

# COMPOSITION OPERATORS BETWEEN GENERALLY WEIGHTED BLOCH SPACES OF POLYDISK

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*Abstract:* Let  $\phi$  be a holomorphic self-map of the open unit polydisk  $U^n$  in  $\mathbb{C}^n$  and  $p, q > 0$ . In this paper, the generally weighted Bloch spaces  $B_{\log}^p(U^n)$  are introduced, and the boundedness and compactness of composition operator  $C_\phi$  from  $B_{\log}^p(U^n)$  to  $B_{\log}^q(U^n)$  are investigated.

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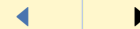
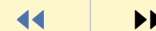
Composition Operators

Haiying Li, Peide Liu and Maofa Wang

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## Composition Operators

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# 1. Introduction

Suppose that  $D$  is a domain in  $\mathbb{C}^n$  and  $\phi$  a holomorphic self-map of  $D$ . We denote by  $H(D)$  the space of all holomorphic functions on  $D$  and define the composition operator  $C_\phi$  on  $H(D)$  by  $C_\phi f = f \circ \phi$ .

The theory of composition operators on various classical spaces, such as Hardy and Bergman spaces on the unit disk  $U$  in the finite complex plane  $\mathbb{C}$  has been studied. However, the multivariable situation remains mysterious. It is well known in [3] and [5] that the restriction of  $C_\phi$  to Hardy or standard weighted Bergman spaces on  $U$  is always bounded by the Littlewood subordination principle. At the same time, Cima, Stanton and Wogen confirmed in [1] that the multivariable situation is much different from the classical case (i.e., the composition operators on the Hardy space of holomorphic functions on the open unit ball of  $\mathbb{C}^2$  as well as on many other spaces of holomorphic functions over a domain of  $\mathbb{C}^n$  can be unbounded, even when  $n = 1$  in [6]). Therefore, it would be of interest to pursue the function-theoretical or geometrical characterizations of those maps  $\phi$  which induce bounded or compact composition operators. In this paper, we will pursue the function-theoretic conditions of those holomorphic self-maps  $\phi$  of  $U^n$  which induce bounded or compact composition operators from a generally weighted  $p$ -Bloch space to a  $q$ -Bloch space with  $p, q > 0$ .

For  $n \in \mathbb{N}$ , we denote by  $U^n$  the open unit polydisk in  $\mathbb{C}^n$  :

$$U^n = \{z = (z_1, z_2, \dots, z_n) : |z_j| < 1, j = 1, 2, \dots, n\},$$

and

$$\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j, \quad |z| = \sqrt{\langle z, z \rangle}$$

for any  $z = (z_1, z_2, \dots, z_n)$ ,  $w = (w_1, w_2, \dots, w_n)$  in  $\mathbb{C}^n$ .  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  is



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said to be an  $n$  multi-index if  $\alpha_i \in \mathbb{N}$ , written by  $\alpha \in \mathbb{N}^n$ . For  $\alpha \in \mathbb{N}^n$ , we write  $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$  and  $z_i^0 = 1$ ,  $1 \leq i \leq n$  for convenience. For  $z, w \in \mathbb{C}^n$ , we denote  $[z, w]_j = z$  when  $j = 0$ ,  $[z, w]_j = w$  when  $j = n$ , and

$$[z, w]_j = (z_1, z_2, \dots, z_{n-j}, w_{n-j+1}, \dots, w_n)$$

when  $j \in \{1, 2, \dots, n-1\}$ . Then  $[z, w]_{n-j} = w$  when  $j = 1$ , and  $[z, w]_{n-j+1} = w$  when  $j = n+1$ , for  $j = 2, 3, \dots, n$ ,

$$[z, w]_{n-j+1} = (z_1, z_2, \dots, z_{j-1}, w_j, \dots, w_n).$$

For any  $a \in \mathbb{C}$  and  $z'_j = (z_1, z_2, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$ , we write

$$(a, z'_j) = (z_1, z_2, \dots, z_{j-1}, a, z_{j+1}, \dots, z_n).$$

Moreover, we adopt the notation  $(z^{[j']})_{j' \in \mathbb{N}}$  for an arbitrary subsequence of  $(z^{[j]})_{j \in \mathbb{N}}$ .

Recall that the Bloch space  $B(U^n)$  is the vector space of all  $f \in H(U^n)$  satisfying

$$b_1(f) = \sup_{z \in U^n} Q_f(z) < \infty,$$

where

$$Q_f(z) = \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{|\langle \nabla f(z), \bar{u} \rangle|}{\sqrt{H(z, u)}}, \quad \nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z) \right)$$

and the Bergman metric  $H : U^n \times \mathbb{C}^n \rightarrow [0, \infty)$  on  $U^n$  is

$$H(z, u) = \sum_{k=1}^n \frac{|u_k|^2}{1 - |z_k|^2}$$

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(for example see [9], [15]). It is easy to verify that both  $|f(0)| + b_1(f)$  and

$$\|f\|_B = |f(0)| + \sup_{z \in U^n} \sum_{k=1}^n \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)$$

are equivalent norms on  $B(U^n)$ . In [10], [12] and [8], some characterizations of the Bloch space  $B(U^n)$  have been given.

In a recent paper [2], a generalized Bloch space has been introduced, the  $p$ -Bloch space: for  $p > 0$ , a function  $f \in H(U^n)$  belongs to the  $p$ -Bloch space  $B^p(U^n)$  if there is some  $M \in [0, \infty)$  such that

$$\sum_{k=1}^n \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)^p \leq M, \quad \forall z \in U^n.$$

The references [13] to [7] studied these spaces and the operators in them.

Dana D. Clahane et al. in [2] proved the following two results:

**Theorem A.** *Let  $\phi$  be a holomorphic self-map of  $U^n$  and  $p, q > 0$ . The following statements are equivalent:*

(a)  $C_\phi$  is a bounded operator from  $B^p(U^n)$  to  $B^q(U^n)$ ;

(b) There is  $M \geq 0$  such that

$$(1.1) \quad \sum_{k, l=1}^n \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \frac{(1 - |z_k|^2)^q}{(1 - |\phi_l(z)|^2)^p} \leq M, \quad \forall z \in U^n.$$

**Theorem B.** *Let  $\phi$  be a holomorphic self-map of  $U^n$  and  $p, q > 0$ . If condition (1.1) and*

$$(1.2) \quad \lim_{\phi(z) \rightarrow \partial U^n} \sum_{k, l=1}^n \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \frac{(1 - |z_k|^2)^q}{(1 - |\phi_l(z)|^2)^p} = 0$$



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hold, then  $C_\phi$  is a compact operator from  $B^p(U^n)$  to  $B^q(U^n)$ .

Now we introduce the generally weighted Bloch space  $B_{\log}^p(U^n)$ .

For  $p > 0$ , a function  $f \in H(U)$  belongs to the generally weighted  $p$ -Bloch space  $B_{\log}^p(U^n)$  if there is some  $M \in [0, \infty)$  such that

$$\sum_{k=1}^n \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)^p \log \frac{2}{1 - |z_k|^2} \leq M, \quad \forall z \in U^n.$$

Its norm in  $B_{\log}^p(U^n)$  is defined by

$$\|f\|_{B_{\log}^p} = |f(0)| + \sup_{z \in U^n} \sum_{k=1}^n \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)^p \log \frac{2}{1 - |z_k|^2}.$$

In this paper, we mainly characterize the boundedness and compactness of the composition operators between  $B_{\log}^p(U^n)$  and  $B_{\log}^q(U^n)$ , and extend some corresponding results in [2] and [11] in several ways.



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## 2. Main Results and their Proofs

First, we have the following lemma:

**Lemma 2.1.** *Let  $f \in B_{\log}^p(U^n)$  and  $z \in U^n$ , then:*

- (a)  $|f(z)| \leq \left(1 + \frac{n}{(1-p)\log 2}\right) \|f\|_{B_{\log}^p}$ , when  $0 < p < 1$ ;
- (b)  $|f(z)| \leq \left(\frac{1}{2\log 2} + \frac{1}{2n\log 2}\right) \sum_{k=1}^n \log \frac{4}{1-|z_k|^2} \|f\|_{B_{\log}^p}$ , when  $p = 1$ ;
- (c)  $|f(z)| \leq \left(\frac{1}{n} + \frac{2^{p-1}}{(p-1)\log 2}\right) \sum_{k=1}^n \frac{1}{(1-|z_k|^2)^{p-1}} \|f\|_{B_{\log}^p}$ , when  $p > 1$ .

*Proof.* Let  $p > 0$ ,  $z \in U^n$ , from the definition of  $\|\cdot\|_{B_{\log}^p}$  we have  $|f(0)| \leq \|f\|_{B_{\log}^p}$  and

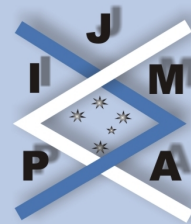
$$(2.1) \quad \left| \frac{\partial f}{\partial z_k}(z) \right| \leq \frac{\|f\|_{B_{\log}^p}}{(1-|z_k|^2)^p \log \frac{2}{1-|z_k|^2}} \leq \frac{\|f\|_{B_{\log}^p}}{(1-|z_k|^2)^p \log 2}$$

for every  $z \in U^n$  and  $k \in \{1, 2, \dots, n\}$ . Notice that

$$\begin{aligned} f(z) - f(0) &= \sum_{k=1}^n f([0, z]_{n-k+1}) - f([0, z]_{n-k}) \\ &= \sum_{k=1}^n z_k \int_0^1 \frac{\partial f([0, (tz_k, z'_k)]_{n-k+1})}{\partial z_k} dt \end{aligned}$$

and then from the inequality (2.1), it follows that

$$|f(z)| \leq |f(0)| + \sum_{k=1}^n \frac{|z_k|}{\log 2} \int_0^1 \frac{\|f\|_{B_{\log}^p}}{(1-|tz_k|^2)^p} dt$$



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$$(2.2) \quad \leq \|f\|_{B_{\log}^p} + \frac{\|f\|_{B_{\log}^p}}{\log 2} \sum_{k=1}^n \int_0^{|z_k|} \frac{1}{(1-t^2)^p} dt.$$

For  $p = 1$ , we have:

$$(2.3) \quad \sum_{k=1}^n \int_0^{|z_k|} \frac{1}{(1-t^2)^p} dt = \sum_{k=1}^n \frac{1}{2} \log \frac{1+|z_k|}{1-|z_k|} \leq \sum_{k=1}^n \frac{1}{2} \log \frac{4}{1-|z_k|^2}.$$

If  $p > 0$  and  $p \neq 1$ , then

$$(2.4) \quad \sum_{k=1}^n \int_0^{|z_k|} \frac{1}{(1-t^2)^p} dt \leq \sum_{k=1}^n \int_0^{|z_k|} \frac{1}{(1-t)^p} dt = \sum_{k=1}^n \frac{1 - (1-|z_k|)^{1-p}}{1-p}.$$

Now for (a), from (2.4),

$$(2.5) \quad \sum_{k=1}^n \int_0^{|z_k|} \frac{1}{(1-t^2)^p} dt \leq \frac{1}{1-p}.$$

From (2.2) and (2.5), it follows that

$$|f(z)| \leq \left(1 + \frac{n}{(1-p)\log 2}\right) \|f\|_{B_{\log}^p}.$$

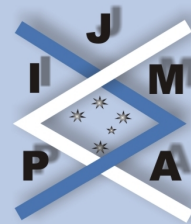
For (b), Since  $\log \frac{4}{1-|z_k|^2} > \log 4 = 2 \log 2$  for each  $k \in \{1, 2, \dots, n\}$ , then

$$(2.6) \quad 1 < \frac{1}{2n \log 2} \sum_{k=1}^n \log \frac{4}{1-|z_k|^2}.$$

Combining (2.2), (2.3) and (2.6) we get

$$|f(z)| \leq \left(\frac{1}{2 \log 2} + \frac{1}{2n \log 2}\right) \sum_{k=1}^n \log \frac{4}{1-|z_k|^2} \|f\|_{B_{\log}^p}.$$





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For (c), from (2.4) we have

$$(2.7) \quad \sum_{k=1}^n \int_0^{|z_k|} \frac{1}{(1-t^2)^p} dt \leq \sum_{k=1}^n \frac{1 - (1 - |z_k|)^{p-1}}{(p-1)(1 - |z_k|)^{p-1}} \\ \leq \sum_{k=1}^n \frac{2^{p-1}}{(p-1)(1 - |z_k|^2)^{p-1}}.$$

By (2.2) and (2.7), we obtain

$$|f(z)| \leq \|f\|_{B_{\log}^p} + \frac{2^{p-1}}{(p-1) \log 2} \sum_{k=1}^n \frac{1}{(1 - |z_k|^2)^{p-1}} \|f\|_{B_{\log}^p} \\ \leq \left( \frac{1}{n} + \frac{2^{p-1}}{(p-1) \log 2} \right) \sum_{k=1}^n \frac{1}{(1 - |z_k|^2)^{p-1}} \|f\|_{B_{\log}^p}.$$

□

**Lemma 2.2.** For  $p > 0$ ,  $l \in \{1, 2, \dots, n\}$  and  $w \in U$ , the function  $f_w^l : \overline{U}^n \rightarrow \mathbb{C}$ ,

$$f_w^l(z) = \int_0^{z_l} \frac{1}{(1 - \bar{w}t)^p \log \frac{2}{1 - \bar{w}t}} dt$$

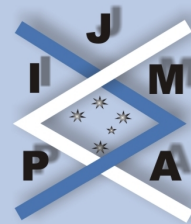
belongs to  $B_{\log}^p(U^n)$ .

*Proof.* Let  $k, l \in \{1, 2, \dots, n\}$  and  $w \in U$ , then

$$(2.8) \quad \frac{\partial f_w^l}{\partial z_k} = 0, \quad \forall z \in U^n, k \neq l$$

and

$$(2.9) \quad \frac{\partial f_w^l}{\partial z_l}(z) = \frac{1}{(1 - \bar{w}z_l)^p \log \frac{2}{1 - \bar{w}z_l}}, \quad \forall z \in U^n.$$



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An easy estimate shows that there is  $0 < M < +\infty$  such that

$$\frac{(1 - |\bar{w}z|)^p \log \frac{2}{1-|\bar{w}z|}}{|1 - \bar{w}z|^p \log \frac{2}{|1-\bar{w}z|}} \leq M, \quad \forall z, w \in U.$$

Therefore, by (2.8) and (2.9), we have

$$\begin{aligned} |f_w^l(0)| + \sum_{k=1}^n \left| \frac{\partial f_w^l}{\partial z_k}(z) \right| (1 - |z_k|^2)^p \log \frac{2}{1 - |z_k|^2} \\ = \frac{(1 - |z_l|^2)^p \log \frac{2}{1-|z_l|^2}}{|1 - \bar{w}z_l|^p \log \frac{2}{1-\bar{w}z_l}|} \\ \leq \frac{(1 - |z_l|^2)^p \log \frac{2}{1-|z_l|^2}}{(1 - |\bar{w}z_l|)^p \log \frac{2}{1-|\bar{w}z_l|}} \cdot \frac{(1 - |\bar{w}z_l|)^p \log \frac{2}{1-|\bar{w}z_l|}}{|1 - \bar{w}z_l|^p \log \frac{2}{|1-\bar{w}z_l|}} \\ \leq \frac{2^p}{pe \log 2} \cdot M < +\infty \end{aligned}$$

and thus  $\{f_w^l : w \in U, l \in \{1, 2, \dots, n\}\} \subset B_{\log}^p(U^n)$ . □

**Theorem 2.3.** *Let  $\phi$  be a holomorphic self-map of the open unit polydisk  $U^n$  and  $p, q > 0$ , then the following statements are equivalent:*

- (a)  $C_\phi$  is a bounded operator from  $B_{\log}^p(U^n)$  and  $B_{\log}^q(U^n)$ ;
- (b) There is an  $M > 0$  such that

$$(2.10) \quad \sum_{k,l=1}^n \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \cdot \frac{(1 - |z_k|^2)^q}{(1 - |\phi_l(z)|^2)^p} \cdot \frac{\log \frac{2}{1-|z_k|^2}}{\log \frac{2}{1-|\phi_l(z)|^2}} \leq M, \quad \forall z \in U^n.$$



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*Proof.* Firstly, assume that (b) is true. By Lemma 2.1, there is a  $C > 0$  such that for all  $f \in B_{\log}^p(U^n)$ ,

$$(2.11) \quad |f(\phi(0))| \leq C \|f\|_{B_{\log}^p}.$$

Then for all  $z \in U^n$ ,

$$\begin{aligned} & \sum_{k=1}^n \left| \frac{\partial(C_\phi f)}{\partial z_k}(z) \right| (1 - |z_k|^2)^q \log \frac{2}{1 - |z_k|^2} \\ & \leq \sum_{l=1}^n \left| \frac{\partial f}{\partial \xi_l}(\phi(z)) \right| (1 - |\phi_l(z)|^2)^p \log \frac{2}{1 - |\phi_l(z)|^2} \\ & \quad \times \sum_{k=1}^n \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \frac{(1 - |z_k|^2)^q}{(1 - |\phi_l(z)|^2)^p} \cdot \frac{\log \frac{2}{1 - |z_k|^2}}{\log \frac{2}{1 - |\phi_l(z)|^2}} \\ & \leq M \|f\|_{B_{\log}^p}, \end{aligned}$$

and (a) is obtained.

Conversely, let  $l \in \{1, 2, \dots, n\}$ , if (a) is true, i.e. there is a  $C \geq 0$  such that

$$(2.12) \quad \|C_\phi f\| \leq C \|f\|_{B_{\log}^p}, \quad \forall f \in B_{\log}^p(U^n),$$

then, by Lemma 2.2 and (2.12), there is a  $Q > 0$  such that

$$\sum_{k=1}^n \left| \sum_{l=1}^n \frac{\partial f_w^l}{\partial \xi_l}(\phi(z)) \cdot \frac{\partial \phi_l}{\partial z_k}(z) \right| (1 - |z_k|^2)^q \log \frac{2}{1 - |z_k|^2} \leq CQ, \quad \forall w \in U, z \in U^n.$$

Letting  $w = \phi(z)$ , and using (2.8) and (2.9), we have

$$\sum_{l,k=1}^n \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \cdot \frac{(1 - |z_k|^2)^q}{(1 - |\phi_l(z)|^2)^p} \cdot \frac{\log \frac{2}{1 - |z_k|^2}}{\log \frac{2}{1 - |\phi_l(z)|^2}} \leq CQ.$$

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**Lemma 2.4.** Let  $\phi : U^n \rightarrow U^n$  be holomorphic and  $p, q > 0$ , then  $C_\phi$  is compact from  $B_{\log}^p(U^n)$  to  $B_{\log}^q(U^n)$  if and only if for any bounded sequence  $(f_j)_{j \in \mathbb{N}}$  in  $B_{\log}^p(U^n)$ , when  $f_j \rightarrow 0$  uniformly on compacta in  $U^n$ , then  $\|C_\phi f_j\|_{B_{\log}^q} \rightarrow 0$  as  $j \rightarrow \infty$ .

*Proof.* Assume that  $C_\phi$  is compact and  $(f_j)_{j \in \mathbb{N}}$  is a bounded sequence in  $B_{\log}^p(U^n)$  with  $f_j \rightarrow 0$  uniformly on compacta in  $U^n$ . If the contrary is true, then there is a subsequence  $(f_{j_m})_{m \in \mathbb{N}}$  and a  $\delta > 0$  such that  $\|C_\phi f_{j_m}\|_{B_{\log}^q} \geq \delta$  for all  $m \in \mathbb{N}$ . Due to the compactness of  $C_\phi$ , we choose a subsequence  $(f_{j_{m_l}} \circ \phi)_{l \in \mathbb{N}}$  of  $(C_\phi f_{j_m})_{m \in \mathbb{N}} = (f_{j_m} \circ \phi)_{m \in \mathbb{N}}$  and some  $g \in B_{\log}^q(U^n)$ , such that

$$(2.13) \quad \lim_{l \rightarrow \infty} \|f_{j_{m_l}} \circ \phi - g\|_{B_{\log}^q} = 0.$$

Since Lemma 2.1 implies that for any compact subset  $K \subset U^n$ , there is a  $C_k \geq 0$  such that

$$(2.14) \quad |f_{j_{m_l}}(\phi(z)) - g(z)| \leq C_k \|f_{j_{m_l}} \circ \phi - g\|_{B_{\log}^q}, \quad \forall l \in \mathbb{N}, z \in K.$$

By (2.13),  $f_{j_{m_l}} \circ \phi - g \rightarrow 0$  uniformly on compact subset in  $U^n$ . Since  $f_{j_{m_l}} \phi(z) \rightarrow 0$  as  $l \rightarrow \infty$  for each  $z \in U^n$ , and by (2.14), then  $g = 0$ ; (2.13) shows

$$\lim_{l \rightarrow \infty} \|C_\phi(f_{j_{m_l}})\|_{B_{\log}^q} = 0,$$

it gives a contradiction.

Conversely, assume that  $(g_j)_{j \in \mathbb{N}}$  is a sequence in  $B_{\log}^p(U^n)$  such that  $\|g_j\|_{B_{\log}^p} \leq M$  for all  $j \in \mathbb{N}$ . Lemma 2.1 implies that if  $(g_j)_{j \in \mathbb{N}}$  is uniformly bounded on any compact subset in  $U^n$  and normal by Montel's theorem, then there is a subsequence  $(g_{j_m})_{m \in \mathbb{N}}$  of  $(g_j)_{j \in \mathbb{N}}$  which converges uniformly on compacta in  $U^n$  to some  $g \in H(U^n)$ . It follows that  $\frac{\partial g_{j_m}}{\partial z_i} \rightarrow \frac{\partial g}{\partial z_i}$  uniformly on compacta in  $U^n$  for each  $l \in$



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$\{1, 2, \dots, n\}$ . Thus  $g \in B_{\log}^p(U^n)$  with  $\|g_{jm} - g\|_{B_{\log}^p} \leq M + \|g\|_{B_{\log}^p} < \infty$  and  $g_{jm} - g$  converges to 0 on compacta in  $U^n$ , so by the hypotheses,  $g_{jm} \circ \phi \rightarrow g \circ \phi$  in  $B_{\log}^q(U^n)$ . Therefore  $C_\phi$  is a compact operator from  $B_{\log}^p(U^n)$  to  $B_{\log}^q(U^n)$ .  $\square$

**Lemma 2.5.** *If for every  $f \in B_{\log}^p(U^n)$ ,  $C_\phi f$  belongs to  $B_{\log}^q(U^n)$ , then  $\phi^\alpha \in B_{\log}^q(U^n)$  for each  $n$ -multi-index  $\alpha$ .*

*Proof.* As is well known, every polynomial  $p_\alpha : \mathbb{C}^n \rightarrow \mathbb{C}$  defined by  $p_\alpha(z) = z^\alpha$  is in  $B_{\log}^p(U^n)$ . Thus, by the assumption  $C_\phi(z^\alpha) = \phi^\alpha \in B_{\log}^q(U^n)$ .  $\square$

**Theorem 2.6.** *Suppose that  $p, q > 0$ ,  $\phi : U^n \rightarrow U^n$  is a holomorphic self-map such that  $\phi_k \in B_{\log}^q(U^n)$  for each  $k \in \{1, 2, \dots, n\}$  and*

$$(2.15) \quad \lim_{\phi(z) \rightarrow \partial U^n} \sum_{k,l=1}^n \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \cdot \frac{(1 - |z_k|^2)^q}{(1 - |\phi_l(z)|^2)^p} \cdot \frac{\log \frac{2}{1 - |z_k|^2}}{\log \frac{2}{1 - |\phi_l(z)|^2}} = 0,$$

then  $C_\phi$  is a compact operator from  $B_{\log}^p(U^n)$  to  $B_{\log}^q(U^n)$ .

*Proof.* Let  $(f_j)_{j \in \mathbb{N}}$  be a sequence in  $B_{\log}^p(U^n)$  with  $f_j \rightarrow 0$  uniformly on compacta in  $U^n$  and

$$(2.16) \quad \|f_j\|_{B_{\log}^p} \leq C, \quad \forall j \in \mathbb{N}.$$

By Lemma 2.4, it suffices to show that

$$(2.17) \quad \lim_{j \rightarrow \infty} \|C_\phi f_j\|_{B_{\log}^q} = 0.$$

Notice that if  $\|\phi_m\|_{B_{\log}^q} = 0$  for all  $m \in \{1, 2, \dots, n\}$ , then  $\phi = 0$  and  $C_\phi$  has finite rank. Therefore, we can assume  $C > 0$  and  $\|\phi_m\|_{B_{\log}^q} > 0$  for some  $m \in$



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$\{1, 2, \dots, n\}$ . Now let  $\varepsilon > 0$ , from (2.15), there is an  $r \in (0, 1)$  such that

$$(2.18) \quad \sum_{k,l=1}^n \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \cdot \frac{(1 - |z_k|^2)^q}{(1 - |\phi_l(z)|^2)^p} \cdot \frac{\log \frac{2}{1 - |z_k|^2}}{\log \frac{2}{1 - |\phi_l(z)|^2}} < \frac{\varepsilon}{2C}$$

for all  $z \in U^n$  satisfying  $d(\phi(z), \partial U^n) < r$ . By using a subsequence and the chain rule for derivatives, (2.16) and (2.18) guarantee that for all such  $z$  and  $j \in \mathbb{N}$ ,

$$\begin{aligned} & \sum_{k=1}^n \left| \frac{\partial(C_\phi f)}{\partial z_k}(z) \right| (1 - |z_k|^2)^q \log \frac{2}{1 - |z_k|^2} \\ & \leq \sum_{l=1}^n \left| \frac{\partial f}{\partial \xi_l}(\phi(z)) \right| (1 - |\phi_l(z)|^2)^p \log \frac{2}{1 - |\phi_l(z)|^2} \\ & \quad \times \sum_{k=1}^n \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \cdot \frac{(1 - |z_k|^2)^q}{(1 - |\phi_l(z)|^2)^p} \cdot \frac{\log \frac{2}{1 - |z_k|^2}}{\log \frac{2}{1 - |\phi_l(z)|^2}} \\ & \leq C \cdot \frac{\varepsilon}{2C} = \frac{\varepsilon}{2}. \end{aligned}$$

To obtain the same estimate in the case  $d(\phi(z), \partial U^n) \geq r$ , let  $E_r = \{w : d(w, \partial U^n) \geq r\}$ . Since  $E_r$  is compact, by the hypothesis,  $(f_j)_{j \in \mathbb{N}}$  and the sequences of partial derivatives  $\left(\frac{\partial f_j}{\partial z_l}\right)_{j \in \mathbb{N}}$  converge to 0 uniformly on  $E_r$ , respectively. Then

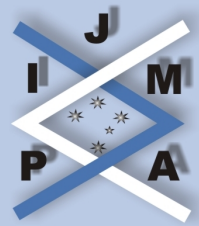
$$\begin{aligned} & \sum_{k=1}^n \left| \frac{\partial(C_\phi f_j)}{\partial z_k}(z) \right| (1 - |z_k|^2)^q \log \frac{2}{1 - |z_k|^2} \\ & \leq \sum_{k=1}^n \left| \frac{\partial f_j}{\partial \xi_l}(\phi(z)) \right| \cdot \sum_{k=1}^n \left| \frac{\partial \phi_l}{\partial z_k}(z) \right| \cdot (1 - |z_k|^2)^q \log \frac{2}{1 - |z_k|^2} \end{aligned}$$

$$\begin{aligned} &\leq \sum_{l=1}^n \left| \frac{\partial f_j}{\partial \xi_l}(\phi(z)) \right| \cdot \|\phi_l\|_{B_{\log}^q} \\ &\leq \sum_{l=1}^n \sup_{w \in E_r} \left| \frac{\partial f_j}{\partial \xi_l}(w) \right| \cdot \|\phi_l\|_{B_{\log}^q} \leq \frac{\varepsilon}{2} \quad (\text{as } j \rightarrow +\infty). \end{aligned}$$

Since  $\{\phi(0)\}$  is compact, we have  $f_j(\phi(0)) \rightarrow 0$  as  $j \rightarrow \infty$ , and  $\|C_\phi f_j\|_{B_{\log}^q} \rightarrow 0$  as  $j \rightarrow \infty$ , thus  $C_\phi$  is a compact operator from  $B_{\log}^p(U^n)$  to  $B_{\log}^q(U^n)$ .  $\square$

**Theorem 2.7.** *Let  $\phi$  be a holomorphic self-map of  $U^n$  and  $p, q > 0$ . If conditions (2.10) and (2.15) hold, then  $C_\phi$  is a compact operator from  $B_{\log}^p(U^n)$  to  $B_{\log}^q(U^n)$ .*

*Proof.* If (2.10) is true, then  $C_\phi$  is bounded from  $B_{\log}^p(U^n)$  to  $B_{\log}^q(U^n)$  by Theorem 2.3, and  $\phi_k \in B_{\log}^q(U^n)$  for each  $k \in \{1, 2, \dots, n\}$  by Lemma 2.5. The proof follows on applying (2.15) and Theorem 2.6.  $\square$



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