journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 8 (2007), Issue 2, Article 57, 9 pp.

A GENERALIZATION OF CONSTANTIN'S INTEGRAL INEQUALITY AND ITS DISCRETE ANALOGUE

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Received 21 March, 2007; accepted 08 May, 2007
Communicated by S.S. Dragomir

ABSTRACT. A generalization of Constantin's integral inequality and its discrete analogy are established. A discrete analogue of Okrasinsky's model for the infiltration phenomena of a fluid is also discussed to convey the usefulness of the discrete inequality obtained.

Key words and phrases: Nonlinear integral inequality, Discrete analogue, Bound on solutions.

2000 *Mathematics Subject Classification.* 26D10, 26D15, 39A12, 45D05.

1. INTRODUCTION

L. Ou-Iang [9] studied the boundedness of solutions for some nonautonomuous second order linear differential equations by means of a nonlinear integral inequality. This integral inequality had been frequently used by authors to obtain global existence, uniqueness and stability properties of various nonlinear differential equations. A number of generalizations and discrete analogues of this inequality and their new applications have appeared in the literature. See, for example, B.G. Pachpatte ([10] – [12]) and the present author [13][14] and the references given therein.

In 1996, A. Constantin [2] established the following interesting alternative result for a generalized Ou-Iang type integral inequality given by B. G. Pachpatte [12]:

Theorem A. *Let* $T > 0, k \geq 0$, *and* $u, f, g \in C([0,T], \mathbb{R}_+), \mathbb{R}_+ = [0,\infty)$. *Further, let* $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing,* $w(r) > 0$ *for* $r > 0$ *and* $\int_{r_0}^{\infty} \frac{ds}{w(s)} = \infty$ *hold for some number* $r_0 > 0$. *Then the integral inequality*

$$u^2(t) \leq k^2 + 2\int_0^t \left\{ f(s)u(s) \left[u(s) + \int_0^s g(\xi) w(u(\xi))\, d\xi \right] \right\} ds, \quad t \in [0,T]$$

The work that is described in this paper was jointly supported by grants from the National Natural Science Foundation of China (No. 50578064), the Natural Science Foundation of Guangdong Province, China (No.06025219), and the Science Foundation of Key Discipline of the State Council Office of Overseas Chinese Affairs of China.

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implies

$$u(t) \le k + \int_0^t f(s) G^{-1} \left\{ G(k) + \int_0^s [f(\xi) + g(\xi)]\, d\xi \right\} ds, \quad t \in [0, T],$$

where G^{-1} denotes the inverse function of G and

$$G(r) := \int_{r_0}^{r} \frac{ds}{w(s) + s}, \quad r \ge r_0,\ 1 > r_0 > 0.$$

Applying the above result and a topological transversality theorem, A. Granas [4] proved a nonlocal existence theorem for a certain class of initial value problems of nonlinear integrodifferential equations. We refer to D. O'Regan and M. Meehan [6] for more existence results obtained by means of topological transversality theorems.

The purpose of the present paper is to obtain a new generalization of Constantin's inequality and its discrete analogue. The integral inequality obtained can be used to study some more general initial value problems by following the same argument as that applied in Constantin [2]. A discrete analogue of W.Okrasinsky's mathematical model for the infiltration phenomena of a fluid (see [7] and [8]) is discussed to convey the usefulness of the discrete inequality given in the paper.

2. NONLINEAR INTEGRAL INEQUALITY

Theorem 2.1. *Let $u, c \in C(\mathbb{R}_+, \mathbb{R}_+)$ with c nondecreasing, and $\varphi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with φ' nonnegative and nondecreasing. Let $f(t,\xi), g(t,\xi), h(t,\xi) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing in t for every ξ fixed. Further, let $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing, $w(r) > 0$ for $r > 0$ and $\int_{r_0}^{\infty} \frac{ds}{w(s)} = \infty$ hold for some number $r_0 > 0$. Then the integral inequality*

$$\begin{aligned} \varphi[u(t)] \le c(t) + \int_0^t \Big\{ & f(t,s) \varphi'[u(s)] \\ & \times \left[u(s) + \int_0^s g(s,\xi) w(u(\xi))\, d\xi \right] + h(t,s) \varphi'[u(s)] \Big\} ds, \quad t \in [0, T], \end{aligned} \tag{2.1}$$

implies

$$\begin{aligned} u(t) \le K(t) + \int_0^t & f(t,s) \\ & \times G^{-1} \left\{ G(K(t)) + \int_0^s [f(t,\xi) + g(t,\xi)]\, d\xi \right\} ds, \quad t \in [0, T], \end{aligned} \tag{2.2}$$

herein

$$K(t) = \varphi^{-1}[c(t)] + \int_0^t h(t,s) ds, \tag{2.3}$$

G^{-1}, φ^{-1}denote the inverse function of G, φ,respectively, and

$$G(r) := \int_{r_0}^{r} \frac{ds}{w(s) + s}, \quad r \ge r_0,\ 1 > r_0 > 0. \tag{2.4}$$

Note that, by Constatin [1] the above function G is positive, strictly increasing and satisfies the condition $G(r) \to \infty$ as $r \to \infty$.

Proof. Letting $t = 0$ in (2.1), we observe that inequality (2.2) holds trivially for $t = 0$. Fixing an arbitrary number $t_0 \in (0, T)$,we define on $[0, t_0]$ a positive function $z(t)$ by

$$z(t) = c(t_0) + \varepsilon + \int_0^t \Big\{ f(t_0, s)\varphi'[u(s)] \times \left[u(s) + \int_0^s g(t_0, \xi) w(u(\xi))\, d\xi\right] + h(t_0, s)\varphi'[u(s)] \Big\} ds, \tag{2.5}$$

where $\varepsilon > 0$ is an arbitrary small constant. By inequality (2.1) we have

$$u(t) \le \varphi^{-1}[z(t)], \quad t \in [0, t_0]. \tag{2.6}$$

From (2.5) we derive by differentiation

$$\begin{aligned} z'(t) &= f(t_0, t)\varphi'[u(t)] \left[u(t) + \int_0^t g(t_0, \xi) w[u(\xi)]\, d\xi\right] + h(t_0, t)\varphi'[u(t)] \\ &\le \varphi'\left[\varphi^{-1}[z(t)]\right] \left\{ f(t_0, t)\left[\varphi^{-1}[z(t)] + \int_0^t g(t_0, \xi) w\left(\varphi^{-1}[z(\xi)]\right) d\xi\right] + h(t_0, t)\right\}, \end{aligned}$$

for $t \in [0, t_0]$, since φ' is nonnegative and nondecreasing. Hence we obtain

$$\begin{aligned} \frac{d}{dt}\varphi^{-1}[z(t)] &= \frac{z'(t)}{\varphi'[\varphi^{-1}[z(t)]]} \\ &\le f(t_0, t)\left[\varphi^{-1}[z(t)] + \int_0^t g(t_0, \xi) w\left(\varphi^{-1}[z(\xi)]\right) d\xi\right] + h(t_0, t),\ t \in [0, t_0], \end{aligned}$$

Integrating both sides of the last relation from 0 to t, we get

$$\begin{aligned} \varphi^{-1}[z(t)] \le \varphi^{-1}[z(0)] &+ \int_0^{t_0} h(t_0, s) ds \\ &+ \int_0^t f(t_0, s)\left[\varphi^{-1}[z(s)] + \int_0^s g(t_0, \xi) w\left(\varphi^{-1}[z(\xi)]\right) d\xi\right] ds,\ t \in [0, t_0]. \end{aligned}$$

Define a function $v(t), 0 \le t \le t_0$, by the right member of the last relation, we have

$$\varphi^{-1}[z(t)] \le v(t), \quad t \in [0, t_0], \tag{2.7}$$

where

$$v(0) = \varphi^{-1}[z(0)] + \int_0^{t_0} h(t_0, s) ds. \tag{2.8}$$

By differentiation we derive

$$\begin{aligned} v'(t) &= f(t_0, t)\left[\varphi^{-1}[z(t)] + \int_0^t g(t_0, \xi) w\left(\varphi^{-1}[z(\xi)]\right) d\xi\right] \\ &\le f(t_0, t)\left[v(t) + \int_0^t g(t_0, \xi) w(v(\xi))\, d\xi\right] = f(t_0, t)\Omega(t), \quad t \in [0, t_0]. \end{aligned} \tag{2.9}$$

where

$$\Omega(t) = \left[v(t) + \int_0^t g(t_0, \xi) w(v(\xi))\, d\xi\right].$$

Hence we have $v(t) \le \Omega(t)$,

$$\Omega(0) = v(0) = \varphi^{-1}[z(0)] + \int_0^{t_0} h(t_0, s) ds, \tag{2.10}$$

and

$$\begin{aligned}\Omega'(t) &= v'(t) + g(t_0,t)w\left(v(t)\right) \\ &\le f(t_0,t)\Omega(t) + g(t_0,t)w\left(\Omega(t)\right), \qquad t \in [0,t_0].\end{aligned}$$

Because $\Omega(t)$, and hence $w\left(\Omega(t)\right)$, is positive on $[0,t_0]$, the last inequality can be rewritten as

$$\frac{\Omega'(t)}{\Omega(t) + w\left(\Omega(t)\right)} \le f(t_0,t) + g(t_0,t), \quad t \in [0,t_0]. \tag{2.11}$$

Integrating both sides of the last relation from 0 to t and in view of the definition of G, we obtain

$$G\left[\Omega(t)\right] - G\left[\Omega(0)\right] \le \int_0^t \left[f(t_0,s) + g(t_0,s)\right] ds, \quad t \in [0,t_0].$$

By (2.10) and the fact that $G(r) \to \infty$ as $r \to \infty$,the last relation yields

$$\Omega(t) \le G^{-1}\left\{G\left[\varphi^{-1}[z(0)] + \int_0^{t_0} h(t_0,s)ds\right] + \int_0^t \left[f(t_0,s) + g(t_0,s)\right]\, ds\right\},\ t \in [0,t_0].$$

Substituting the last relation into (2.9), then integrating from 0 to t, we derive for $t \in [0,t_0]$ that

$$\begin{aligned}u(t) \le \varphi^{-1}\left[c(t_0) + \varepsilon\right] &+ \int_0^{t_0} h(t_0,s)ds \\ &+ \int_0^t f(t_0,s)G^{-1}\left\{G\left[\varphi^{-1}[c(t_0)+\varepsilon] + \int_0^{t_0} h(t_0,s)ds\right]\right. \\ &\qquad\qquad \left. + \int_0^s \left[f(t_0,\xi) + g(t_0,\xi)\right]d\xi\right\}ds,\end{aligned}$$

where we used the relation $u(t) \le \varphi^{-1}[z(t)] \le v(t) \le \Omega(t)$.

Taking $t = t_0$ and letting $\varepsilon \to 0$, from the last relation we have

$$u(t_0) \le K(t_0) + \int_0^{t_0} f(t_0,s)G^{-1}\left\{G\left[K(t_0)\right] + \int_0^s \left[f(t_0,\xi) + g(t_0,\xi)\right]\, d\xi\right\}\, ds,$$

where $K(t)$ is defined by (2.3). This means that the desired inequality (2.2) is valid when $t = t_0$. Since the choice of t_0 from $(0,T]$ is arbitrary, the proof of Theorem 2.1 is complete. □

If $w(r) = r$ holds in Theorem 2.1, the inequality (2.11) can be replaced by the following sharper relation

$$\frac{\Omega'(t)}{\Omega(t)} \le f(t_0,t) + g(t_0,t), \quad t \in [0,t_0],$$

and functions G,G^{-1} can be replaced by $H(r) = \ln\left(r/r_0\right)$, $H^{-1}(\eta) = r_0e^{\eta}$, respectively. Hence we derive the following:

Corollary 2.2. *Under the conditions of Theorem 2.1, the integral inequality*

$$\begin{aligned}\varphi[u(t)] \le c(t) + \int_0^t &\left\{ f(t,s)\varphi'[u(s)] \right. \\ &\left. \times \left[u(s) + \int_0^s g(s,\xi)u(\xi)d\xi\right] + h(t,s)\varphi'[u(s)]\right\}ds, \quad t \in [0,T],\end{aligned} \tag{2.12}$$

implies

$$u(t) \le \left(\varphi^{-1}[c(t)] + \int_0^t h(t,s)ds\right) \times \left(1 + \int_0^t f(t,s)\left\{\exp\int_0^s [f(t,\xi)+g(t,\xi)]\,d\xi\right\}ds\right), \quad t\in[0,T]. \tag{2.13}$$

If $\varphi(\eta) = \eta^p$, $p > 1$, $c(t) = k^p \ge 0$ and $f(t,s), g(t,s), h(t,s)$ do not depend on the variable t, by Theorem 2.1 we have the following:

Corollary 2.3. *Let $p > 1, k \ge 0$ be constants and $u, f, g \in C([0,T],\mathbb{R}_+)$. Then the integral inequality*

$$u^p(t) \le k^p + p\int_0^t \left\{f(s)u^{p-1}(s)\left[u(s) + \int_0^s g(\xi)w\left(u(\xi)\right)\,d\xi\right]\right\}ds, \quad t\in[0,T]$$

implies

$$u(t) \le k + \int_0^t f(s)G^{-1}\left\{G\left(k\right) + \int_0^s [f(\xi)+g(\xi)]\,d\xi\right\}ds, \quad t\in[0,T].$$

Remark 2.4. Clearly, Constantin's Theorem A is the special case $p = 2$ of the last result.

3. DISCRETE ANALOGUE

In this section we will establish a discrete analogue of Theorem 2.1. Denote by $\mathbb{N}$ the set of nonnegative integers and let $\mathbb{N}_0 = \{n \in \mathbb{N}\colon n \le M\}$ for some natural number M. For simplicity, we denote by $K(P,Q)$ the class of functions defined on set P with range in set Q. For a function $u \in K(\mathbb{N},\mathbb{R})$, $\mathbb{R} = (-\infty,\infty)$, we define the forward difference operator Δ by $\Delta u(n) = u(n+1) - u(n)$.

As usual, we suppose that the empty sum and empty product are zero and one, respectively . For instance,

$$\sum_{s=0}^{-1} p(s) = 0 \quad \text{and} \quad \prod_{s=0}^{-1} p(s) = 1$$

hold for any function $p(n)$, $n\in\mathbb{N}$.

Theorem 3.1. *Let the functions w, φ be as defined in Theorem 2.1 and $u, c \in K(\mathbb{N},\mathbb{R}_+)$ with $c(n)$ nondecreasing. Further, let $f(n,s), g(n,s), h(n,s) \in K(\mathbb{N}\times\mathbb{N},\mathbb{R}_+)$ be nondecreasing with respect to n for every s fixed. Then the discrete inequality*

$$\varphi[u(n)] \le c(n) + \sum_{s=0}^{n-1}\left\{f(n,s)\varphi'[u(s)] \times \left[u(s) + \sum_{\xi=0}^{s-1} g(s,\xi)w\left(u(\xi)\right)\right] + h(n,s)\varphi'[u(s)]\right\}, \quad n\in\mathbb{N}_0, \tag{3.1}$$

implies

$$u(n) \le L(n) + \sum_{s=0}^{n-1} f(n,s)G^{-1}\left\{G[L(n)] + \sum_{\xi=0}^{s-1}[f(n,\xi)+g(n,\xi)]\right\},\ n\in\mathbb{N}_0, \tag{3.2}$$

where G, G^{-1} are as defined in Theorem 2.1 and

$$L(n) := \varphi^{-1}[c(n)] + \sum_{s=0}^{n-1} h(n,s). \tag{3.3}$$

Proof. Fixing an arbitrary positive integer $m \in (0, M)$, we define on the set $J := \{0, 1, \dots, m\}$ a positive function $z(n) \in K(J, (0,\infty))$ by

$$z(n) = c(m) + \varepsilon + \sum_{s=0}^{n-1}\Bigg\{ f(m,s)\varphi'[u(s)] \times \left[u(s) + \sum_{\xi=0}^{s-1} g(m,\xi)w\left(u(\xi)\right)\right] + h(m,s)\varphi'[u(s)]\Bigg\},$$

where ε is an arbitrary positive constant, then $z(0) = c(m) + \varepsilon > 0$ and by (3.1) we have

$$u(n) \le \varphi^{-1}[z(n)], \quad n \in J. \tag{3.4}$$

Using the last relation, we derive

$$\begin{aligned}\Delta z(n) &= f(m,n)\varphi'[u(n)]\left[u(n) + \sum_{s=0}^{n-1} g(m,s)w\left(u(s)\right)\right] + h(m,n)\varphi'[u(n)] \\ &\le \varphi'\left[\varphi^{-1}[z(n)]\right] \\ &\quad \times \left\{ f(m,n)\left[\varphi^{-1}[z(n)] + \sum_{s=0}^{n-1} g(m,s)w\left(\varphi^{-1}[z(s)]\right)\right] + h(m,n)\right\}, \quad n \in J.\end{aligned}$$

By the mean value theorem and the last relation, we obtain

$$\begin{aligned}\Delta\varphi^{-1}[z(n)] &\le \frac{\Delta z(n)}{\varphi'\left[\varphi^{-1}[z(n)]\right]} \\ &\le f(m,n)\left[\varphi^{-1}[z(n)] + \sum_{s=0}^{n-1} g(m,s)w\left(\varphi^{-1}[z(s)]\right)\right] + h(m,n), \quad n \in J,\end{aligned}$$

since $\varphi'^{-1}$and $z(n)$ are nondecreasing. Substituting $n = \xi$ in the last relation and then summing over $\xi = 0, 1, 2, \dots, n-1$, we obtain

$$\begin{aligned}\varphi^{-1}[z(n)] \le \varphi^{-1}[z(0)] &+ \sum_{\xi=0}^{m-1} h(m,\xi) \\ &+ \sum_{\xi=0}^{n-1} f(m,\xi)\left[\varphi^{-1}[z(\xi)] + \sum_{s=0}^{\xi-1} g(m,s)w\left(\varphi^{-1}[z(s)]\right)\right],\end{aligned}$$

where $n \in J$, since $h(n,s)$ is nonnegative and $m \ge n$ holds. Now, defining by $v(n)$ the right member of the last relation, we have

$$v(0) = \varphi^{-1}[z(0)] + \sum_{\xi=0}^{m-1} h(m,\xi) = \varphi^{-1}[c(m)+\varepsilon] + \sum_{\xi=0}^{m-1} h(m,\xi)$$

and

$$\varphi^{-1}[z(n)] \le v(n), \quad n \in J. \tag{3.5}$$

By (3.5) we easily derive

$$\Delta v(n) = f(m,n)\left[\varphi^{-1}[z(n)] + \sum_{s=0}^{n-1} g(m,s)w\left(\varphi^{-1}[z(s)]\right)\right]$$
$$\leq f(m,n)\left[v(n) + \sum_{s=0}^{n-1} g(m,s)w\left(v(s)\right)\right], \quad n \in J,$$

or

$$\Delta v(n) \leq f(m,n)y(n), \quad n \in J, \tag{3.6}$$

where

$$y(n) := v(n) + \sum_{s=0}^{n-1} g(m,s)w\left(v(s)\right), \quad n \in J.$$

Clearly, $y(0) = v(0)$ holds and by (3.6) we have

$$\Delta y(n) \leq \Delta v(n) + g(m,n)w\left(v(n)\right) \leq \left[f(m,n) + g(m,n)\right]\left[y(n) + w\left(y(n)\right)\right],$$

i.e.,

$$\frac{\Delta y(n)}{y(n) + w\left(y(n)\right)} \leq f(m,n) + g(m,n), \quad n \in J.$$

Because $y(n), w(r)$ are positive and nondecreasing, we have

$$\int_{y(0)}^{y(n)} \frac{ds}{s + w(s)} \leq \sum_{s=0}^{n-1} \frac{\Delta y(s)}{y(s) + w(y(s))} \leq \sum_{s=0}^{n-1} [f(m,s) + g(m,s)],$$

or

$$G[y(n)] - G[y(0)] \leq \sum_{s=0}^{n-1} [f(m,s) + g(m,s)], \quad n \in J.$$

Since $G(r) \to \infty$ as $r \to \infty$, the last relation yields

$$y(n) \leq G^{-1}\left\{G\left[\varphi^{-1}[c(m)+\varepsilon] + \sum_{\xi=0}^{m-1} h(m,\xi)\right] + \sum_{s=0}^{n-1} [f(m,s) + g(m,s)]\right\},\ n \in J.$$

Substituting this relation into (3.6), setting $n = s$ and then summing over $s = 0, 1, \dots, n-1$, we have

$$v(n) \leq v(0) + \sum_{s=0}^{n-1} f(m,s)$$
$$\times G^{-1}\left\{G\left[\varphi^{-1}[c(m)+\varepsilon] + \sum_{\xi=0}^{m-1} h(m,\xi)\right] + \sum_{\xi=0}^{s-1} [f(m,\xi) + g(m,\xi)]\right\},\ n \in J.$$

Because $u(n) \leq \varphi^{-1}[z(n)] \leq v(n)$, $n \in J$, by letting $n = m$ and $\varepsilon \to 0$ in the last relation, we obtain

$$u(m) \leq L(m) + \sum_{s=0}^{m-1} f(m,s)G^{-1}\left\{G[L(m)] + \sum_{\xi=0}^{s-1} [f(m,\xi) + g(m,\xi)]\right\}.$$

This means that the desired inequality (3.2) is valid when $n = m$. Since $m \in (0, M)$ is chosen arbitrarily and by (3.1), inequality (3.2) holds also for $n = 0$. Thus the proof of Theorem 3.1 is complete. $\square$

The following result is a special case of Theorem 3.1 when $\varphi(\eta) = \eta^p, w(r) = r$:

Corollary 3.2. *Under the conditions of Theorem 3.1,the discrete inequality*

$$u^p(n) \leq c^p(n) + p\sum_{s=0}^{n-1}\left\{f(n,s)u(s)\left[u(s) + \sum_{\xi=0}^{s-1} g(s,\xi)u(\xi)\right] + h(n,s)u(s)\right\}, \quad n \in \mathbb{N}_0, \tag{3.7}$$

where $p > 1$ *is a real number, implies that*

$$u(n) \leq \left[c(n) + \sum_{s=0}^{n-1} h(n,s)\right] \times \left\{1 + \sum_{s=0}^{n-1} f(n,s)\exp\sum_{\xi=0}^{s-1}[f(n,\xi) + g(n,\xi)]\right\}, \quad n \in \mathbb{N}_0. \tag{3.8}$$

Note that, the particular case of Theorem 3.1 when $\varphi(\eta) = \eta^2, c(n) \equiv k^2 > 0$ and the functions $f(n,s), g(n,s), h(n,s)$ are independent of the variable n, yields a discrete analogue of the Constantin integral inequality.

4. DISCRETE MODEL OF INFILTRATION

The mathematical model of the infiltration phenomena of a fluid due to Okrasinsky [7] was studied in [2] (see, also [8]):

$$u^2(t) = L + \int_0^t P(t-s)u(s)ds, \quad t \in \mathbb{R}_+, \tag{4.1}$$

where $L > 0$ is a constant, $P \in C(\mathbb{R}_+, \mathbb{R}_+)$ and u denotes the height of the percolating fluid above the horizontal impervious base, multiplied by a positive number. This model describes the infiltration phenomena of a fluid from a cylindrical reservoir into an isotropic homogeneous porous medium. Under the condition "P is differentiable and nondecreasing", Constantin obtained the existence and uniqueness of a solution $u \in C^1(\mathbb{R}_+, (0,\infty))$ of equation (4.1). Some known results for equation (4.1) are also given in Constantin [3] and Lipovan [5].

We note here that, although the conclusions given therein are correct, the derivation of them has a small defect. Actually, since function P depends on both variables t, s, the integral inequality given in the lemma of [2] is not applicable. However, using our Theorem 2.1 and by following the same argument as used in [2] these conclusions can be reproved very easily.

Now we consider the discrete analogue of equation (4.1) without a differentiability requirement on the function P :

$$u^2(n) = L + \sum_{s=0}^{n-1} P(n,s)u(s), \quad n \in \mathbb{N}, \tag{4.2}$$

where $L > 0$ is a constant, $u, P \in K(\mathbb{N}, \mathbb{R}_+)$ with P nondecreasing. The unique positive solution to equation (4.2) can be obtained by successive substitution. For instance, by letting $n = 0, 1, 2$ successively in (4.2), we obtain

$$u(0) = \sqrt{L}\,, \quad u(1) = \sqrt{L + P(1)u(0)}\,, \quad u(2) = \sqrt{L + P(1)u(1) + P(2)u(0)}.$$

An application of Corollary 2.3 with $f(n,s) = g(n,s) \equiv 0, h(n,s) = P(n-s)$ to (4.1) yields an upper bound on $u(n)$ of the form

$$u(n) \leq \sqrt{L} + \sum_{s=0}^{n-1} P(n-s), \quad n \in \mathbb{N}.$$

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