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CERTAIN SUBCLASSES OF $p-$VALENT MEROMORPHIC FUNCTIONS INVOLVING CERTAIN OPERATOR

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ABSTRACT. In this paper, a new subclass $\sum_{p,\beta}^{\alpha}(\eta,\delta,\mu,\lambda)$ of $p-$valent meromorphic functions defined by certain integral operator is introduced. Some interesting properties of this class are obtained.

Key words and phrases: Analytic, $p-$valent, Meromorphic, Integral operator.

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1. INTRODUCTION

Let $\sum_p$ be the class of functions f of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and $p-$valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U\backslash\{0\}$.

Similar to [1], we define the following family of integral operators $Q^{\alpha}_{\beta,p} : \sum_p \to \sum_p$ $(\alpha \geq 0, \beta > -1; p \in \mathbb{N})$ as follows:

(i)

$$Q^{\alpha}_{\beta,p} f(z) = \binom{\alpha+\beta-1}{\beta-1} \alpha z^{-(p+\beta)} \int_0^z (1 - \frac{t}{z})^{\alpha-1} t^{\beta+p-1} f(t) dt \tag{1.2}$$

$$(\alpha > 0;\ \beta > -1;\ p \in \mathbb{N};\ f \in \Sigma_p); \tag{1.3}$$

and

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(ii)

(1.4) $$Q^0_{\beta,p}f(z) = f(z) \quad (\text{for } \alpha = 0;\ \beta > -1;\ p \in \mathbb{N};\ f \in \Sigma_p).$$

From (1.2) and (1.4), we have

(1.5) $$Q^\alpha_{\beta,p}f(z) = z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\beta+\alpha)} a_{k-p} z^{k-p}$$

(1.6) $$(\alpha \geq 0;\ \beta > -1;\ p \in \mathbb{N};\ f \in \Sigma_p).$$

Using the relation (1.5), it is easy to show that

(1.7) $$z(Q^\alpha_{\beta,p}f(z))' = (\alpha+\beta-1)Q^{\alpha-1}_{\beta,p}f(z) - (\alpha+\beta+p-1)Q^\alpha_{\beta,p}f(z).$$

Definition 1.1. *Let* $\sum^\alpha_{p,\beta}(\eta,\delta,\mu,\lambda)$ *be the class of functions* $f \in \sum_p$ *which satisfy:*

(1.8) $$\mathrm{Re}\left\{(1-\lambda)\left(\frac{Q^\alpha_{\beta,p}f(z)}{Q^\alpha_{\beta,p}g(z)}\right)^\mu + \lambda \frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)}\left(\frac{Q^\alpha_{\beta,p}f(z)}{Q^\alpha_{\beta,p}g(z)}\right)^{\mu-1}\right\} > \eta,$$

where $g \in \sum_p$ *satisfies the following condition:*

(1.9) $$\mathrm{Re}\left\{\frac{Q^\alpha_{\beta,p}g(z)}{Q^{\alpha-1}_{\beta,p}g(z)}\right\} > \delta \quad (0 \leq \delta < 1; z \in U),$$

and η *and* μ *are real numbers such that* $0 \leq \eta < 1$, $\mu > 0$ *and* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}\{\lambda\} > 0$.

To establish our main results we need the following lemmas.

Lemma 1.1 ([2])**.** *Let* Ω *be a set in the complex plane* $\mathbb{C}$ *and let the function* $\psi : \mathbb{C}^2 \to \mathbb{C}$ *satisfy the condition* $\psi(ir_2, s_1) \notin \Omega$ *for all real* $r_2, s_1 \leq -\frac{1+r_2^2}{2}$. *If* q *is analytic in* U *with* $q(0) = 1$ *and* $\psi(q(z), zq'(z)) \in \Omega$, $z \in U$, *then* $\mathrm{Re}\{q(z)\} > 0$ $(z \in U)$.

Lemma 1.2 ([3])**.** *If* q *is analytic in* U *with* $q(0) = 1$, *and if* $\lambda \in C\backslash\{0\}$ *with* $\mathrm{Re}\{\lambda\} \geq 0$, *then* $\mathrm{Re}\{q(z) + \lambda zq'(z)\} > \alpha$ $(0 \leq \alpha < 1)$ *implies* $\mathrm{Re}\{q(z)\} > \alpha + (1-\alpha)(2\gamma - 1)$, *where* γ *is given by*

$$\gamma = \gamma(\mathrm{Re}\,\lambda) = \int_0^1 (1 + t^{\mathrm{Re}\{\lambda\}})^{-1} dt$$

which is increasing function of $\mathrm{Re}\{\lambda\}$ *and* $\frac{1}{2} \leq \gamma < 1$. *The estimate is sharp in the sense that the bound cannot be improved.*

For real or complex numbers a, b and c $(c \neq 0, -1, -2, \ldots)$, the Gauss hypergeometric function is defined by

(1.10) $$ {}_2F_1(a,b;c;z) = 1 + \frac{a \cdot b}{c}\frac{z}{1!} + \frac{a(a+1) \cdot b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots .$$

We note that the series (1.10) converges absolutely for $z \in U$ and hence represents an analytic function in U (see, for details, [4, Ch. 14]). Each of the identities (asserted by Lemma 1.3 below) is fairly well known (cf., e.g., [4, Ch. 14]).

Lemma 1.3 ([4])**.** *For real or complex numbers* a, b *and* c $(c \neq 0, -1, -2, \ldots)$,

(1.11) $$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z)$$
$$(\mathrm{Re}(c) > \mathrm{Re}(b) > 0);$$

(1.12) $$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right); $$

(1.13) $$ {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z); $$

and

(1.14) $$ {}_2F_1\left(1,1;2;\frac{1}{2}\right) = 2\ln 2. $$

2. MAIN RESULTS

Unless otherwise mentioned, we assume throughout this paper that

$$ \alpha \geq 0;\ \beta > -1;\ \alpha+\beta \neq 1;\ \mu > 0;\ 0 \leq \eta < 1;\ p \in \mathbb{N} \text{ and } \lambda \geq 0. $$

Theorem 2.1. *Let* $f \in \sum_{p,\beta}^{\alpha}(\eta,\delta,\mu,\lambda)$. *Then*

(2.1) $$ \operatorname{Re}\left(\frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha} g(z)}\right)^{\mu} > \frac{2\mu\eta(\alpha+\beta-1)+\lambda\delta}{2\mu(\alpha+\beta-1)+\lambda\delta}, \qquad (z \in U), $$

where the function $g \in \sum_p$ *satisfies the condition* (1.9).

Proof. Let $\gamma = \frac{2\mu\eta(\alpha+\beta-1)+\lambda\delta}{2\mu(\alpha+\beta-1)+\lambda\delta}$, and we define the function q by

(2.2) $$ q(z) = \frac{1}{1-\gamma}\left[\left(\frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha} g(z)}\right)^{\mu} - \gamma\right]. $$

Then q is analytic in U and $q(0)=1$. If we set

(2.3) $$ h(z) = \frac{Q_{\beta,p}^{\alpha} g(z)}{Q_{\beta,p}^{\alpha-1} g(z)}, $$

then by the hypothesis (1.9), $\operatorname{Re}\{h(z)\} > \delta$. Differentiating (2.2) with respect to z and using the identity (1.7), we have

(2.4) $$ (1-\lambda)\left(\frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha} g(z)}\right)^{\mu} + \lambda\frac{Q_{\beta,p}^{\alpha-1} f(z)}{Q_{\beta,p}^{\alpha-1} g(z)}\left(\frac{Q_{\beta,p}^{\alpha} f(z)}{Q_{\beta,p}^{\alpha} g(z)}\right)^{\mu-1} = [(1-\gamma)q(z)+\gamma] + \frac{\lambda(1-\gamma)}{\mu(\alpha+\beta-1)} zq'(z)h(z). $$

Let us define the function $\psi(r,s)$ by

(2.5) $$ \psi(r,s) = [(1-\gamma)r+\gamma] + \frac{\lambda(1-\gamma)}{\mu(\alpha+\beta-1)} sh(z). $$

Using (2.5) and the fact that $f \in \sum_{p,\beta}^{\alpha}(\eta,\delta,\mu,\lambda)$, we obtain

$$ \{\psi(q(z), zq'(z)); z \in U\} \subset \Omega = \{w \in C : \operatorname{Re}(w) > \eta\}. $$

Now for all real $r_2, s_1 \leq -\frac{1+r_2^2}{2}$, we have

$$\begin{aligned}\operatorname{Re}\{\psi(ir_2, s_1)\} &= \gamma + \frac{\lambda(1-\gamma)s_1}{\mu(\alpha+\beta-1)} \operatorname{Re} h(z) \\ &\leq \gamma - \frac{\lambda(1-\gamma)\delta(1+r_2^2)}{2\mu(\alpha+\beta-1)} \\ &\leq \gamma - \frac{\lambda(1-\gamma)\delta}{2\mu(\alpha+\beta-1)} = \eta.\end{aligned}$$

Hence for each $z \in U,\ \psi(ir_2, s_1) \notin \Omega$. Thus by Lemma 1.1, we have $\operatorname{Re}\{q(z)\} > 0\ (z \in U)$ and hence

$$\operatorname{Re}\left(\frac{Q^{\alpha}_{\beta,p} f(z)}{Q^{\alpha}_{\beta,p} g(z)}\right)^{\mu} > \gamma \quad (z \in U).$$

This proves Theorem 2.1. □

Corollary 2.2. *Let the functions f and g be in $\sum_p$ and let g satisfy the condition (1.9). If $\lambda \geq 1$ and*

$$\operatorname{Re}\left\{(1-\lambda)\frac{Q^{\alpha}_{\beta,p} f(z)}{Q^{\alpha}_{\beta,p} g(z)} + \lambda\frac{Q^{\alpha-1}_{\beta,p} f(z)}{Q^{\alpha-1}_{\beta,p} g(z)}\right\} > \eta \quad (0 \leq \eta < 1; z \in U), \tag{2.6}$$

then

$$\operatorname{Re}\left\{\frac{Q^{\alpha-1}_{\beta,p} f(z)}{Q^{\alpha-1}_{\beta,p} g(z)}\right\} > \gamma = \frac{\eta[2(\alpha+\beta-1)+\delta]+\delta(\lambda-1)}{2(\alpha+\beta-1)+\delta\lambda} \quad (z \in U). \tag{2.7}$$

Proof. We have

$$\lambda\frac{Q^{\alpha-1}_{\beta,p} f(z)}{Q^{\alpha-1}_{\beta,p} g(z)} = \left\{(1-\lambda)\frac{Q^{\alpha}_{\beta,p} f(z)}{Q^{\alpha}_{\beta,p} g(z)} + \lambda\frac{Q^{\alpha-1}_{\beta,p} f(z)}{Q^{\alpha-1}_{\beta,p} g(z)}\right\} + (\lambda-1)\frac{Q^{\alpha}_{\beta,p} f(z)}{Q^{\alpha}_{\beta,p} g(z)} \quad (z \in U).$$

Since $\lambda \geq 1$, making use of (2.6) and (2.1) (for $\mu = 1$), we deduce that

$$\operatorname{Re}\left\{\frac{Q^{\alpha-1}_{\beta,p} f(z)}{Q^{\alpha-1}_{\beta,p} g(z)}\right\} > \gamma = \frac{\eta[2(\alpha+\beta-1)+\delta]+\delta(\lambda-1)}{2(\alpha+\beta-1)+\delta\lambda} \quad (z \in U).$$

□

Corollary 2.3. *Let $\lambda \in C\backslash\{0\}$ with $\operatorname{Re}\{\lambda\} \geq 0$. If $f \in \sum_p$ satisfies the following condition:*

$$\operatorname{Re}\{(1-\lambda)(z^p Q^{\alpha}_{\beta,p} f(z))^{\mu} + \lambda z^p Q^{\alpha-1}_{\beta,p} f(z)(z^p Q^{\alpha}_{\beta,p} f(z))^{\mu-1}\} > \eta \quad (z \in U),$$

then

$$\operatorname{Re}\{(z^p Q^{\alpha}_{\beta,p} f(z))^{\mu}\} > \frac{2\mu\eta(\alpha+\beta-1)+\operatorname{Re}\{\lambda\}}{2\mu(\alpha+\beta-1)+\operatorname{Re}\{\lambda\}} \quad (z \in U). \tag{2.8}$$

Further, if $\lambda \geq 1$ and $f \in \sum_p$ satisfies

$$\operatorname{Re}\{(1-\lambda)z^p Q^{\alpha}_{\beta,p} f(z) + \lambda z^p Q^{\alpha-1}_{\beta,p} f(z)\} > \eta \quad (z \in U), \tag{2.9}$$

then

$$\operatorname{Re}\{z^p Q^{\alpha-1}_{\beta,p} f(z)\} > \frac{2\eta(\alpha+\beta-1)+\lambda-1}{2(\alpha+\beta-1)+\lambda} \quad (z \in U). \tag{2.10}$$

Proof. The results (2.8) and (2.10) follow by putting $g(z) = z^{-p}$ in Theorem 2.1 and Corollary 2.2, respectively. □

Remark 1. Choosing α, λ and μ appropriately in Corollary 2.3, we have,

(i) *For* $\alpha = 0, \beta \neq 1$ *and* $\lambda = 1$ *in Corollary 2.3, we have*:

$$\operatorname{Re}\left\{\frac{1}{\beta-1}\left[\beta+p-1+\frac{zf'(z)}{f(z)}\right](z^p f(z))^{\mu}\right\} > \eta \quad (z \in U),$$

which implies that

$$\operatorname{Re}\{z^p f(z)\}^{\mu} > \frac{2\mu\eta(\beta-1)+1}{2\mu(\beta-1)+1} \quad (z \in U).$$

(ii) *For* $\alpha = 0, \beta \neq 1, \mu = 1$ *and* $\lambda \in C\backslash\{0\}$ *with* $\operatorname{Re}\{\lambda\} \geq 0$ *in Corollary 2.3, we have*

$$\operatorname{Re}\left\{(1+\frac{\lambda p}{\beta-1})z^p f(z) + \frac{\lambda}{\beta-1}z^{p+1}f'(z)\right\} > \eta,$$

which *implies that*

$$\operatorname{Re}\{z^p f(z)\} > \frac{2\eta(\beta-1)+\operatorname{Re}\{\lambda\}}{2(\beta-1)+\operatorname{Re}\{\lambda\}} \quad (z \in U).$$

(iii) *Replacing* f *by* $-\frac{zf'}{p}$ *in the result* (ii)*, we have:*

$$-\operatorname{Re}\left\{\left[1+\tfrac{\lambda}{\beta-1}(p+1)\right]\frac{z^{p+1}f'(z)}{p} + \tfrac{\lambda}{p(\beta-1)}z^{p+1}f''(z)\right\} > \eta \quad (0 \leq \eta < 1; z \in U),$$

which implies that

$$-\operatorname{Re}\left\{\frac{z^{p+1}f'(z)}{p}\right\} > \frac{2\eta(\beta-1)+\operatorname{Re}\{\lambda\}}{2(\beta-1)+\operatorname{Re}\{\lambda\}} \quad (z \in U).$$

Theorem 2.4. *Let* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\{\lambda\} > 0$. *If* $f \in \sum_p$ *satisfies the following condition:*

(2.11) $$\operatorname{Re}\{(1-\lambda)(z^p Q^{\alpha}_{\beta,p}f(z))^{\mu} + \lambda z^p Q^{\alpha-1}_{\beta,p}f(z)(z^p Q^{\alpha}_{\beta,p}f(z))^{\mu-1}\} > \eta \quad (z \in U),$$

then

(2.12) $$\operatorname{Re}\{(z^p Q^{\alpha}_{\beta,p}f(z))^{\mu}\} > \eta + (1-\eta)(2\rho-1),$$

where

(2.13) $$\rho = \frac{1}{2}\,{}_2F_1\left(1,1;\frac{\mu(\alpha+\beta-1)}{\operatorname{Re}\{\lambda\}}+1;\frac{1}{2}\right).$$

Proof. Let

(2.14) $$q(z) = (z^p Q^{\alpha}_{\beta,p}f(z))^{\mu}.$$

Then q is analytic with $q(0) = 1$. Differentiating (2.14) with respect to z and using the identity (1.7), we have

$$(1-\lambda)(z^p Q^{\alpha}_{\beta,p}f(z))^{\mu} + \lambda z^p Q^{\alpha-1}_{\beta,p}f(z)(z^p Q^{\alpha}_{\beta,p}f(z))^{\mu-1} = q(z) + \frac{\lambda}{\mu(\alpha+\beta-1)}zq'(z),$$

so that by the hypothesis (2.11), we have

$$\operatorname{Re}\left\{q(z) + \frac{\lambda}{\mu(\alpha+\beta-1)}zq'(z)\right\} > \eta \quad (z \in U).$$

In view of Lemma 1.2, this implies that

$$\operatorname{Re}\{q(z)\} > \eta + (1-\eta)(2\rho-1),$$

where

$$\rho=\rho(\mathrm{Re}\{\lambda\})=\int_0^1\left(1+t^{\frac{\mathrm{Re}\{\lambda\}}{\mu(\alpha+\beta-1)}}\right)^{-1}dt.$$

Putting $\mathrm{Re}\{\lambda\}=\lambda_1>0$, we have

$$\rho=\int_0^1\left(1+t^{\frac{\lambda_1}{\mu(\alpha+\beta-1))}}\right)^{-1}dt=\frac{\mu(\alpha+\beta-1)}{\lambda_1}\int_0^1(1+u)^{-1}u^{\frac{\mu(\alpha+\beta-1)}{\lambda_1}-1}du$$

Using (1.11), (1.12), (1.13) and (1.14), we obtain

$$\rho=\frac{1}{2}\ {}_2F_1\left(1,1;\frac{\mu(\alpha+\beta-1)}{\lambda_1}+1;\frac{1}{2}\right).$$

This completes the proof of Theorem 2.1. □

Corollary 2.5. *Let $\lambda\in\mathbb{R}$ with $\lambda\geq 1$. If $f\in\sum_p$ satisfies*

(2.15) $$\mathrm{Re}\left\{(1-\lambda)z^pQ^{\alpha}_{\beta,p}f(z)+\lambda z^pQ^{\alpha-1}_{\beta,p}f(z)\right\}>\eta\quad(z\in U),$$

then

$$\mathrm{Re}\{z^pQ^{\alpha-1}_{\beta,p}f(z)\}>\eta+(1-\eta)(2\rho_1-1)\left(1-\frac{1}{\lambda}\right)\quad(z\in U),$$

where

$$\rho_1=\frac{1}{2}\ {}_2F_1\left(1,1;\frac{(\alpha+\beta-1)}{\lambda}+1;\frac{1}{2}\right).$$

Proof. The result follows by using the identity

(2.16) $$\lambda z^pQ^{\alpha-1}_{\beta,p}f(z)=(1-\lambda)z^pQ^{\alpha}_{\beta,p}f(z)+\lambda z^pQ^{\alpha-1}_{\beta,p}f(z)+(\lambda-1)z^pQ^{\alpha}_{\beta,p}f(z).$$

□

Remark 2. We note that, for $\alpha=0,\beta=2$ and $\lambda=\mu>0$ in Corollary 2.3, that is, if

(2.17) $$\mathrm{Re}\left\{(1-\lambda)(z^pf(z))^{\lambda}+\lambda(z^{p+1}f(z))'(z^pf(z))^{\lambda-1}\right\}>\eta\quad(z\in U),$$

then (2.8) *implies that*

(2.18) $$\mathrm{Re}\{(z^pf(z))^{\lambda}\}>\frac{2\eta+1}{3}\quad(z\in U),$$

whereas, if $f\in\sum_p$ satisfies the condition (2.17) *then by using Theorem 2.4, we have*

$$\mathrm{Re}\{(z^pf(z))^{\lambda}\}>2(1-\ln 2)\eta+(2\ln 2-1)\quad(z\in U),$$

which is better than (2.18).

Theorem 2.6. *Suppose that the functions f and g are in $\sum_p$ and g satisfies the condition* (1.9). *If*

(2.19) $$\mathrm{Re}\left\{\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)}-\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right\}>-\frac{(1-\eta)\delta}{2(\alpha+\beta-1)}\quad(z\in U),$$

for some η $(0\leq\eta<1)$, then

(2.20) $$\mathrm{Re}\left\{\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right\}>\eta\quad(z\in U)$$

and

(2.21) $$\mathrm{Re}\left\{\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)}\right\}>\frac{\eta[2(\alpha+\beta-1)+\delta]-\delta}{2(\alpha+\beta-1)}\quad(z\in U).$$

Proof. Let

$$q(z) = \frac{1}{1-\eta}\left[\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)} - \eta\right]. \tag{2.22}$$

Then q is analytic in U with $q(0) = 1$. Setting

$$\phi(z) = \frac{Q^{\alpha}_{\beta,p}g(z)}{Q^{\alpha-1}_{\beta,p}g(z)} \quad (z \in U), \tag{2.23}$$

we observe that from (1.9), we have $\mathrm{Re}\{\phi(z)\} > \delta$ $(0 \le \delta < 1)$ in U. A simple computation shows that

$$\begin{aligned}\frac{(1-\eta)zq'(z)}{\alpha+\beta-1}\phi(z) &= \frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)} - \frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)} \\ &= \psi(q(z), zq'(z)),\end{aligned}$$

where

$$\psi(r,s) = \frac{(1-\eta)s\phi(z)}{\alpha+\beta-1}.$$

Using the hypothesis (2.19), we obtain

$$\{\psi(q(z), zq'(z)); z \in U\} \subset \Omega = \left\{w \in \mathbb{C} : \mathrm{Re}\, w > -\frac{(1-\eta)\delta}{2(\alpha+\beta-1)}\right\}.$$

Now, for all real $r_2,\ s_1 \le -\frac{1+r_2^2}{2}$, we have

$$\begin{aligned}\mathrm{Re}\left\{\psi(ir_2, s_1)\right\} &= \frac{s_1(1-\eta)\,\mathrm{Re}\{\phi(z)\}}{\alpha+\beta-1} \\ &\le \frac{-(1-\eta)\delta(1+r_2^2)}{2(\alpha+\beta-1)} \\ &\le \frac{-(1-\eta)\delta}{2(\alpha+\beta-1)}.\end{aligned}$$

This shows that $\psi(ir_2, s_1) \notin \Omega$ for each $z \in U$. Hence by Lemma 1.1, we have $\mathrm{Re}\{q(z)\} > 0$ $(z \in U)$. This proves (2.20). The proof of (2.21) follows by using (2.20) and (2.21) in the identity:

$$\mathrm{Re}\left\{\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)}\right\} = \mathrm{Re}\left\{\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha-1}_{\beta,p}g(z)} - \frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right\} - \mathrm{Re}\left\{\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}g(z)}\right\}.$$

This completes the proof of Theorem 2.6. □

Remark 3.

(i) For $\alpha = 0$ and $g(z) = z^{-p}$ in Theorem 2.6, we have

$$\mathrm{Re}\left\{z^{p+1}f'(z) + pz^p f(z)\right\} > \frac{-(1-\eta)\delta}{2} \quad (z \in U),$$

which *implies that*

$$\mathrm{Re}\left\{z^p f(z)\right\} > \eta \quad (z \in U)$$

and

$$\mathrm{Re}\left\{z^{p+1}f'(z) + (p+\beta-1)z^p f(z)\right\} > \frac{\eta[2(\beta-1)+\delta]-\delta}{2} \quad (z \in U).$$

(ii) *Putting* $\alpha = 0, \beta = 2$ *in Theorem 2.6, we get that, if*

$$\mathrm{Re}\left\{\frac{zf'(z)+(p+1)f(z)}{zg'(z)+(p+1)g(z)} - \frac{f(z)}{g(z)}\right\} > \frac{-(1-\eta)\delta}{2} \quad (z \in U),$$

then

$$\mathrm{Re}\left\{\frac{f(z)}{g(z)}\right\} > \eta \quad (z \in U)$$

and

$$\mathrm{Re}\left\{\frac{zf'(z)+(p+1)f(z)}{zg'(z)+(p+1)g(z)}\right\} > \frac{\eta(2+\delta)-\delta}{2} \quad (z \in U).$$

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