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ON SHORT SUMS OF CERTAIN MULTIPLICATIVE FUNCTIONS

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ABSTRACT. We use the recent theory of integer points close to a smooth curve developed by Huxley-Sargos and Filaseta-Trifonov to get an asymptotic formula for short sums of a class of multiplicative functions.

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1. INTRODUCTION AND NOTATION

Let $k \geqslant 2$ be an integer. A positive integer n is said to be $k-$*free* (resp. $k-$*full*) if, for any prime $p \mid n$, the $p-$adic valuation $v_p(n)$ of n satisfies $v_p(n) < k$ (resp. $v_p(n) \geqslant k$), and we use the terms *squarefree* or *squarefull* when $k = 2$. We denote by μ_k the multiplicative function defined by

$$\mu_k(n) := \begin{cases} 1, & \text{if } n \text{ is } k-\text{free} \\ 0, & \text{otherwise.} \end{cases}$$

Obtaining gap results for $k-$free (or $k-$full) numbers is a very famous problem in analytic number theory (see [1] and the references). The best estimation in this direction has been obtained by Filaseta and Trifonov ([1]) who showed that, for x sufficiently large, any interval of the type $\left]x; x + cx^{1/(2k+1)} \log x\right]$ ($c := c(k) > 0$) contains a $k-$free number.

A dual problem is to get an asymptotic formula for μ_k. This requires estimations for short sums of multiplicative functions, but such results are still relatively rare in the literature (see [2, 6]). In this paper, we are motivated by finding asymptotic results for short sums of the following class of arithmetical functions: define $\mathcal{M}$ to be the set of multiplicative functions f verifying $0 \leqslant f(n) \leqslant 1$ for any positive integer n and $f(p) = 1$ for any prime number p. If

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$f \in \mathcal{M}$, we set

$$\mathcal{P}(f) := \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{l=1}^{\infty} \frac{f\left(p^l\right)}{p^l}\right).$$

We prove:

Theorem 1.1. *Let $\varepsilon > 0$ and $x, y \geqslant 2$ be real numbers such that $x^{6\varepsilon} \leqslant y \leqslant x$. If $f \in \mathcal{M}$, we have as $x \to +\infty$:*

$$\sum_{x<n\leqslant x+y} f(n) = y\mathcal{P}(f) + O\left(x^{1/6+\varepsilon} + y^{1/2}\right).$$

Moreover, if $y \leqslant x^{1/3}$, then:

$$\sum_{x<n\leqslant x+y} f(n) = y\mathcal{P}(f) + O\left(x^{1/7+\varepsilon} + x^{1/21+\varepsilon} y^{2/3}\right).$$

For example, using this, we get:

$$\sum_{x<n\leqslant x+y} \mu_2(n) = \frac{6y}{\pi^2} + O\left(x^{1/6+\varepsilon} + y^{1/2}\right)$$

with $x^{6\varepsilon} \leqslant y \leqslant x$. The proof of this theorem uses a convolution argument, and the new theory of integer points close to a curve (see [1, 3, 4]) arises as the crucial point to estimate the difference of integer parts.

In what follows, $a, b, d, k, l, m, n, q, B, N$ will always denote positive integers, p will be a prime number and $[x]$ is the integral part of x. For any $X, Y > 0$, the notation $X \ll Y$ means there exists $C > 0$ such that $X \leqslant CY$. the notation $X \asymp Y$ means $X \ll Y$ and $Y \ll X$ simultaneously.

If f, g are two arithmetical functions, the Dirichlet convolution product $f * g$ is defined by

$$(f * g)(n) := \sum_{d|n} f(d)\, g(n/d).$$

$\mu(n)$ is the Möbius function, $\tau(n) := \sum_{d|n} 1$ the classical divisor function, and, more generally, $\tau_{(k)}(n)$ is the arithmetical function defined by

$$\tau_{(k)}(n) := \sum_{d^k|n} 1.$$

If φ is any smooth function and if $\delta > 0$ is any real number, we set

$$\mathcal{R}(\varphi, N, \delta) := \left|\{n \in]N; 2N] \cap \mathbb{Z},\ \exists\, m \in \mathbb{Z},\ |\varphi(n) - m| \leqslant \delta\}\right|.$$

Thus, if δ is sufficiently small, $\mathcal{R}(\varphi, N, \delta)$ counts the number of integer points close to the curve $y = \varphi(x)$ when $N < x \leqslant 2N$, and then

$$\mathcal{R}(\varphi, N, \delta) = \sum_{N<n\leqslant 2N} \left([\varphi(n) + \delta] - [\varphi(n) - \delta]\right).$$

We will therefore use the following principle: *let M be a large real number, $\varphi(n)$ and $\delta(n)$ two functions such that $\delta := \max_{n\leqslant M} \delta(n)$ satisfies $0 < \delta \leqslant 1/4$. Then we have*

$$\sum_{n\leqslant M} \left([\varphi(n) + \delta(n)] - [\varphi(n)]\right) \ll \max_{N\leqslant M} \mathcal{R}(\varphi, N, \delta) \log M. \tag{1.1}$$

2. AUXILIARY RESULTS

We collect here the lemmas we will use to show the theorem. The first was proved by Huxley and Sargos in [3, 4] using divided differences and the reduction principle, and the second is due to a remarkable work by Filaseta and Trifonov ([1, Theorem 7]) who used divided differences and a very useful polynomial identity:

Lemma 2.1. *Let $\delta > 0$ be a real number and $\varphi \in C^k\left(]N;2N] \mapsto \mathbb{R}\right)$ such that there exists a real number $\lambda_k > 0$ such that*

$$\left|\varphi^{(k)}(x)\right| \asymp \lambda_k$$

for any real number $x \in]N;2N]$. Then:

(i) *If $k \geqslant 2$,*

$$\mathcal{R}(\varphi, N, \delta) \ll N\lambda_k^{\frac{2}{k(k+1)}} + N\delta^{\frac{2}{k(k-1)}} + \left(\delta\lambda_k^{-1}\right)^{1/k} + 1.$$

(ii) *Moreover, if $k \geqslant 5$ and $\left|\varphi^{(k-1)}(x)\right| \asymp \lambda_{k-1} = N\lambda_k$ for any real number $x \in]N;2N]$, then:*

$$\mathcal{R}(\varphi, N, \delta) \ll N\lambda_k^{\frac{2}{k(k+1)}} + N\delta^{\frac{2}{(k-1)(k-2)}} + \left(\delta\lambda_{k-1}^{-1}\right)^{\frac{1}{k-1}} + 1.$$

Lemma 2.2. *Let $k \geqslant 2$ be an integer and $x, c_0 > 0$, $\delta \geqslant 0$ be real numbers (c_0 sufficiently small) satisfying $N^{k-1}\delta \leqslant c_0$ and $N \leqslant x^{1/k}$. Then we have:*

$$\mathcal{R}\left(\frac{x}{n^k}, N, \delta\right) \ll x^{\frac{1}{2k+1}} + x^{\frac{1}{6k+3}}\delta N^{\frac{6k^2+k-1}{6k+3}}.$$

Lemma 2.3. *Let $f \in \mathcal{M}$ and $z \geqslant 1$ any real number. Then:*

(i)

$$\sum_{\substack{n \leqslant z \\ n \text{ squarefull}}} 1 < 3z^{1/2} \quad \text{and} \quad \sum_{\substack{n > z \\ n \text{ squarefull}}} \frac{1}{n} < 8z^{-1/2}.$$

(ii)

$$\sum_{n \leqslant z} |(f * \mu)(n)| < 4z^{1/2} \quad \text{and} \quad \sum_{n > z} \frac{|(f * \mu)(n)|}{n} < 8z^{-1/2}.$$

Proof. (i) Since every squarefull number n can be written in a unique way as $n = a^2b^3$ with b squarefree, we have:

$$\sum_{\substack{n \leqslant z \\ n \text{ squarefull}}} 1 \leqslant \sum_{b \leqslant z^{1/3}} \sum_{a \leqslant \sqrt{zb^{-3}}} 1 < z^{1/2} \sum_{b=1}^{\infty} b^{-3/2} = \zeta\left(\frac{3}{2}\right) z^{1/2}$$

and the well-known inequality $\zeta(\sigma) \leqslant \sigma/(\sigma - 1)$ gives the first part of the result. In the same way, let $Z > z$ be any real number. We have:

$$\begin{aligned}\sum_{\substack{z < n \leqslant Z \\ n \text{ squarefull}}} \frac{1}{n} &\leqslant \sum_{b \leqslant z^{1/3}} \frac{1}{b^3} \sum_{\sqrt{zb^{-3}} < a \leqslant \sqrt{Zb^{-3}}} \frac{1}{a^2} + \sum_{z^{1/3} < b \leqslant Z^{1/3}} \frac{1}{b^3} \sum_{a \leqslant \sqrt{Zb^{-3}}} \frac{1}{a^2} \\ &< 2z^{-1/2} \sum_{b \leqslant z^{1/3}} \frac{1}{b^{3/2}} + \frac{\pi^2}{6} \sum_{b > z^{1/3}} \frac{1}{b^3} \\ &\leqslant 6z^{-1/2} + \frac{\pi^2}{6} z^{-2/3} < 8z^{-1/2}.\end{aligned}$$

(ii) We set $g := f * \mu$. The hypothesis $f(p) = 1$ implies $|g(p)| = 0$ and, using multiplicativity, $|g(n_1)| = 0$ for any positive squarefree integer $n_1 > 1$. Since any positive integer n can be written in a unique way as $n = n_1 n_2$ with n_1 squarefree, n_2 squarefull and $(n_1, n_2) = 1$, we deduce that $|g(n)| \neq 0$ if either $n = 1$ or $n > 1$ is squarefull. The result follows by using (i) and the fact that $|g(n)| \leqslant 1$ for any positive integer n.

□

Lemma 2.4. *Let $k \geqslant 1$ be an integer and $\varepsilon > 0$ be a fixed real number. Then, for any positive integer d, we have:*

$$\tau_{(k)}(d) \leqslant \left(\frac{2}{e\varepsilon \log 2}\right)^{2^{1/\varepsilon}} d^{\varepsilon/k}.$$

Proof. We set $c(\varepsilon) := 2^{2^{1/\varepsilon}} (e\varepsilon \log 2)^{-2^{1/\varepsilon}}$. The bound $\tau(d) \leqslant c(\varepsilon) d^{\varepsilon}$ is well-known (see [5]). Since any positive integer n can be represented in a unique way as $n = qm^k$ with q a positive $k-$free integer, we have

$$\tau_{(k)}(d) = \tau(m) \leqslant c(\varepsilon) m^{\varepsilon} \leqslant c(\varepsilon) d^{\varepsilon/k}.$$

□

3. PROOF OF THE THEOREM

Let $\varepsilon > 0$ be a fixed real number. We take $g := f * \mu$ again, and we have:

$$\begin{aligned}
\sum_{x<n\leqslant x+y} f(n) &= \sum_{x<n\leqslant x+y} \sum_{d|n} g(d) \\
&= \sum_{d\leqslant x+y} g(d) \left(\left[\frac{x+y}{d}\right] - \left[\frac{x}{d}\right] \right) \\
&= \sum_{d\leqslant y} + \sum_{y<d\leqslant x+y} := \Sigma_1 + \Sigma_2.
\end{aligned}$$

(1) Using Lemma 2.3, we have:

$$\begin{aligned}
\Sigma_1 &= y \sum_{d\leqslant y} \frac{g(d)}{d} + O\left(\sum_{d\leqslant y} |g(d)| \right) \\
&= y \sum_{d=1}^{\infty} \frac{g(d)}{d} + O\left(y \sum_{d>y} \frac{|g(d)|}{d} \right) + O\left(y^{1/2}\right) \\
&= y \prod_p \left(1 + \sum_{l=1}^{\infty} \frac{f(p^l) - f(p^{l-1})}{p^l} \right) + O\left(y^{1/2}\right) \\
&= y \prod_p \left(1 + \left(1 - \frac{1}{p}\right) \sum_{l=1}^{\infty} \frac{f(p^l)}{p^l} - \frac{1}{p} \right) + O\left(y^{1/2}\right) \\
&= y\mathcal{P}(f) + O\left(y^{1/2}\right).
\end{aligned}$$

(2) Writing again $d = a^2 b^3$ with $\mu_2(b) = 1$, we have:

$$|\Sigma_2| \leqslant \sum_{\substack{y<d\leqslant x+y \\ d \text{ squarefull}}} |g(d)| \left(\left[\frac{x+y}{d}\right] - \left[\frac{x}{d}\right] \right)$$

$$\begin{aligned}
&\leqslant \sum_{b\leqslant (x+y)^{1/3}} \sum_{\sqrt{\frac{y}{b^3}}<a\leqslant\sqrt{\frac{x+y}{b^3}}} \left(\left[\frac{(x+y)\,b^{-3}}{a^2}\right]-\left[\frac{xb^{-3}}{a^2}\right]\right)\\
&= \sum_{b\leqslant (x+y)^{1/3}} \sum_{xb^{-3}<d\leqslant (x+y)b^{-3}} \sum_{\substack{a^2|d\\ \sqrt{\frac{y}{b^3}}<a\leqslant\sqrt{\frac{x+y}{b^3}}}} 1\\
&\leqslant \sum_{b\leqslant (x+y)^{1/3}} \sum_{xb^{-3}<d\leqslant (x+y)b^{-3}} \tau_{(2)}(d)\\
&\leqslant c(\varepsilon)\,(2x)^{\varepsilon/2} \sum_{b\leqslant (x+y)^{1/3}} \left(\left[\frac{x+y}{b^3}\right]-\left[\frac{x}{b^3}\right]\right)\\
&\ll x^{\varepsilon} \max_{1\leqslant B\leqslant (x+y)^{1/3}} \mathcal{R}\left(\frac{x}{b^3},B,\frac{y}{B^3}\right),
\end{aligned}$$

where we used (1.1), Lemma 2.4 (with $k=2$) and the inequality $\log x \leqslant 2\,(e\varepsilon)^{-1}\,x^{\varepsilon/2}$. Now Lemma 2.1$(i)$ with $k=3$ and $\lambda_3 = xB^{-6}$ gives

$$|\Sigma_2| \ll x^{\varepsilon}\left(x^{1/6}+y^{1/3}\right) \ll x^{1/6+\varepsilon}+y^{1/2}$$

since $y \geqslant x^{6\varepsilon}$.

(3) We suppose now that $y\leqslant x^{1/3}$. One can improve the former estimation by using Lemma 2.2 instead of Lemma 2.1. The hypothesis $N^{k-1}\delta\leqslant c_0$ compels us to be more careful:

$$\begin{aligned}
&\sum_{b\leqslant (x+y)^{1/3}} \left(\left[\frac{x+y}{b^3}\right]-\left[\frac{x}{b^3}\right]\right)\\
&\quad= \sum_{b\leqslant c_0^{-1}y} \left(\left[\frac{x+y}{b^3}\right]-\left[\frac{x}{b^3}\right]\right) + \sum_{c_0^{-1}y<b\leqslant (x+y)^{1/3}} \left(\left[\frac{x+y}{b^3}\right]-\left[\frac{x}{b^3}\right]\right)\\
&\quad\ll \left\{\max_{B\leqslant c_0^{-1}y} \mathcal{R}\left(\frac{x}{b^3},B,\frac{y}{B^3}\right) + \max_{c_0^{-1}y<B\leqslant (x+y)^{1/3}} \mathcal{R}\left(\frac{x}{b^3},B,\frac{y}{B^3}\right)\right\}\log x.
\end{aligned}$$

Lemma 2.1(ii) with $k=6$ for the first sum and Lemma 2.2 with $k=3$ for the second yield:

$$\sum_{b\leqslant (x+y)^{1/3}} \left(\left[\frac{x+y}{b^3}\right]-\left[\frac{x}{b^3}\right]\right) \ll \left\{x^{-1/5}y^{6/5}+y^{4/5}+x^{1/7}+x^{1/21}y^{2/3}\right\}\log x$$

and one easily checks that

$$x^{-1/5}y^{6/5}+y^{4/5} \ll x^{1/21}y^{2/3}$$

if $y\leqslant x^{1/3}$. The proof of the theorem is complete.

REFERENCES

[1] M. FILASETA AND O. TRIFONOV, The distribution of fractional parts with applications to gap results in Number Theory, *Proc. London Math. Soc.*, **73** (1996), 241–278.

[2] A. HILDEBRAND, Multiplicative functions in short intervals, *Can. J. Math.*, **3** (1987), 646–672.

[3] M.N. HUXLEY AND P. SARGOS, Points entiers au voisinage d'une courbe plane de classe C^n, *Acta Arith.*, **69** (1995), 359–366.

[4] M.N. HUXLEY AND P. SARGOS, Points entiers au voisinage d'une courbe plane de classe C^n II, l*es prépublications de l'Institut Elie Cartan de Nancy* (1997).

[5] D.S. MITRINOVIĆ AND J. SÁNDOR (in cooperation with B. Crstici), *Handbook of Number Theory*, Kluwer Academic Publisher (1996).

[6] P. SHIU, A Brun-Titchmarsh theorem for multiplicative functions, *J. Reine Angew. Math.*, **313** (1980), 161–170.