

## A NOTE ON THE TRACE INEQUALITY FOR PRODUCTS OF HERMITIAN MATRIX POWER

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volume 3, issue 5, article 70,  
2002.

*Received 30 August, 2002;  
accepted 16 October, 2002.*

*Communicated by: L. Toth*

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Abstract

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## Abstract

We use the recent theory of integer points close to a smooth curve developed by Huxley-Sargos and Filaseta-Trifonov to get an asymptotic formula for short sums of a class of multiplicative functions.

*2000 Mathematics Subject Classification:* 11N37, 11P21

*Key words:* Multiplicative Functions, Short Sums, Integer points close to a curve

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# 1. Introduction and Notation

Let  $k \geq 2$  be an integer. A positive integer  $n$  is said to be  $k$ -free (resp.  $k$ -full) if, for any prime  $p \mid n$ , the  $p$ -adic valuation  $v_p(n)$  of  $n$  satisfies  $v_p(n) < k$  (resp.  $v_p(n) \geq k$ ), and we use the terms *squarefree* or *squarefull* when  $k = 2$ . We denote by  $\mu_k$  the multiplicative function defined by

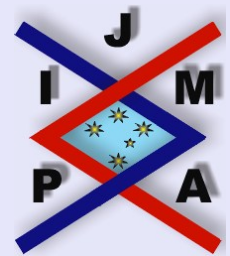
$$\mu_k(n) := \begin{cases} 1, & \text{if } n \text{ is } k\text{-free} \\ 0, & \text{otherwise.} \end{cases}$$

Obtaining gap results for  $k$ -free (or  $k$ -full) numbers is a very famous problem in analytic number theory (see [1] and the references). The best estimation in this direction has been obtained by Filaseta and Trifonov ([1]) who showed that, for  $x$  sufficiently large, any interval of the type  $]x; x + cx^{1/(2k+1)} \log x]$  ( $c := c(k) > 0$ ) contains a  $k$ -free number.

A dual problem is to get an asymptotic formula for  $\mu_k$ . This requires estimations for short sums of multiplicative functions, but such results are still relatively rare in the literature (see [2, 6]). In this paper, we are motivated by finding asymptotic results for short sums of the following class of arithmetical functions: define  $\mathcal{M}$  to be the set of multiplicative functions  $f$  verifying  $0 \leq f(n) \leq 1$  for any positive integer  $n$  and  $f(p) = 1$  for any prime number  $p$ . If  $f \in \mathcal{M}$ , we set

$$\mathcal{P}(f) := \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \sum_{l=1}^{\infty} \frac{f(p^l)}{p^l}\right).$$

We prove:



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**Theorem 1.1.** Let  $\varepsilon > 0$  and  $x, y \geq 2$  be real numbers such that  $x^{6\varepsilon} \leq y \leq x$ . If  $f \in \mathcal{M}$ , we have as  $x \rightarrow +\infty$  :

$$\sum_{x < n \leq x+y} f(n) = y\mathcal{P}(f) + O(x^{1/6+\varepsilon} + y^{1/2}).$$

Moreover, if  $y \leq x^{1/3}$ , then:

$$\sum_{x < n \leq x+y} f(n) = y\mathcal{P}(f) + O(x^{1/7+\varepsilon} + x^{1/21+\varepsilon}y^{2/3}).$$

For example, using this, we get:

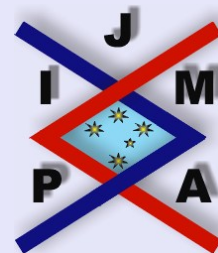
$$\sum_{x < n \leq x+y} \mu_2(n) = \frac{6y}{\pi^2} + O(x^{1/6+\varepsilon} + y^{1/2})$$

with  $x^{6\varepsilon} \leq y \leq x$ . The proof of this theorem uses a convolution argument, and the new theory of integer points close to a curve (see [1, 3, 4]) arises as the crucial point to estimate the difference of integer parts.

In what follows,  $a, b, d, k, l, m, n, q, B, N$  will always denote positive integers,  $p$  will be a prime number and  $[x]$  is the integral part of  $x$ . For any  $X, Y > 0$ , the notation  $X \ll Y$  means there exists  $C > 0$  such that  $X \leq CY$ . the notation  $X \asymp Y$  means  $X \ll Y$  and  $Y \ll X$  simultaneously.

If  $f, g$  are two arithmetical functions, the Dirichlet convolution product  $f * g$  is defined by

$$(f * g)(n) := \sum_{d|n} f(d)g(n/d).$$



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$\mu(n)$  is the Möbius function,  $\tau(n) := \sum_{d|n} 1$  the classical divisor function, and, more generally,  $\tau_{(k)}(n)$  is the arithmetical function defined by

$$\tau_{(k)}(n) := \sum_{d^k|n} 1.$$

If  $\varphi$  is any smooth function and if  $\delta > 0$  is any real number, we set

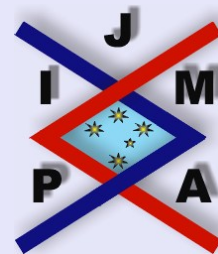
$$\mathcal{R}(\varphi, N, \delta) := |\{n \in ]N; 2N] \cap \mathbb{Z}, \exists m \in \mathbb{Z}, |\varphi(n) - m| \leq \delta\}|.$$

Thus, if  $\delta$  is sufficiently small,  $\mathcal{R}(\varphi, N, \delta)$  counts the number of integer points close to the curve  $y = \varphi(x)$  when  $N < x \leq 2N$ , and then

$$\mathcal{R}(\varphi, N, \delta) = \sum_{N < n \leq 2N} ([\varphi(n) + \delta] - [\varphi(n) - \delta]).$$

We will therefore use the following principle: *let  $M$  be a large real number,  $\varphi(n)$  and  $\delta(n)$  two functions such that  $\delta := \max_{n \leq M} \delta(n)$  satisfies  $0 < \delta \leq 1/4$ . Then we have*

$$(1.1) \quad \sum_{n \leq M} ([\varphi(n) + \delta(n)] - [\varphi(n)]) \ll \max_{N \leq M} \mathcal{R}(\varphi, N, \delta) \log M.$$



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## 2. Auxiliary Results

We collect here the lemmas we will use to show the theorem. The first was proved by Huxley and Sargos in [3, 4] using divided differences and the reduction principle, and the second is due to a remarkable work by Filaseta and Trifonov ([1, Theorem 7]) who used divided differences and a very useful polynomial identity:

**Lemma 2.1.** *Let  $\delta > 0$  be a real number and  $\varphi \in C^k([N; 2N] \mapsto \mathbb{R})$  such that there exists a real number  $\lambda_k > 0$  such that*

$$|\varphi^{(k)}(x)| \asymp \lambda_k$$

for any real number  $x \in [N; 2N]$ . Then:

(i) If  $k \geq 2$ ,

$$\mathcal{R}(\varphi, N, \delta) \ll N\lambda_k^{\frac{2}{k(k+1)}} + N\delta^{\frac{2}{k(k-1)}} + (\delta\lambda_k^{-1})^{1/k} + 1.$$

(ii) Moreover, if  $k \geq 5$  and  $|\varphi^{(k-1)}(x)| \asymp \lambda_{k-1} = N\lambda_k$  for any real number  $x \in [N; 2N]$ , then:

$$\mathcal{R}(\varphi, N, \delta) \ll N\lambda_k^{\frac{2}{k(k+1)}} + N\delta^{\frac{2}{(k-1)(k-2)}} + (\delta\lambda_{k-1}^{-1})^{\frac{1}{k-1}} + 1.$$

**Lemma 2.2.** *Let  $k \geq 2$  be an integer and  $x, c_0 > 0, \delta \geq 0$  be real numbers ( $c_0$  sufficiently small) satisfying  $N^{k-1}\delta \leq c_0$  and  $N \leq x^{1/k}$ . Then we have:*

$$\mathcal{R}\left(\frac{x}{n^k}, N, \delta\right) \ll x^{\frac{1}{2k+1}} + x^{\frac{1}{6k+3}}\delta N^{\frac{6k^2+k-1}{6k+3}}.$$



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**Lemma 2.3.** Let  $f \in \mathcal{M}$  and  $z \geq 1$  any real number. Then:

(i)

$$\sum_{\substack{n \leq z \\ n \text{ squarefull}}} 1 < 3z^{1/2} \quad \text{and} \quad \sum_{\substack{n > z \\ n \text{ squarefull}}} \frac{1}{n} < 8z^{-1/2}.$$

(ii)

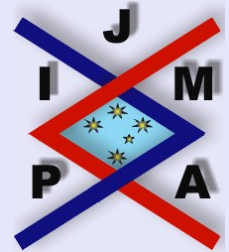
$$\sum_{n \leq z} |(f * \mu)(n)| < 4z^{1/2} \quad \text{and} \quad \sum_{n > z} \frac{|(f * \mu)(n)|}{n} < 8z^{-1/2}.$$

*Proof.* (i) Since every squarefull number  $n$  can be written in a unique way as  $n = a^2b^3$  with  $b$  squarefree, we have:

$$\sum_{\substack{n \leq z \\ n \text{ squarefull}}} 1 \leq \sum_{b \leq z^{1/3}} \sum_{a \leq \sqrt{zb^{-3}}} 1 < z^{1/2} \sum_{b=1}^{\infty} b^{-3/2} = \zeta\left(\frac{3}{2}\right) z^{1/2}$$

and the well-known inequality  $\zeta(\sigma) \leq \sigma/(\sigma - 1)$  gives the first part of the result. In the same way, let  $Z > z$  be any real number. We have:

$$\begin{aligned} \sum_{\substack{z < n \leq Z \\ n \text{ squarefull}}} \frac{1}{n} &\leq \sum_{b \leq z^{1/3}} \frac{1}{b^3} \sum_{\sqrt{zb^{-3}} < a \leq \sqrt{Zb^{-3}}} \frac{1}{a^2} + \sum_{z^{1/3} < b \leq Z^{1/3}} \frac{1}{b^3} \sum_{a \leq \sqrt{Zb^{-3}}} \frac{1}{a^2} \\ &< 2z^{-1/2} \sum_{b \leq z^{1/3}} \frac{1}{b^{3/2}} + \frac{\pi^2}{6} \sum_{b > z^{1/3}} \frac{1}{b^3} \\ &\leq 6z^{-1/2} + \frac{\pi^2}{6} z^{-2/3} < 8z^{-1/2}. \end{aligned}$$



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(ii) We set  $g := f * \mu$ . The hypothesis  $f(p) = 1$  implies  $|g(p)| = 0$  and, using multiplicativity,  $|g(n_1)| = 0$  for any positive squarefree integer  $n_1 > 1$ . Since any positive integer  $n$  can be written in a unique way as  $n = n_1 n_2$  with  $n_1$  squarefree,  $n_2$  squarefull and  $(n_1, n_2) = 1$ , we deduce that  $|g(n)| \neq 0$  if either  $n = 1$  or  $n > 1$  is squarefull. The result follows by using (i) and the fact that  $|g(n)| \leq 1$  for any positive integer  $n$ .

□

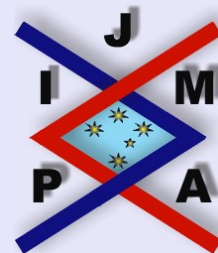
**Lemma 2.4.** *Let  $k \geq 1$  be an integer and  $\varepsilon > 0$  be a fixed real number. Then, for any positive integer  $d$ , we have:*

$$\tau_{(k)}(d) \leq \left( \frac{2}{e\varepsilon \log 2} \right)^{2^{1/\varepsilon}} d^{\varepsilon/k}.$$

*Proof.* We set  $c(\varepsilon) := 2^{2^{1/\varepsilon}} (e\varepsilon \log 2)^{-2^{1/\varepsilon}}$ . The bound  $\tau(d) \leq c(\varepsilon) d^\varepsilon$  is well-known (see [5]). Since any positive integer  $n$  can be represented in a unique way as  $n = qm^k$  with  $q$  a positive  $k$ -free integer, we have

$$\tau_{(k)}(d) = \tau(m) \leq c(\varepsilon) m^\varepsilon \leq c(\varepsilon) d^{\varepsilon/k}.$$

□



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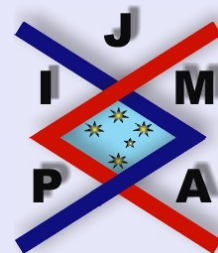
### 3. Proof of the Theorem

Let  $\varepsilon > 0$  be a fixed real number. We take  $g := f * \mu$  again, and we have:

$$\begin{aligned} \sum_{x < n \leq x+y} f(n) &= \sum_{x < n \leq x+y} \sum_{d|n} g(d) \\ &= \sum_{d \leq x+y} g(d) \left( \left[ \frac{x+y}{d} \right] - \left[ \frac{x}{d} \right] \right) \\ &= \sum_{d \leq y} + \sum_{y < d \leq x+y} := \Sigma_1 + \Sigma_2. \end{aligned}$$

1. Using Lemma 2.3, we have:

$$\begin{aligned} \Sigma_1 &= y \sum_{d \leq y} \frac{g(d)}{d} + O \left( \sum_{d \leq y} |g(d)| \right) \\ &= y \sum_{d=1}^{\infty} \frac{g(d)}{d} + O \left( y \sum_{d > y} \frac{|g(d)|}{d} \right) + O(y^{1/2}) \\ &= y \prod_p \left( 1 + \sum_{l=1}^{\infty} \frac{f(p^l) - f(p^{l-1})}{p^l} \right) + O(y^{1/2}) \\ &= y \prod_p \left( 1 + \left( 1 - \frac{1}{p} \right) \sum_{l=1}^{\infty} \frac{f(p^l)}{p^l} - \frac{1}{p} \right) + O(y^{1/2}) \\ &= y \mathcal{P}(f) + O(y^{1/2}). \end{aligned}$$



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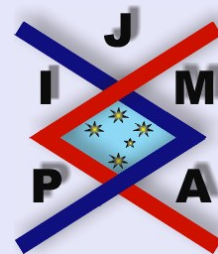
2. Writing again  $d = a^2 b^3$  with  $\mu_2(b) = 1$ , we have:

$$\begin{aligned}
 |\Sigma_2| &\leq \sum_{\substack{y < d \leq x+y \\ d \text{ squarefull}}} |g(d)| \left( \left[ \frac{x+y}{d} \right] - \left[ \frac{x}{d} \right] \right) \\
 &\leq \sum_{b \leq (x+y)^{1/3}} \sum_{\sqrt{\frac{y}{b^3}} < a \leq \sqrt{\frac{x+y}{b^3}}} \left( \left[ \frac{(x+y)b^{-3}}{a^2} \right] - \left[ \frac{xb^{-3}}{a^2} \right] \right) \\
 &= \sum_{b \leq (x+y)^{1/3}} \sum_{xb^{-3} < d \leq (x+y)b^{-3}} \sum_{\substack{a^2 | d \\ \sqrt{\frac{y}{b^3}} < a \leq \sqrt{\frac{x+y}{b^3}}} } 1 \\
 &\leq \sum_{b \leq (x+y)^{1/3}} \sum_{xb^{-3} < d \leq (x+y)b^{-3}} \tau_{(2)}(d) \\
 &\leq c(\varepsilon) (2x)^{\varepsilon/2} \sum_{b \leq (x+y)^{1/3}} \left( \left[ \frac{x+y}{b^3} \right] - \left[ \frac{x}{b^3} \right] \right) \\
 &\ll x^\varepsilon \max_{1 \leq B \leq (x+y)^{1/3}} \mathcal{R} \left( \frac{x}{b^3}, B, \frac{y}{B^3} \right),
 \end{aligned}$$

where we used (1.1), Lemma 2.4 (with  $k = 2$ ) and the inequality  $\log x \leq 2(\varepsilon)^{-1} x^{\varepsilon/2}$ . Now Lemma 2.1(i) with  $k = 3$  and  $\lambda_3 = xB^{-6}$  gives

$$|\Sigma_2| \ll x^\varepsilon (x^{1/6} + y^{1/3}) \ll x^{1/6+\varepsilon} + y^{1/2}$$

since  $y \geq x^{6\varepsilon}$ .



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3. We suppose now that  $y \leq x^{1/3}$ . One can improve the former estimation by using Lemma 2.2 instead of Lemma 2.1. The hypothesis  $N^{k-1}\delta \leq c_0$  compels us to be more careful:

$$\begin{aligned} & \sum_{b \leq (x+y)^{1/3}} \left( \left[ \frac{x+y}{b^3} \right] - \left[ \frac{x}{b^3} \right] \right) \\ &= \sum_{b \leq c_0^{-1}y} \left( \left[ \frac{x+y}{b^3} \right] - \left[ \frac{x}{b^3} \right] \right) + \sum_{c_0^{-1}y < b \leq (x+y)^{1/3}} \left( \left[ \frac{x+y}{b^3} \right] - \left[ \frac{x}{b^3} \right] \right) \\ &\ll \left\{ \max_{B \leq c_0^{-1}y} \mathcal{R} \left( \frac{x}{b^3}, B, \frac{y}{B^3} \right) + \max_{c_0^{-1}y < B \leq (x+y)^{1/3}} \mathcal{R} \left( \frac{x}{b^3}, B, \frac{y}{B^3} \right) \right\} \log x. \end{aligned}$$

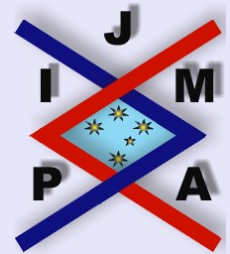
Lemma 2.1(ii) with  $k = 6$  for the first sum and Lemma 2.2 with  $k = 3$  for the second yield:

$$\begin{aligned} & \sum_{b \leq (x+y)^{1/3}} \left( \left[ \frac{x+y}{b^3} \right] - \left[ \frac{x}{b^3} \right] \right) \\ &\ll \{ x^{-1/5}y^{6/5} + y^{4/5} + x^{1/7} + x^{1/21}y^{2/3} \} \log x \end{aligned}$$

and one easily checks that

$$x^{-1/5}y^{6/5} + y^{4/5} \ll x^{1/21}y^{2/3}$$

if  $y \leq x^{1/3}$ . The proof of the theorem is complete.



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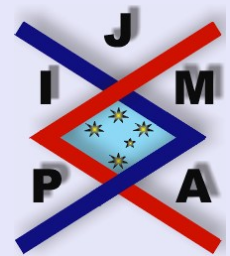
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