi_1(z) + \overline{\phi_2(z)}$ be a function where $\phi_1(z) \equiv \phi_1(a_1, b_1; c_1; z)$ and $\phi_2(z) \equiv \phi_2(a_2, b_2; c_2; z)$ are the hypergeometric functions defined by

$$\phi_1(z) := zF(a_1, b_1; c_1; z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^n, \tag{1.7}$$

$$\phi_2(z) := zF(a_2, b_2; c_2; z) - 1 = \sum_{n=1}^{\infty} \frac{(a_2)_n (b_2)_n}{(c_2)_n (1)_n} z^n, \quad a_2 b_2 < c_2. \tag{1.8}$$

It was surprising to discover the use of hypergeometric functions in the proof of the Bieberbach conjecture by L. de Branges [3] in 1985. This discovery has prompted renewed interests in these classes of functions. For example, see [7], [8], and [9].

However, connections between the theory of harmonic univalent functions and hypergeometric functions have not yet been explored. The purpose of this paper is to uncover some of the connections. In particular, we will investigate the convolution multipliers $f\widetilde{*}(\phi_1 + \overline{\phi_2})$, where ϕ_1, ϕ_2 are as defined by (1.7) and (1.8) and f is a harmonic starlike univalent (or harmonic convex univalent) function in Δ.

2. MAIN RESULTS

We need the following sufficient condition.

Lemma 2.1 ([4, 10]). *For $f = h + \overline{g}$ with h and g of the form (1.1), if*

$$\sum_{n=2}^{\infty} n\,|A_n| + \sum_{n=1}^{\infty} n\,|B_n| \le 1, \tag{2.1}$$

then $f \in S_H^*$.

Theorem 2.2. *If $a_j, b_j > 0$, $c_j > a_j + b_j + 1$ for $j = 1, 2,$, then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ to be harmonic univalent in Δ and $G \in S_H^*$, is that*

$$\left(1 + \frac{a_1 b_1}{c_1 - a_1 - b_1 - 1}\right) F(a_1, b_1; c_1; 1) + \frac{a_2 b_2}{c_2 - a_2 - b_2 - 1} F(a_2, b_2; c_2; 1) \le 2. \tag{2.2}$$

Proof. In order to prove that G is locally univalent and sense-preserving in Δ, we only need to show that $|\phi_1'(z)| > |\phi_2'(z)|\,,\, z \in \Delta$. In view of (1.7), (1.3), (1.4) and (1.5) we have

$$\begin{aligned}
|\phi_1'(z)| &= \left| 1 + \sum_{n=2}^{\infty} n \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^{n-1} \right| \\
&> 1 - \sum_{n=2}^{\infty} (n-1) \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} \\
&= 1 - \frac{a_1 b_1}{c_1} \sum_{n=1}^{\infty} \frac{(a_1+1)_{n-1}(b_1+1)_{n-1}}{(c_1+1)_{n-1}(1)_{n-1}} - \sum_{n=1}^{\infty} \frac{(a_1)_n (b_1)_n}{(c_1)_n (1)_n} \\
&= 2 - \frac{a_1 b_1}{c_1} \cdot \frac{\Gamma(c_1+1)\Gamma(c_1 - a_1 - b_1 - 1)}{\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)} - \frac{\Gamma(c_1)\Gamma(c_1 - a_1 - b_1)}{\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)} \\
&= 2 - \left(\frac{a_1 b_1}{c_1 - a_1 - b_1 - 1} + 1 \right) F(a_1, b_1; c_1; 1).
\end{aligned}$$

Again, using (2.2), (1.5), (1.3), and (1.8) in turn, to the above mentioned inequality, we have

$$\begin{aligned}|\phi_1'(z)| &\geq \frac{a_2b_2}{c_2-a_2-b_2-1}F(a_2,b_2;c_2;1)\\ &= \frac{a_2b_2}{c_2}\frac{\Gamma(c_2+1)\Gamma(c_2-a_2-b_2-1)}{\Gamma(c_2-a_2)\Gamma(c_2-b_2)}\\ &= \sum_{n=0}^{\infty}\frac{(a_2)_{n+1}(b_2)_{n+1}}{(c_2)_{n+1}(1)_n}\\ &> \sum_{n=1}^{\infty} n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}|z|^{n-1}\\ &\geq \left|\sum_{n=1}^{\infty} n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}z^{n-1}\right| = |\phi_2'(z)|.\end{aligned}$$

To show that G is univalent in Δ, we assume that $z_1, z_2 \in \Delta$ so that $z_1 \neq z_2$. Since Δ is simply connected and convex, we have $z(t) = (1-t)z_1 + tz_2 \in \Delta$, where $0 \leq t \leq 1$. Then we can write

$$F(z_2) - F(z_1) = \int_0^1 \left[(z_2-z_1)\,\phi_1'(z(t)) + \overline{(z_2-z_1)\,\phi_2'(z(t))}\right]dt$$

so that

$$\begin{aligned}\text{(2.3)}\qquad \operatorname{Re}\frac{F(z_2)-F(z_1)}{z_2-z_1} &= \int_0^1 \operatorname{Re}\left[\left(\phi_1'(z(t)) + \frac{\overline{z_2-z_1}}{z_2-z_1}\right)\overline{\phi_2'(z(t))}\right]dt\\ &> \int_0^1 [\operatorname{Re}\phi_1'(z(t)) - |\phi_2'(z(t))|]dt\end{aligned}$$

On the other hand,

$$\begin{aligned}&\operatorname{Re}\phi_1'(z) - |\phi_2'(z)|\\ &\geq 1 - \sum_{n=2}^{\infty} n\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|z|^{n-1} - \sum_{n=1}^{\infty} n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}|z|^{n-1}\\ &> 1 - \sum_{n=2}^{\infty}(n-1+1)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} - \sum_{n=1}^{\infty} n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\\ &= 2 - \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-2}} - \sum_{n=0}^{\infty}\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} - \frac{a_2b_2}{c_2}\sum_{n=1}^{\infty}\frac{(a_2+1)_{n-1}(b_2+1)_{n-1}}{(c_2+1)_{n-1}(1)_{n-1}}\\ &= 2 - \left(1 + \frac{a_1b_1}{c_1-a_1-b_1-1}\right)F(a_1,b_1;c_1;1) - \frac{a_2b_2}{c_2-a_2-b_2-1}F(a_2,b_2;c_2;1)\\ &\geq 0,\ \text{by (2.2).}\end{aligned}$$

Thus (2.3) and the above inequality lead to $F(z_1) \neq F(z_2)$ and hence F is univalent in Δ. In order to prove that $G \in S_H^*$,using Lemma 2.1, we only need to prove that

$$\text{(2.4)}\qquad \sum_{n=2}^{\infty} n\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \leq 1.$$

Writing $n = n - 1 + 1$, the left hand side of (2.4) reduces to

$$\frac{a_1b_1}{c_1}\sum_{n=0}^{\infty}\frac{(a_1+1)_n(b_1+1)_n}{(c_1+1)_n(1)_n} + \left[\sum_{n=0}^{\infty}\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} - 1\right] + \frac{a_2b_2}{c_2}\sum_{n=0}^{\infty}\frac{(a_2+1)_n(b_2+1)_n}{(c_2+1)_n(1)_n}$$
$$= F(a_1,b_1;c_1;1)\left(\frac{a_1b_1}{c_1-a_1-b_1-1}+1\right) + \frac{a_2b_2}{c_2-a_2-b_2-1}F(a_2,b_2;c_2;1) - 1.$$

The last expression is bounded above by 1 provided that (2.2) is satisfied. This completes the proof. □

Lemma 2.3 ([5, 10]). *For $f = h + \overline{g}$ with h and g of the form (1.1), if*

$$\sum_{n=2}^{\infty} n^2 |A_n| + \sum_{n=1}^{\infty} n^2 |B_n| \le 1,$$

then $f \in K_H$.

Theorem 2.4. *If $a_j, b_j > 0$, $c_j > a_j + b_j + 2$, for $j = 1, 2$ then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ to be harmonic univalent in Δ and $G \in K_H$, is that*

$$\left(1 + \frac{3a_1b_1}{c_1-a_1-b_1-1} + \frac{(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2}\right)F(a_1,b_1;c_1;1)$$
$$+ \left(\frac{a_2b_2}{c_2-a_2-b_2-1} + \frac{(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2}\right)F(a_2,b_2;c_2;1) \le 2. \tag{2.5}$$

Proof. The proof of the first part is similar to that of Theorem 2.2 and so it is omitted. In view of Lemma 2.3, we only need to show that

$$\sum_{n=2}^{\infty} n^2 \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} n^2 \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \le 1.$$

That is,

$$\sum_{n=0}^{\infty}(n+2)^2\frac{(a_1)_{n+1}(b_1)_{n+1}}{(c_1)_{n+1}(1)_{n+1}} + \sum_{n=0}^{\infty}(n+1)^2\frac{(a_2)_{n+1}(b_2)_{n+1}}{(c_2)_{n+1}(1)_{n+1}} \le 1. \tag{2.6}$$

But,

$$\sum_{n=0}^{\infty}(n+2)^2\frac{(a_1)_{n+1}(b_1)_{n+1}}{(c_1)_{n+1}(1)_{n+1}}$$
$$= \sum_{n=0}^{\infty}(n+1)\frac{(a_1)_{n+1}(b_1)_{n+1}}{(c_1)_{n+1}(1)_n} + 2\sum_{n=0}^{\infty}\frac{(a_1)_{n+1}(b_1)_{n+1}}{(c_1)_{n+1}(1)_n} + \sum_{n=0}^{\infty}\frac{(a_1)_{n+1}(b_1)_{n+1}}{(c_1)_{n+1}(1)_{n+1}}$$
$$= \left[\frac{(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2} + \frac{3a_1b_1}{c_1-a_1-b_1-1} + 1\right]F(a_1,b_1;c_1;1) - 1,$$

and

$$\sum_{n=0}^{\infty}(n+1)^2\frac{(a_2)_{n+1}(b_2)_{n+1}}{(c_2)_{n+1}(1)_{n+1}}$$
$$=\sum_{n=1}^{\infty}\frac{(a_2)_{n+1}(b_2)_{n+1}}{(c_2)_{n+1}(1)_{n-1}}+\sum_{n=0}^{\infty}\frac{(a_2)_{n+1}(b_2)_{n+1}}{(c_2)_{n+1}(1)_{n}}$$
$$=\left[\frac{(a_2)_2(b_2)_2}{(c_2-a_2-b_1-2)_2}+\frac{a_2b_2}{c_1-a_1-b_1-1}\right]F(a_2,b_2;c_2;1)-1.$$

Thus, (2.6) is equivalent to

$$F(a_1,b_1;c_1;1)\left(\frac{(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2}+\frac{3a_1b_1}{c_1-a_1-b_1-1}+1\right)-1$$
$$+F(a_2,b_2;c_2;1)\left(\frac{(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2}+\frac{a_2b_2}{c_2-a_2-b_2-1}\right)\le 1$$

which is true because of the hypothesis. □

Denote by S^*_{RH} and K_{RH}, respectively, the subclasses of S^*_H and K_H consisting of functions $f=h+\overline{g}$ so that h and g are of the form

$$h(z)=z-\sum_{n=2}^{\infty}A_nz^n,\quad g(z)=\sum_{n=1}^{\infty}B_nz^n,\quad A_n\ge 0,\ B_n\ge 0,\ B_1<1. \tag{2.7}$$

Lemma 2.5 ([4, 10]). *Let $f=h+\overline{g}$ be given by (2.7). Then*

(i) $f\in S^*_{RH}\Leftrightarrow\sum_{n=2}^{\infty}nA_n+\sum_{n=1}^{\infty}nB_n\le 1,$

(ii) $f\in K_{RH}\Leftrightarrow\sum_{n=2}^{\infty}n^2A_n+\sum_{n=1}^{\infty}n^2B_n\le 1.$

Theorem 2.6. *Let $a_j,b_j>0,\ c_j>a_j+b_j+1$, for $j=1,2$ and $a_2b_2<c_2$. If*

$$G_1(z)=z\left(2-\frac{\phi_1(z)}{z}\right)+\overline{\phi_2(z)} \tag{2.8}$$

then

(i) $G_1\in S^*_{RH}\Leftrightarrow$*(2.2) holds*

(ii) $G_1\in K_{RH}\Leftrightarrow$*(2.5) holds.*

Proof. (i) We observe that

$$G_1(z)=z-\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}z^n+\overline{\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}z^n},$$

and $S^*_{RH}\subset S^*_H$. In view of Theorem 2.2, we only need to show the necessary condition for G_1 to be in S^*_{RH}. If $G_1\in S^*_{RH}$, then G_1 satisfies the inequality in Lemma 2.5(i) and the result in (i) follows from Lemma 2.5(i). The proof of (ii) is similar because $K_{RH}\subset K_H$, and by using Lemma 2.5(ii) and Theorem 2.4. □

Theorem 2.7. *Let $a_j,b_j>0,\ c_j>a_j+b_j+1$, for $j=1,2$ and $a_2b_2<c_2$. A necessary and sufficient condition such that $f\widetilde{*}(\phi_1+\overline{\phi_2})\in S^*_{RH}$ for $f\in S^*_{RH}$ is that*

$$F(a_1,b_1;c_1;1)+F(a_2,b_2;c_2;1)\le 3, \tag{2.9}$$

where ϕ_1, ϕ_2 are as defined, respectively, by (1.7) and (1.8).

Proof. Let $f = h + \overline{g} \in S^*_{RH}$, where h and g are given by (2.7). Then

$$\begin{aligned}\left(f\tilde{*}(\phi_1 + \overline{\phi_2})\right)(z) &= h(z) * \phi_1(z) + \overline{g(z) * \phi_2(z)} \\ &= z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n z^n + \overline{\sum_{n=1}^{\infty} \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} B_n z^n}.\end{aligned}$$

In view of Lemma 2.5(i), we need to prove that $f\tilde{*}(\phi_1 + \overline{\phi_2}) \in S^*_{RH}$ if and only if

$$\sum_{n=2}^{\infty} n \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n + \sum_{n=1}^{\infty} n \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} B_n \leq 1. \tag{2.10}$$

As an application of Lemma 2.5(i), we have

$$|A_n| \leq \frac{1}{n}, \quad |B_n| \leq \frac{1}{n}.$$

Therefore, the left side of (2.10) is bounded above by

$$\sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} = F(a_1, b_1; c_1; 1) + F(a_2, b_2; c_2; 1) - 2.$$

The last expression is bounded above by 1 if and only if (2.9) is satisfied. This proves (2.10) and results follow. □

Theorem 2.8. *If $a_j, b_j > 0$ and $c_j > a_j + b_j$ for $j = 1, 2$, then a sufficient condition for a function*

$$G_2(z) = \int_0^z F(a_1, b_1; c_1; t)dt + \overline{\int_0^z [F(a_2, b_2; c_2; t) - 1]dt}$$

to be in S^*_H is that

$$F(a_1, b_1; c_1; 1) + F(a_2, b_2; c_2; 1) \leq 3.$$

Proof. In view of Lemma 2.1, the function

$$G_2(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} z^n + \overline{\sum_{n=2}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} z^n}$$

is in S^*_H if

$$\sum_{n=2}^{\infty} n \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} + \sum_{n=2}^{\infty} n \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} \leq 1.$$

That is, if

$$\sum_{n=1}^{\infty} \frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} + \sum_{n=1}^{\infty} \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \leq 1.$$

Equivalently, $G \in S^*_H$ if

$$F(a_1, b_1; c_1; 1) + F(a_2, b_2; c_2; 1) \leq 3.$$

□

Theorem 2.9. *If $a_1, b_1 > -1,\ c_1 > 0,\ a_1 b_1 < 0,\ a_2 > 0,\ b_2 > 0,$ and $c_j > a_j + b_j + 1,\ j = 1, 2,$ then*

$$G_2(z) = \int_0^z F(a_1, b_1; c_1; t)dt + \overline{\int_0^z [F(a_2, b_2; c_2; t) - 1]dt}$$

is in S^*_H if and only if $F(a_1, b_1; c_1; 1) - F(a_2, b_2; c_2; 1) + 1 \geq 0$.

Proof. Applying Lemma 2.5(i) to

$$G_2(z) = z - \frac{|a_1 b_1|}{c_1} \sum_{n=2}^{\infty} \frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_n} z^n + \overline{\sum_{n=2}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} z^n},$$

it suffices to show that

$$\frac{|a_1 b_1|}{c_1} \sum_{n=2}^{\infty} n \frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_n} + \sum_{n=2}^{\infty} n \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} \leq 1.$$

Or equivalently

$$\sum_{n=0}^{\infty} \frac{(a_1+1)_n (b_1+1)_n}{(c_1+1)_n (1)_{n+1}} + \frac{c_1}{|a_1 b_1|} \sum_{n=1}^{\infty} \frac{(a_2)_n (b_2)_n}{(c_2)_n (1)_n} \leq \frac{c_1}{|a_1 b_1|}.$$

But, this is equivalent to

$$\frac{c_1}{a_1 b_1} \sum_{n=1}^{\infty} \frac{(a_1)_n (b_1)_n}{(c_1)_n (1)_n} + \frac{c_1}{|a_1 b_1|} \sum_{n=1}^{\infty} \frac{(a_2)_n (b_2)_n}{(c_2)_n (1)_n} \leq \frac{c_1}{|a_1 b_1|}.$$

That is,

$$F(a_1, b_1; c_1; 1) - F(a_2, b_2; c_2; 1) \geq -1.$$

This completes the proof of the theorem. □

Remark 2.10. Comparable results to Theorems 2.7, 2.8, 2.9 for harmonic convex functions may also be obtained. The proofs and results are similar and hence are omitted.

In particular, the results parallel to Theorems 2.2, 2.4, 2.6 to 2.9 may also be obtained for the incomplete beta function $\varphi(a, c; z)$ as defined by (1.6). If

$$\psi_1(z) := z\varphi(a_1, c_1; z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}}{(c_1)_{n-1}} z^n,$$

$$\psi_2(z) := z\varphi(a_2, c_2; z) - 1 = \sum_{n=1}^{\infty} \frac{(a_2)_n}{(c_2)_n} z^n, \; a_2 < c_2,$$

then

$$\psi_1(z) + \overline{\psi_2(z)} \equiv \phi_1(z) + \overline{\phi_2(z)},$$

whenever $b_1 = 1,\ b_2 = 1$.

Note that

$$\psi_1(1) = F(a_1, 1; c_1; 1) = \frac{c_1}{(c_1 - a_1)} \text{ and } \psi_2(1) = F(a_2, 1; c_2; 1) - 1 = \frac{a_2}{(c_2 - a_2)}.$$

As an illustration, we close this section with the incomplete beta function analog to some of the earlier results.

Theorem 2.2′. *If $a_j > 0$ and $c_j > a_j + 2$ for $j = 1, 2$, then a sufficient condition for $\psi_1 + \overline{\psi_2}$ to be harmonic univalent in Δ with $\psi_1 + \overline{\psi_2} \in S_H^*$ is*

$$\frac{c_1(c_1 - 2)}{(c_1 - a_1)(c_1 - a_1 - 2)} + \frac{a_2^2}{(c_2 - a_2)(c_2 - a_2 - 2)} \leq 2.$$

Theorem 2.4′. *If* $a_j > 0$ *and* $c_j > a_j + 3$ *for* $j = 1, 2$ *, then a sufficient condition for* $\psi_1 + \overline{\psi_2}$ *to be harmonic univalent in* Δ *with* $\psi_1 + \overline{\psi_2} \in K_H$ *is*

$$\frac{c_1}{(c_1 - a_1)} \left[1 + \frac{3a_1}{c_1 - a_1 - 2} + \frac{2a_2}{(c_1 - a_1 - 3)_2}\right] + \frac{a_2}{(c_2 - a_2)} \left[\frac{a_2}{c_2 - a_2 - 2} + \frac{2(a_2)_2}{(c_2 - a_2 - 3)_2}\right] \leq 2.$$

Theorem 2.7′. *A necessary and sufficient condition such that* $f \tilde{*} (\psi_1 + \overline{\psi_2}) \in S^*_{RH}$ *for* $f \in S^*_{RH}$ *is that*

$$\frac{c_1}{(c_1 - a_1)} + \frac{a_2}{(c_2 - a_2)} \leq 1.$$

Theorem 2.9′. *If* $a_1 > -1,\ c_1 > 0,\ a_1 < 0,\ a_2 > 0,\ c_j > a_j + 1$ *for* $j = 1, 2,$ *and* $c_j > a_j + b_j + 1,\ j = 1, 2,$ *then*

$$\int_0^z \varphi\left(a_1, c_1; t\right) dt + \overline{\int_0^z \left[\varphi\left(a_2, c_2; t\right) - 1\right] dt}$$

is in S^*_H *if and only if*

$$\frac{c_1 - 1}{c_1 - a_1 - 1} \geq \frac{a_2}{c_2 - a_2 - 1}.$$

2.1. **Positive Order.** We say that f of the form (1.1) is harmonic starlike of order $\alpha,\ 0 \leq \alpha \leq 1$,for $|z| = r$ if $\frac{\partial}{\partial\theta}\left(\arg\ f(re^{i\theta})\right) \geq \alpha, \quad |z| = r$. Denote by $S^*_H(\alpha)$and $S^*_{RH}(\alpha)$ the subclasses of $S^*_H$and S^*_{RH}, respectively, that are starlike of order α. Also, denote by $K_H(\alpha)$ and $K_{RH}(\alpha)$ the subclasses of $K_H$and K_{RH}, respectively, that are convex of order α. Most of our results can also be rewritten for functions of positive order by using similar techniques. For instance, using the results in [4] we have the following:

Theorem 2.11. *If* $a_j, b_j > 0$ *and* $c_j > a_j + 1,\ a_2 b_2 < c_2$ *for* $j = 1, 2,$ *then* $\phi_1 + \overline{\phi_2}$ *is harmonic univalent in* Δ *with* $\phi_1 + \overline{\phi_2} \in S^*_H(\alpha),\ 0 \leq \alpha \leq 1$ *if*

$$\left(1 - \alpha + \frac{a_1 b_1}{c_1 - a_1 - b_1 - 1}\right) F(a_1, b_1; c_1; 1) + \left(\alpha + \frac{a_2 b_2}{c_2 - a_2 - b_2 - 1}\right) F(a_2, b_2; c_2; 1) \leq 2(1 - \alpha).$$

REFERENCES

[1] J. CLUNIE AND T. SHEIL-SMALL, Harmonic univalent functions, *Ann. Acad. Aci., Fenn. Ser.* A. I. Math., **9** (1984), 3–25.

[2] B.C. CARLSON AND D.B. SHAFFER, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, **15** (1984), 737–745.

[3] L. DE BRANGES, A proof of the Bieberbach conjecture, *Acta Math.*, **154** (1985), 137–152.

[4] J.M. JAHANGIRI, Harmonic functions starlike in the unit disk, *J. Math. Anal. Appl.*, **235** (1999), 470–477.

[5] J.M. JAHANGIRI, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, *Ann. Univ. Mariae Curie – Sklodowska Sect A.*, A **52** (1998), 57–66.

[6] E. MERKES AND B.T. SCOTT, Starlike hypergeometric functions, *Proc. American Math Soc.*, **12** (1961), 885–888.

[7] S.S. MILLER AND P.T. MOCANU, Univalence of Gaussian and confluent hypergeometric functions, *Proc. Amer. Math. Soc.*, **110**(2) (1990), 333–342.

[8] St. RUSCHEWEYH AND V. SINGH, On the order of starlikeness of hypergeometric functions, *J. Math. Anal. Appl.*, **113** (1986), 1–11

[9] H. SILVERMAN, Starlike and convexity properties for hypergeometric functions, *J. Math. Anal. Appl.*, **172** (1993), 574–581.

[10] H. SILVERMAN AND E.M. SILVIA, Subclasses of harmonic univalent functions, *New Zealand J. Math.*, **28** (1999), 275–284.

[11] H.M. SRIVASTAVA AND H.L. MANOCHA, *A Treatise on Generating Functions*, Ellis Horwood Limited and John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1984.