COMPOSITION OPERATORS BETWEEN GENERALLY WEIGHTED BLOCH SPACES OF POLYDISK

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Abstract: Let ϕ be a holomorphic self-map of the open unit polydisk U^n in $\mathbb{C}^n$ and $p,\ q > 0$. In this paper, the generally weighted Bloch spaces $B^p_{\log}(U^n)$ are introduced, and the boundedness and compactness of composition operator C_ϕ from $B^p_{\log}(U^n)$ to $B^q_{\log}(U^n)$ are investigated.

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1. Introduction

Suppose that D is a domain in $\mathbb{C}^n$ and ϕ a holomorphic self-map of D. We denote by $H(D)$ the space of all holomorphic functions on D and define the composition operator C_ϕ on $H(D)$ by $C_\phi f = f \circ \phi$.

The theory of composition operators on various classical spaces, such as Hardy and Bergman spaces on the unit disk U in the finite complex plane $\mathbb{C}$ has been studied. However, the multivariable situation remains mysterious. It is well known in [3] and [5] that the restriction of C_ϕ to Hardy or standard weighted Bergman spaces on U is always bounded by the Littlewood subordination principle. At the same time, Cima, Stanton and Wogen confirmed in [1] that the multivariable situation is much different from the classical case (i.e., the composition operators on the Hardy space of holomorphic functions on the open unit ball of $\mathbb{C}^2$ as well as on many other spaces of holomorphic functions over a domain of $\mathbb{C}^n$ can be unbounded, even when $n = 1$ in [6]). Therefore, it would be of interest to pursue the function-theoretical or geometrical characterizations of those maps ϕ which induce bounded or compact composition operators. In this paper, we will pursue the function-theoretic conditions of those holomorphic self-maps ϕ of U^n which induce bounded or compact composition operators from a generally weighted $p-$Bloch space to a $q-$Bloch space with $p,\ q > 0$.

For $n \in \mathbb{N}$, we denote by U^n the open unit polydisk in $\mathbb{C}^n$:

$$U^n = \{z = (z_1, z_2, \ldots, z_n) : |z_j| < 1, j = 1, 2, \ldots, n\},$$

and

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w_j}, \quad |z| = \sqrt{\langle z, z\rangle}$$

for any $z = (z_1, z_2, \ldots, z_n)$, $w = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is

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said to be an n multi-index if $\alpha_i \in \mathbb{N}$, written by $\alpha \in \mathbb{N}^n$. For $\alpha \in \mathbb{N}^n$, we write $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ and $z_i^0 = 1$, $1 \le i \le n$ for convenience. For $z, w \in \mathbb{C}^n$, we denote $[z,w]_j = z$ when $j = 0$, $[z,w]_j = w$ when $j = n$, and

$$[z,w]_j = (z_1, z_2, \dots, z_{n-j}, w_{n-j+1}, \dots, w_n)$$

when $j \in \{1, 2, \dots, n-1\}$. Then $[z,w]_{n-j} = w$ when $j = 1$, and $[z,w]_{n-j+1} = w$ when $j = n+1$, for $j = 2, 3, \dots, n$,

$$[z,w]_{n-j+1} = (z_1, z_2, \dots, z_{j-1}, w_j, \dots, w_n).$$

For any $a \in \mathbb{C}$ and $z_j' = (z_1, z_2, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$, we write

$$(a, z_j') = (z_1, z_2, \dots, z_{j-1}, a, z_{j+1}, \dots, z_n).$$

Moreover, we adopt the notation $(z^{[j']})_{j' \in \mathbb{N}}$ for an arbitrary subsequence of $(z^{[j]})_{j \in \mathbb{N}}$.

Recall that the Bloch space $B(U^n)$ is the vector space of all $f \in H(U^n)$ satisfying

$$b_1(f) = \sup_{z \in U^n} Q_f(z) < \infty,$$

where

$$Q_f(z) = \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{|\langle \nabla f(z), \bar{u} \rangle|}{\sqrt{H(z,u)}}, \qquad \nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z) \right)$$

and the Bergman metric $H : U^n \times \mathbb{C}^n \to [0, \infty)$ on U^n is

$$H(z,u) = \sum_{k=1}^{n} \frac{|u_k|^2}{1 - |z_k|^2}$$

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(for example see [9], [15]). It is easy to verify that both $|f(0)| + b_1(f)$ and

$$\|f\|_B = |f(0)| + \sup_{z \in U^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)$$

are equivalent norms on $B(U^n)$. In [10], [12] and [8], some characterizations of the Bloch space $B(U^n)$ have been given.

In a recent paper [2], a generalized Bloch space has been introduced, the $p-$Bloch space: for $p > 0$, a function $f \in H(U^n)$ belongs to the $p-$Bloch space $B^p(U^n)$ if there is some $M \in [0, \infty)$ such that

$$\sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)^p \leq M, \qquad \forall z \in U^n.$$

The references [13] to [7] studied these spaces and the operators in them.

Dana D. Clahane et al. in [2] proved the following two results:

Theorem A. *Let ϕ be a holomorphic self-map of U^n and $p, q > 0$. The following statements are equivalent:*

(a) C_ϕ is a bounded operator from $B^p(U^n)$ to $B^q(U^n)$;

(b) There is $M \geq 0$ such that

$$\sum_{k,\ l=1}^{n} \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \frac{(1 - |z_k|^2)^q}{(1 - |\phi_l(z)|^2)^p} \leq M, \qquad \forall z \in U^n. \tag{1.1}$$

Theorem B. *Let ϕ be a holomorphic self-map of U^n and $p,\ q > 0$. If condition (1.1) and*

$$\lim_{\phi(z) \to \partial U^n} \sum_{k,\ l=1}^{n} \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \frac{(1 - |z_k|^2)^q}{(1 - |\phi_l(z)|^2)^p} = 0 \tag{1.2}$$

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hold, then C_ϕ is a compact operator from $B^p(U^n)$ to $B^q(U^n)$.

Now we introduce the generally weighted Bloch space $B^p_{\log}(U^n)$.

For $p > 0$, a function $f \in H(U)$ belongs to the generally weighted $p-$Bloch space $B^p_{\log}(U^n)$ if there is some $M \in [0, \infty)$ such that

$$\sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)^p \log \frac{2}{1 - |z_k|^2} \leq M, \qquad \forall z \in U^n.$$

Its norm in $B^p_{\log}(U^n)$ is defined by

$$\|f\|_{B^p_{\log}} = |f(0)| + \sup_{z \in U^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)^p \log \frac{2}{1 - |z_k|^2}.$$

In this paper, we mainly characterize the boundedness and compactness of the composition operators between $B^p_{\log}(U^n)$ and $B^q_{\log}(U^n)$, and extend some corresponding results in [2] and [11] in several ways.

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2. Main Results and their Proofs

First, we have the following lemma:

Lemma 2.1. *Let $f \in B^p_{\log}(U^n)$ and $z \in U^n$, then:*

(a) $|f(z)| \le \left(1 + \frac{n}{(1-p)\log 2}\right) \|f\|_{B^p_{\log}}$, *when* $0 < p < 1$;

(b) $|f(z)| \le \left(\frac{1}{2\log 2} + \frac{1}{2n\log 2}\right) \sum_{k=1}^n \log \frac{4}{1-|z_k|^2} \|f\|_{B^p_{\log}}$, *when* $p = 1$;

(c) $|f(z)| \le \left(\frac{1}{n} + \frac{2^{p-1}}{(p-1)\log 2}\right) \sum_{k=1}^n \frac{1}{(1-|z_k|^2)^{p-1}} \|f\|_{B^p_{\log}}$, *when* $p > 1$.

Proof. Let $p > 0,\ z \in U^n$, from the definition of $\|\cdot\|_{B^p_{\log}}$ we have $|f(0)| \le \|f\|_{B^p_{\log}}$ and

$$\left|\frac{\partial f}{\partial z_k}(z)\right| \le \frac{\|f\|_{B^p_{\log}}}{(1-|z_k|^2)^p \log \frac{2}{1-|z_k|^2}} \le \frac{\|f\|_{B^p_{\log}}}{(1-|z_k|^2)^p \log 2} \tag{2.1}$$

for every $z \in U^n$ and $k \in \{1, 2, \ldots, n\}$. Notice that

$$\begin{aligned} f(z) - f(0) &= \sum_{k=1}^n f([0,z]_{n-k+1}) - f([0,z]_{n-k}) \\ &= \sum_{k=1}^n z_k \int_0^1 \frac{\partial f([0,(tz_k, z'_k)]_{n-k+1})}{\partial z_k} dt \end{aligned}$$

and then from the inequality (2.1), it follows that

$$|f(z)| \le |f(0)| + \sum_{k=1}^n \frac{|z_k|}{\log 2} \int_0^1 \frac{\|f\|_{B^p_{\log}}}{(1-|tz_k|^2)^p} dt$$

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$$\leq \|f\|_{B^p_{\log}} + \frac{\|f\|_{B^p_{\log}}}{\log 2} \sum_{k=1}^{n} \int_0^{|z_k|} \frac{1}{(1-t^2)^p} dt. \tag{2.2}$$

For $p = 1$, we have:

$$\sum_{k=1}^{n} \int_0^{|z_k|} \frac{1}{(1-t^2)^p} dt = \sum_{k=1}^{n} \frac{1}{2} \log \frac{1+|z_k|}{1-|z_k|} \leq \sum_{k=1}^{n} \frac{1}{2} \log \frac{4}{1-|z_k|^2}. \tag{2.3}$$

If $p > 0$ and $p \neq 1$, then

$$\sum_{k=1}^{n} \int_0^{|z_k|} \frac{1}{(1-t^2)^p} dt \leq \sum_{k=1}^{n} \int_0^{|z_k|} \frac{1}{(1-t)^p} dt = \sum_{k=1}^{n} \frac{1-(1-|z_k|)^{1-p}}{1-p}. \tag{2.4}$$

Now for (a), from (2.4),

$$\sum_{k=1}^{n} \int_0^{|z_k|} \frac{1}{(1-t^2)^p} dt \leq \frac{1}{1-p}. \tag{2.5}$$

From (2.2) and (2.5), it follows that

$$|f(z)| \leq \left(1 + \frac{n}{(1-p)\log 2}\right) \|f\|_{B^p_{\log}}.$$

For (b), Since $\log \frac{4}{1-|z_k|^2} > \log 4 = 2\log 2$ for each $k \in \{1, 2, \ldots, n\}$, then

$$1 < \frac{1}{2n \log 2} \sum_{k=1}^{n} \log \frac{4}{1-|z_k|^2}. \tag{2.6}$$

Combining (2.2), (2.3) and (2.6) we get

$$|f(z)| \leq \left(\frac{1}{2\log 2} + \frac{1}{2n\log 2}\right) \sum_{k=1}^{n} \log \frac{4}{1-|z_k|^2} \|f\|_{B^p_{\log}}.$$

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For (c), from (2.4) we have

$$\sum_{k=1}^{n}\int_0^{|z_k|}\frac{1}{(1-t^2)^p}dt \le \sum_{k=1}^{n}\frac{1-(1-|z_k|)^{p-1}}{(p-1)(1-|z_k|)^{p-1}} \le \sum_{k=1}^{n}\frac{2^{p-1}}{(p-1)(1-|z_k|^2)^{p-1}}. \tag{2.7}$$

By (2.2) and (2.7), we obtain

$$|f(z)| \le \|f\|_{B^p_{\log}} + \frac{2^{p-1}}{(p-1)\log 2}\sum_{k=1}^{n}\frac{1}{(1-|z_k|^2)^{p-1}}\|f\|_{B^p_{\log}}$$
$$\le \left(\frac{1}{n}+\frac{2^{p-1}}{(p-1)\log 2}\right)\sum_{k=1}^{n}\frac{1}{(1-|z_k|^2)^{p-1}}\|f\|_{B^p_{\log}}.$$

$\square$

Lemma 2.2. *For $p>0,\ l\in\{1,2,\dots,n\}$ and $w\in U$, the function $f_w^l:\overline{U^n}\to\mathbb{C}$,*

$$f_w^l(z)=\int_0^{z_l}\frac{1}{(1-\overline{w}t)^p\log\frac{2}{1-\overline{w}t}}dt$$

belongs to $B^p_{\log}(U^n)$.

Proof. Let $k,l\in\{1,2,\dots,n\}$ and $w\in U$, then

$$\frac{\partial f_w^l}{\partial z_k}=0, \qquad \forall z\in U^n, k\neq l \tag{2.8}$$

and

$$\frac{\partial f_w^l}{\partial z_l}(z)=\frac{1}{(1-\overline{w}z_l)^p\log\frac{2}{1-\overline{w}z_l}}, \qquad \forall z\in U^n. \tag{2.9}$$

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An easy estimate shows that there is $0 < M < +\infty$ such that

$$\frac{(1-|\overline{w}z|)^p \log \frac{2}{1-|\overline{w}z|}}{|1-\overline{w}z|^p \log \frac{2}{|1-\overline{w}z|}} \leq M, \qquad \forall z, w \in U.$$

Therefore, by (2.8) and (2.9), we have

$$\begin{aligned}
|f_w^l(0)| &+ \sum_{k=1}^{n} \left| \frac{\partial f_w^l}{\partial z_k}(z) \right| (1-|z_k|^2)^p \log \frac{2}{1-|z_k|^2} \\
&= \frac{(1-|z_l|^2)^p \log \frac{2}{1-|z_l|^2}}{|1-\overline{w}z_l|^p \left| \log \frac{2}{1-\overline{w}z_l} \right|} \\
&\leq \frac{(1-|z_l|^2)^p \log \frac{2}{1-|z_l|^2}}{(1-|\overline{w}z_l|)^p \log \frac{2}{1-|\overline{w}z_l|}} \cdot \frac{(1-|\overline{w}z_l|)^p \log \frac{2}{1-|\overline{w}z_l|}}{|1-\overline{w}z_l|^p \log \frac{2}{|1-\overline{w}z_l|}} \\
&\leq \frac{2^p}{pe \log 2} \cdot M < +\infty
\end{aligned}$$

and thus $\{f_w^l : w \in U,\ l \in \{1, 2, \dots, n\}\} \subset B_{\log}^p(U^n)$. □

Theorem 2.3. *Let ϕ be a holomorphic self-map of the open unit polydisk U^n and $p, q > 0$, then the following statements are equivalent:*

(a) C_ϕ is a bounded operator from $B_{\log}^p(U^n)$ and $B_{\log}^q(U^n)$;

(b) There is an $M > 0$ such that

$$\sum_{k,l=1}^{n} \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \cdot \frac{(1-|z_k|^2)^q}{(1-|\phi_l(z)|^2)^p} \cdot \frac{\log \frac{2}{1-|z_k|^2}}{\log \frac{2}{1-|\phi_l(z)|^2}} \leq M, \qquad \forall z \in U^n. \tag{2.10}$$

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Proof. Firstly, assume that (b) is true. By Lemma 2.1, there is a $C > 0$ such that for all $f \in B^p_{\log}(U^n)$,

$$|f(\phi(0))| \leq C\|f\|_{B^p_{\log}}. \tag{2.11}$$

Then for all $z \in U^n$,

$$\begin{aligned}\sum_{k=1}^{n}\left|\frac{\partial(C_\phi f)}{\partial z_k}(z)\right|&(1-|z_k|^2)^q \log\frac{2}{1-|z_k|^2}\\ &\leq \sum_{l=1}^{n}\left|\frac{\partial f}{\partial \xi_l}(\phi(z))\right|(1-|\phi_l(z)|^2)^p \log\frac{2}{1-|\phi_l(z)|^2}\\ &\qquad\times\sum_{k=1}^{n}\left|\frac{\partial \phi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^q}{(1-|\phi_l(z)|^2)^p}\cdot\frac{\log\frac{2}{1-|z_k|^2}}{\log\frac{2}{1-|\phi_l(z)|^2}}\\ &\leq M\|f\|_{B^p_{\log}},\end{aligned}$$

and (a) is obtained.

Conversely, let $l \in \{1, 2, \ldots, n\}$, if (a) is true, i.e. there is a $C \geq 0$ such that

$$\|C_\phi f\| \leq C\|f\|_{B^p_{\log}}, \qquad \forall f \in B^p_{\log}(U^n), \tag{2.12}$$

then, by Lemma 2.2 and (2.12), there is a $Q > 0$ such that

$$\sum_{k=1}^{n}\left|\sum_{l=1}^{n}\frac{\partial f_w^l}{\partial \xi_l}(\phi(z))\cdot\frac{\partial \phi_l}{\partial z_k}(z)\right|(1-|z_k|^2)^q \log\frac{2}{1-|z_k|^2} \leq CQ, \qquad \forall w \in U, z \in U^n.$$

Letting $w = \phi(z)$, and using (2.8) and (2.9), we have

$$\sum_{l,k=1}^{n}\left|\frac{\partial \phi_l}{\partial z_k}(z)\right|\cdot\frac{(1-|z_k|^2)^q}{(1-|\phi_l(z)|^2)^p}\cdot\frac{\log\frac{2}{1-|z_k|^2}}{\log\frac{2}{1-|\phi_l(z)|^2}} \leq CQ.$$

□

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Lemma 2.4. *Let $\phi : U^n \to U^n$ be holomorphic and $p,\ q > 0$, then C_ϕ is compact from $B^p_{\log}(U^n)$ to $B^q_{\log}(U^n)$ if and only if for any bounded sequence $(f_j)_{j\in\mathbb{N}}$ in $B^p_{\log}(U^n)$, when $f_j \to 0$ uniformly on compacta in U^n, then $\|C_\phi f_j\|_{B^q_{\log}} \to 0$ as $j \to \infty$.*

Proof. Assume that C_ϕ is compact and $(f_j)_{j\in\mathbb{N}}$ is a bounded sequence in $B^p_{\log}(U^n)$ with $f_j \to 0$ uniformly on compacta in U^n. If the contrary is true, then there is a subsequence $(f_{jm})_{m\in\mathbb{N}}$ and a $\delta > 0$ such that $\|C_\phi f_{jm}\|_{B^q_{\log}} \geq \delta$ for all $m \in \mathbb{N}$. Due to the compactness of C_ϕ, we choose a subsequence $(f_{jml} \circ \phi)_{l\in\mathbb{N}}$ of $(C_\phi f_{jm})_{m\in\mathbb{N}} = (f_{jm} \circ \phi)_{m\in\mathbb{N}}$ and some $g \in B^p_{\log}(U^n)$, such that

$$\lim_{l\to\infty} \|f_{jml} \circ \phi - g\|_{B^q_{\log}} = 0. \tag{2.13}$$

Since Lemma 2.1 implies that for any compact subset $K \subset U^n$, there is a $C_k \geq 0$ such that

$$|f_{jml}(\phi(z)) - g(z)| \leq C_k \|f_{jml} \circ \phi - g\|_{B^q_{\log}}, \qquad \forall l \in \mathbb{N},\ z \in K. \tag{2.14}$$

By (2.13), $f_{jml} \circ \phi - g \to 0$ uniformly on compact subset in U^n. Since $f_{jml}\phi(z) \to 0$ as $l \to \infty$ for each $z \in U^n$, and by (2.14), then $g = 0$; (2.13) shows

$$\lim_{l\to\infty} \|C_\phi(f_{jml})\|_{B^q_{\log}} = 0,$$

it gives a contradiction.

Conversely, assume that $(g_j)_{j\in\mathbb{N}}$ is a sequence in $B^p_{\log}(U^n)$ such that $\|g_j\|_{B^p_{\log}} \leq M$ for all $j \in \mathbb{N}$. Lemma 2.1 implies that if $(g_j)_{j\in\mathbb{N}}$ is uniformly bounded on any compact subset in U^n and normal by Montel's theorem, then there is a subsequence $(g_{jm})_{m\in\mathbb{N}}$ of $(g_j)_{j\in\mathbb{N}}$ which converges uniformly on compacta in U^n to some $g \in H(U^n)$. It follows that $\frac{\partial g_{jm}}{\partial z_l} \to \frac{\partial g}{\partial z_l}$ uniformly on compacta in U^n for each $l \in$

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$\{1, 2, \ldots, n\}$. Thus $g \in B^p_{\log}(U^n)$ with $\|g_{jm} - g\|_{B^p_{\log}} \le M + \|g\|_{B^p_{\log}} < \infty$ and $g_{jm} - g$ converges to 0 on compacta in U^n, so by the hypotheses, $g_{jm} \circ \phi \to g \circ \phi$ in $B^q_{\log}(U^n)$. Therefore C_ϕ is a compact operator from $B^p_{\log}(U^n)$ to $B^q_{\log}(U^n)$. □

Lemma 2.5. *If for every $f \in B^p_{\log}(U^n)$, $C_\phi f$ belongs to $B^q_{\log}(U^n)$, then $\phi^\alpha \in B^q_{\log}(U^n)$ for each n-multi-index α.*

Proof. As is well known, every polynomial $p_\alpha : \mathbb{C}^n \to \mathbb{C}$ defined by $p_\alpha(z) = z^\alpha$ is in $B^p_{\log}(U^n)$. Thus, by the assumption $C_\phi(z^\alpha) = \phi^\alpha \in B^q_{\log}(U^n)$. □

Theorem 2.6. *Suppose that $p,\ q > 0$, $\phi : U^n \to U^n$ is a holomorphic self-map such that $\phi_k \in B^q_{\log}(U^n)$ for each $k \in \{1, 2, \ldots, n\}$ and*

$$\lim_{\phi(z)\to\partial U^n} \sum_{k,l=1}^{n} \left|\frac{\partial\phi_l}{\partial z_k}(z)\right| \cdot \frac{(1-|z_k|^2)^q}{(1-|\phi_l(z)|^2)^p} \cdot \frac{\log\frac{2}{1-|z_k|^2}}{\log\frac{2}{1-|\phi_l(z)|^2}} = 0, \tag{2.15}$$

then C_ϕ is a compact operator from $B^p_{\log}(U^n)$ to $B^q_{\log}(U^n)$.

Proof. Let $(f_j)_{j\in\mathbb{N}}$ be a sequence in $B^p_{\log}(U^n)$ with $f_j \to 0$ uniformly on compacta in U^n and

$$\|f_j\|_{B^p_{\log}} \le C, \qquad \forall j \in \mathbb{N}. \tag{2.16}$$

By Lemma 2.4, it suffices to show that

$$\lim_{j\to\infty} \|C_\phi f_j\|_{B^q_{\log}} = 0. \tag{2.17}$$

Notice that if $\|\phi_m\|_{B^q_{\log}} = 0$ for all $m \in \{1, 2, \ldots, n\}$, then $\phi = 0$ and C_ϕ has finite rank. Therefore, we can assume $C > 0$ and $\|\phi_m\|_{B^q_{\log}} > 0$ for some $m \in$

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$\{1, 2, \ldots, n\}$. Now let $\varepsilon > 0$, from (2.15), there is an $r \in (0,1)$ such that

$$\sum_{k,l=1}^{n} \left|\frac{\partial \phi_l}{\partial z_k}(z)\right| \cdot \frac{(1-|z_k|^2)^q}{(1-|\phi_l(z)|^2)^p} \cdot \frac{\log \frac{2}{1-|z_k|^2}}{\log \frac{2}{1-|\phi_l(z)|^2}} < \frac{\varepsilon}{2C} \tag{2.18}$$

for all $z \in U^n$ satisfying $d(\phi(z), \partial U^n) < r$. By using a subsequence and the chain rule for derivatives, (2.16) and (2.18) guarantee that for all such z and $j \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{k=1}^{n} \left|\frac{\partial (C_\phi f)}{\partial z_k}(z)\right| (1-|z_k|^2)^q \log \frac{2}{1-|z_k|^2} \\
&\qquad \le \sum_{l=1}^{n} \left|\frac{\partial f}{\partial \xi_l}(\phi(z))\right| (1-|\phi_l(z)|^2)^p \log \frac{2}{1-|\phi_l(z)|^2} \\
&\qquad\qquad \times \sum_{k=1}^{n} \left|\frac{\partial \phi_l}{\partial z_k}(z)\right| \cdot \frac{(1-|z_k|^2)^q}{(1-|\phi_l(z)|^2)^p} \cdot \frac{\log \frac{2}{1-|z_k|^2}}{\log \frac{2}{1-|\phi_l(z)|^2}} \\
&\qquad \le C \cdot \frac{\varepsilon}{2C} = \frac{\varepsilon}{2}.
\end{aligned}$$

To obtain the same estimate in the case $d(\phi(z), \partial U^n) \ge r$, let $E_r = \{w : d(w, \partial U^n) \ge r\}$. Since E_r is compact, by the hypothesis, $(f_j)_{j\in\mathbb{N}}$ and the sequences of partial derivatives $\left(\frac{\partial f_j}{\partial z_l}\right)_{j\in\mathbb{N}}$ converge to 0 uniformly on E_r, respectively. Then

$$\begin{aligned}
&\sum_{k=1}^{n} \left|\frac{\partial (C_\phi f_j)}{\partial z_k}(z)\right| (1-|z_k|^2)^q \log \frac{2}{1-|z_k|^2} \\
&\qquad \le \sum_{k=1}^{n} \left|\frac{\partial f_j}{\partial \xi_l}(\phi(z))\right| \cdot \sum_{k=1}^{n} \left|\frac{\partial \phi_l}{\partial z_k}(z)\right| \cdot (1-|z_k|^2)^q \log \frac{2}{1-|z_k|^2}
\end{aligned}$$

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$$\leq \sum_{l=1}^{n} \left| \frac{\partial f_j}{\partial \xi_l}(\phi(z)) \right| \cdot \|\phi_l\|_{B^q_{\log}}$$

$$\leq \sum_{l=1}^{n} \sup_{w \in E_r} \left| \frac{\partial f_j}{\partial \xi_l}(w) \right| \cdot \|\phi_l\|_{B^q_{\log}} \leq \frac{\varepsilon}{2} \quad (\text{as } j \to +\infty).$$

Since $\{\phi(0)\}$ is compact, we have $f_j(\phi(0)) \to 0$ as $j \to \infty$, and $\|C_\phi f_j\|_{B^q_{\log}} \to 0$ as $j \to \infty$, thus C_ϕ is a compact operator from $B^p_{\log}(U^n)$ to $B^q_{\log}(U^n)$. □

Theorem 2.7. *Let ϕ be a holomorphic self-map of U^n and $p,\ q > 0$. If conditions (2.10) and (2.15) hold, then C_ϕ is a compact operator from $B^p_{\log}(U^n)$ to $B^q_{\log}(U^n)$.*

Proof. If (2.10) is true, then C_ϕ is bounded from $B^p_{\log}(U^n)$ to $B^q_{\log}(U^n)$ by Theorem 2.3, and $\phi_k \in B^q_{\log}(U^n)$ for each $k \in \{1, 2, \dots, n\}$ by Lemma 2.5. The proof follows on applying (2.15) and Theorem 2.6. □

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