Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 4, Article 77, 2003

NOTES ON QI TYPE INTEGRAL INEQUALITIES

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Received 01 July, 2003; accepted 08 October, 2003
Communicated by F. Qi

ABSTRACT. By utilizing a reversed Hölder inequality and a reversed convolution inequality, some new integral inequalities which originate from an open problem posed in [F. Qi, *Several integral inequalities*, J. Inequal. Pure Appl. Math. **1**(2) (2000), Art. 19. Available online at http://jipam.vu.edu.au/v1n2.html/001_00.html. RGMIA Res. Rep. Coll. **2** (1999), no. 7, Art. 9, 1039–1042. Available online at http://rgmia.vu.edu.au/v2n7.html] are established.

Key words and phrases: Qi type integral inequality, Reversed Hölder inequality, Reversed convolution inequality.

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 39B62, 60E15.

1. INTRODUCTION

In [6] F. Qi posed the following open problem:
Under what conditions does the inequality

$$\int_a^b \left[f(x)\right]^t dx \geq \left(\int_a^b f(x)dx\right)^{t-1} \tag{1.1}$$

hold for $t > 1$?

In response, affirmative answers, generalizations, reversed forms, and interpretations of inequality (1.1) have been obtained by several mathematicians.

The paper [9] first gave an affirmative answer to this open problem using the integral version of Jensen's inequality and a lemma of convexity. The second affirmative answer to (1.1) was

ISSN (electronic): 1443-5756

The author would like to thank the anonymous referees for their valuable help to renew the first section, add and correct several references, and improve the typesetting of this paper.

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given by Towghi in [8]. Motivated by (1.1), Pogány [5] proved the following

$$\int_a^b \left[f(x)\right]^\alpha dx \ge \left(\int_a^b f(x)dx\right)^\beta \tag{1.2}$$

and its reversed form under assumptions of the bounds, depending on $b-a$, α and β, and convexity of f, which contains an answer to the above open problem and some reversed forms of (1.1).

In [3, 4], by employing a functional inequality which is an abstract generalization of the classical Jessen's inequality, the following functional inequality (1.4) was established, from which inequality (1.1), some integral inequalities, and an interesting discrete inequality involving sums can be deduced:

Let $\mathcal{L}$ be a linear vector space of real-valued functions, p and q be two real numbers such that $p \ge q \ge 1$. Assume that f and g are two positive functions in $\mathcal{L}$ and G is a positive linear form on $\mathcal{L}$ such that $G(g) > 0$, $fg \in \mathcal{L}$, and $gf^p \in \mathcal{L}$. If

$$[G(gf)]^{p-q} \ge [G(g)]^{p-1}, \tag{1.3}$$

then

$$G(gf^p) \ge [G(gf)]^q. \tag{1.4}$$

Very recently, Csiszár and Móri [1] interpreted inequality (1.2) in terms of moments as

$$E(X^\alpha) \ge C(EX)^\beta, \tag{1.5}$$

where $C = (b-a)^{\beta-1}$ and $X = f$ is a random variable. To demonstrate the power of the convexity method in probability theory, among other things, they found sharp conditions on the range of X, under which (1.5) or the converse inequality holds for fixed $0 < \beta < \alpha$. Hence, the results by Pogány [5] were improved slightly by a factor of at least $\left(\frac{3}{2}\right)^{\frac{\alpha}{\alpha-\beta}}$.

In this short note, by utilizing the reversed Hölder inequality in [2] and a reversed convolution inequality in [7], we establish some new Qi type integral inequalities which extend related results in references.

2. Two Lemmas

In order to prove our main results, the following two lemmas are necessary.

Lemma 2.1 ([2]). *For two positive functions f and g satisfying $0 < m \le \frac{f}{g} \le M < \infty$ on the set X, and for $p > 1$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\left(\int_X f d\mu\right)^{\frac{1}{p}} \left(\int_X g d\mu\right)^{\frac{1}{q}} \le A_{p,q}\left(\frac{m}{M}\right) \int_X f^{\frac{1}{p}} g^{\frac{1}{q}} d\mu, \tag{2.1}$$

where

$$A_{p,q}(t) = \frac{1}{p^{\frac{1}{p}} q^{\frac{1}{q}}} \cdot \frac{1-t}{t^{\frac{1}{pq}} \left(1-t^{\frac{1}{p}}\right)^{\frac{1}{p}} \left(1-t^{\frac{1}{q}}\right)^{\frac{1}{q}}}. \tag{2.2}$$

Inequality (2.1) is called the reverse Hölder inequality.

Lemma 2.2 ([7]). *For two positive functions f and g satisfying $0 < m \le \frac{f^p}{g^q} \le M < \infty$ on the set X, and for $p > 1$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\left(\int_X f^p d\mu\right)^{\frac{1}{p}} \left(\int_X g^q d\mu\right)^{\frac{1}{q}} \le \left(\frac{M}{m}\right)^{\frac{1}{pq}} \int_X fg d\mu. \tag{2.3}$$

3. MAIN RESULTS AND PROOFS

In this section, we will state our main results and give their proofs as follows.

Using Lemma 2.1 and an estimation due to Lars-Erik Persson, we obtain:

Theorem 3.1. *If* $0 < m \le f \le M < \infty$ *on* $[a,b]$*, then*

$$\int_{[a,b]} f^{\frac{1}{p}} d\mu \ge B \left(\int_{[a,b]} f d\mu \right)^{\frac{1}{p}-1}, \tag{3.1}$$

where $B = m[\mu(b) - \mu(a)]^{1+\frac{1}{q}} \left(\frac{m}{M}\right)^{\frac{1}{pq}}$ *and* $p > 1$*,* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*.*

Proof. In Lemma 2.1, putting $g \equiv 1$ yields

$$[\mu(b) - \mu(a)]^{\frac{1}{q}} \left(\int_{[a,b]} f d\mu \right)^{\frac{1}{p}} \le A_{p,q} \left(\frac{m}{M}\right) \int_{[a,b]} f^{\frac{1}{p}} d\mu, \tag{3.2}$$

and so

$$\int_{[a,b]} f^{\frac{1}{p}} d\mu \ge \frac{[\mu(b) - \mu(a)]^{\frac{1}{q}}}{A_{p,q}\left(\frac{m}{M}\right)} \left(\int_{[a,b]} f d\mu \right)^{\frac{1}{p}-1} \left(\int_{[a,b]} f d\mu \right). \tag{3.3}$$

Since $0 < m \le f$, we have

$$\int_{[a,b]} f^{\frac{1}{p}} d\mu \ge \frac{m\,[\mu(b) - \mu(a)]^{1+\frac{1}{q}}}{A_{p,q}\left(\frac{m}{M}\right)} \left(\int_{[a,b]} f d\mu \right)^{\frac{1}{p}-1}. \tag{3.4}$$

Now, using the estimation $A_{p,q}(t) < t^{-\frac{1}{pq}}$ due to Lars-Erik Persson (see [7]), we obtain inequality (3.1). □

Corollary 3.2. *Let* $p > 1$ *and* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. If*

$$m\left(\frac{m}{M}\right)^{\frac{1}{pq}} = \frac{1}{[\mu(b) - \mu(a)]^{1+\frac{1}{q}}} \tag{3.5}$$

and $0 < m \le f \le M < \infty$ *on* $[a,b]$*, then*

$$\int_{[a,b]} f^{\frac{1}{p}} d\mu \ge \left(\int_{[a,b]} f d\mu \right)^{\frac{1}{p}-1}. \tag{3.6}$$

Using Lemma 2.1, Lemma 2.2 and the estimation due to Lars-Erik Persson, we have the following:

Theorem 3.3. *Let* $p > 1$ *and* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. If* $0 < m^{\frac{1}{p}} \le f \le M^{\frac{1}{p}} < \infty$ *on* $[a,b]$*, then*

$$\left(\int_{[a,b]} f^{\frac{1}{p}} d\mu \right)^{p} \ge [\mu(b) - \mu(a)]^{\frac{p+1}{q}} \left(\frac{m}{M}\right)^{\frac{2}{pq}} \left(\int_{[a,b]} f^{p} d\mu \right)^{\frac{1}{p}}. \tag{3.7}$$

Proof. Putting $g \equiv 1$ into Lemma 2.2, we obtain

$$\left(\int_{[a,b]} f^{p} d\mu \right)^{\frac{1}{p}} [\mu(b) - \mu(a)]^{\frac{1}{q}} \le \left(\frac{m}{M}\right)^{-\frac{1}{pq}} \int_{[a,b]} f d\mu. \tag{3.8}$$

Therefore

$$\left(\int_{[a,b]} f^{p} d\mu \right)^{\frac{1}{p}} \le \left(\frac{m}{M}\right)^{-\frac{1}{pq}} [\mu(b) - \mu(a)]^{-\frac{1}{q}} \int_{[a,b]} f d\mu. \tag{3.9}$$

Again, substituting $g \equiv 1$ in the reverse Hölder inequality in Lemma 2.1 leads to

$$\left(\int_{[a,b]} f d\mu\right)^{\frac{1}{p}} \leq [\mu(b)-\mu(a)]^{-\frac{1}{q}} A_{p,q}\left(\frac{m^{\frac{1}{p}}}{M^{\frac{1}{p}}}\right)\int_{[a,b]} f^{\frac{1}{p}} d\mu,$$

and so,

$$\int_{[a,b]} f d\mu \leq [\mu(b)-\mu(a)]^{-\frac{p}{q}} A_{p,q}^{p}\left(\frac{m^{\frac{1}{p}}}{M^{\frac{1}{p}}}\right)\left(\int_{[a,b]} f^{\frac{1}{p}} d\mu\right)^{p}. \tag{3.10}$$

Combining (3.9) with (3.10), we obtain

$$\left(\int_{[a,b]} f^{\frac{1}{p}} d\mu\right)^{p} \geq [\mu(b)-\mu(a)]^{\frac{p+1}{q}}\left(\frac{m}{M}\right)^{\frac{1}{pq}} A_{p,q}^{-p}\left(\frac{m^{\frac{1}{p}}}{M^{\frac{1}{p}}}\right)\left(\int_{[a,b]} f^{p} d\mu\right)^{\frac{1}{p}}. \tag{3.11}$$

Then, by using the estimation $A_{p,q}(t) < t^{-\frac{1}{pq}}$ due to Lars-Erik Persson (see [7]), we obtain inequality (3.7). □

Corollary 3.4. *If* $0 < m^{\frac{1}{p}} \leq f \leq M^{\frac{1}{p}} < \infty$ *on* $[a,b]$ *and* $\frac{m}{M} = [\mu(b)-\mu(a)]^{\frac{-p(p+1)}{2}}$ *for* $p > 1$, *then*

$$\left(\int_{[a,b]} f^{\frac{1}{p}} d\mu\right)^{p} \geq \left(\int_{[a,b]} f^{p} d\mu\right)^{\frac{1}{p}}. \tag{3.12}$$

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