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CERTAIN CLASSES OF ANALYTIC FUNCTIONS INVOLVING SĂLĂGEAN OPERATOR

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ABSTRACT. Using Sălăgean differential operator, we study new subclasses of analytic functions. Coefficient inequalities and distortion theorems and extreme points of these classes are studied. Furthermore, integral means inequalities are obtained for the fractional derivatives of these classes.

Key words and phrases: Sălăgean operator, coefficient inequalities, distortion inequalities, extreme points, integral means, fractional derivative.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the open disc $\mathbb{U} = \{z : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of analytic and univalent functions $f(z)$ in $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the class of starlike functions of order α and the class of convex functions of order α, respectively, that is,

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, 0 \leqq \alpha < 1, z \in \mathbb{U} \right\}$$

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and

$$\mathcal{K}(\alpha)=\left\{f\in\mathcal{A}:\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,0\leqq\alpha<1,z\in\mathbb{U}\right\}.$$

For $f(z)\in\mathcal{A}$, Sălăgean [1] introduced the following operator which is called the Sălăgean operator:

$$D^0f(z)=f(z)$$
$$D^1f(z)=Df(z)=zf'(z)$$
$$D^nf(z)=D(D^{n-1}f(z))\qquad(n\in\mathbb{N}=1,2,3,...).$$

We note that,

$$D^nf(z)=z+\sum_{j=2}^{\infty}j^na_jz^j\qquad(n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}).$$

Let $\mathcal{N}_{m,n}(\alpha,\beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\mathrm{Re}\left\{\frac{D^mf(z)}{D^nf(z)}\right\}>\beta\left|\frac{D^mf(z)}{D^nf(z)}-1\right|+\alpha$$

for some $0\leqq\alpha<1$, $\beta\geq 0$, $m\in\mathbb{N}$, $n\in\mathbb{N}_0$ and all $z\in\mathbb{U}$. Also let $\mathcal{M}^s_{m,n}(\alpha,\beta)$ $(s=0,1,2,\dots)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the condition:

$$f(z)\in\mathcal{M}^s_{m,n}(\alpha,\beta)\Leftrightarrow D^sf(z)\in\mathcal{N}_{m,n}(\alpha,\beta).$$

It is easy to see that if $s=0$, then $\mathcal{M}^0_{m,n}(\alpha,\beta)\equiv\mathcal{N}_{m,n}(\alpha,\beta)$. Furthermore, special cases of our classes are the following:

(i) $\mathcal{N}_{1,0}(\alpha,0)=\mathcal{S}^*(\alpha)$ and $\mathcal{N}_{2,1}(\alpha,0)=\mathcal{K}(\alpha)$ which were studied by Silverman [2].
(ii) $\mathcal{N}_{1,0}(\alpha,\beta)=\mathcal{SD}(\alpha,\beta)$ and $\mathcal{M}^1_{1,0}(\alpha,\beta)=\mathcal{KD}(\alpha,\beta)$ which were studied by Shams at all [3].
(iii) $\mathcal{N}_{m,n}(\alpha,0)=\mathcal{K}_{m,n}(\alpha)$ and $\mathcal{M}^s_{m,n}(\alpha,0)=\mathcal{M}^s_{m,n}(\alpha)$ which were studied by Eker and Owa [4].

Therefore, our present paper is a generalization of these papers. In view of the coefficient inequalities for $f(z)$ to be in the classes $\mathcal{N}_{m,n}(\alpha,\beta)$ and $\mathcal{M}^s_{m,n}(\alpha,\beta)$, we introduce two subclasses $\widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$ and $\widetilde{\mathcal{M}}^s_{m,n}(\alpha,\beta)$. Some distortion inequalities for $f(z)$ and some integral means inequalities for fractional calculus of $f(z)$ in the above classes are discussed in this paper.

2. COEFFICIENT INEQUALITIES FOR CLASSES $\mathcal{N}_{m,n}(\alpha,\beta)$ AND $\mathcal{M}^s_{m,n}(\alpha,\beta)$

Theorem 2.1. *If $f(z)\in\mathcal{A}$ satisfies*

$$\sum_{j=2}^{\infty}\Psi(m,n,j,\alpha,\beta)\,|a_j|\leqq 2(1-\alpha)\tag{2.1}$$

where

$$\Psi(m,n,j,\alpha,\beta)=|j^m-j^n-\alpha j^n|+(j^m+j^n-\alpha j^n)+2\beta\,|j^m-j^n|\tag{2.2}$$

for some $\alpha(0\leqq\alpha<1)$, $\beta\geq 0$, $m\in\mathbb{N}$ and $n\in\mathbb{N}_0$,then $f(z)\in\mathcal{N}_{m,n}(\alpha,\beta)$.

Proof. Suppose that (2.1) is true for $\alpha(0\leqq\alpha<1)$, $\beta\geq 0$, $m\in\mathbb{N}$, $n\in\mathbb{N}_0$. For $f(z)\in\mathcal{A}$, let us define the function $F(z)$ by

$$F(z)=\frac{D^mf(z)}{D^nf(z)}-\beta\left|\frac{D^mf(z)}{D^nf(z)}-1\right|-\alpha.$$

It suffices to show that

$$\left|\frac{F(z)-1}{F(z)+1}\right| < 1 \qquad (z \in \mathbb{U}).$$

We note that

$$\begin{aligned}
\left|\frac{F(z)-1}{F(z)+1}\right| &= \left|\frac{D^m f(z) - \beta e^{i\theta}\left|D^m f(z) - D^n f(z)\right| - \alpha D^n f(z) - D^n f(z)}{D^m f(z) - \beta e^{i\theta}\left|D^m f(z) - D^n f(z)\right| - \alpha D^n f(z) + D^n f(z)}\right| \\
&= \left|\frac{-\alpha + \sum_{j=2}^{\infty}(j^m - j^n - \alpha j^n) a_j z^{j-1} - \beta e^{i\theta}\left|\sum_{j=2}^{\infty}(j^m - j^n) a_j z^{j-1}\right|}{(2-\alpha) + \sum_{j=2}^{\infty}(j^m + j^n - \alpha j^n) a_j z^{j-1} - \beta e^{i\theta}\left|\sum_{j=2}^{\infty}(j^m - j^n) a_j z^{j-1}\right|}\right| \\
&\leqq \frac{\alpha + \sum_{j=2}^{\infty}\left|j^m - j^n - \alpha j^n\right| |a_j||z|^{j-1} + \beta |e|^{i\theta}\sum_{j=2}^{\infty}|j^m - j^n||a_j||z|^{j-1}}{(2-\alpha) - \sum_{j=2}^{\infty}\left(j^m + j^n - \alpha j^n\right)|a_j||z|^{j-1} - \beta |e|^{i\theta}\sum_{j=2}^{\infty}|j^m - j^n||a_j||z|^{j-1}} \\
&\leqq \frac{\alpha + \sum_{j=2}^{\infty}\left|j^m - j^n - \alpha j^n\right| |a_j| + \beta \sum_{j=2}^{\infty}|j^m - j^n||a_j|}{(2-\alpha) - \sum_{j=2}^{\infty}\left(j^m + j^n - \alpha j^n\right)|a_j| - \beta \sum_{j=2}^{\infty}|j^m - j^n||a_j|}.
\end{aligned}$$

The last expression is bounded above by 1, if

$$\begin{aligned}
\alpha + \sum_{j=2}^{\infty}\left|j^m - j^n - \alpha j^n\right| |a_j| + \beta \sum_{j=2}^{\infty}|j^m - j^n||a_j| \\
\leqq (2-\alpha) - \sum_{j=2}^{\infty}\left(j^m + j^n - \alpha j^n\right)|a_j| - \beta \sum_{j=2}^{\infty}|j^m - j^n||a_j|
\end{aligned}$$

which is equivalent to our condition (2.1). This completes the proof of our theorem. □

By using Theorem 2.1, we have:

Theorem 2.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{j=2}^{\infty} j^s \Psi(m,n,j,\alpha,\beta)\,|a_j| \leqq 2(1-\alpha),$$

where $\Psi(m,n,j,\alpha,\beta)$ is defined by (2.2) for some $\alpha (0 \leqq \alpha < 1)$, $\beta \geq 0$, $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, then $f(z) \in \mathcal{M}^s_{m,n}(\alpha,\beta)$.

Proof. From

$$f(z) \in \mathcal{M}^s_{m,n}(\alpha,\beta) \Leftrightarrow D^s f(z) \in \mathcal{N}_{m,n}(\alpha,\beta),$$

replacing a_j by $j^s a_j$ in Theorem 2.1, we have the theorem. □

Example 2.1. The function $f(z)$ given by

$$f(z) = z + \sum_{j=2}^{\infty} \frac{2(2+\delta)(1-\alpha)\epsilon_j}{(j+\delta)(j+1+\delta)\Psi(m,n,j,\alpha,\beta)} z^j = z + \sum_{j=2}^{\infty} A_j z^j$$

with

$$A_j = \frac{2(2+\delta)(1-\alpha)\epsilon_j}{(j+\delta)(j+1+\delta)\Psi(m,n,j,\alpha,\beta)}$$

belongs to the class $\mathcal{N}_{m,n}(\alpha,\beta)$ for $\delta > -2$, $0 \leqq \alpha < 1$, $\beta \geq 0$, $\epsilon_j \in \mathbb{C}$ and $|\epsilon_j| = 1$. Because, we know that

$$\begin{aligned}\sum_{j=2}^{\infty} \Psi(m,n,j,\alpha,\beta)\,|A_j| &\leqq \sum_{j=2}^{\infty} \frac{2(2+\delta)(1-\alpha)}{(j+\delta)(j+1+\delta)} \\ &= \sum_{j=2}^{\infty} 2(2+\delta)(1-\alpha) \sum_{j=2}^{\infty} \left(\frac{1}{j+\delta} - \frac{1}{j+1+\delta}\right) \\ &= 2(1-\alpha).\end{aligned}$$

Example 2.2. The function $f(z)$ given by

$$f(z) = z + \sum_{j=2}^{\infty} \frac{2(2+\delta)(1-\alpha)\epsilon_j}{j^s(j+\delta)(j+1+\delta)\Psi(m,n,j,\alpha,\beta)} z^j = z + \sum_{j=2}^{\infty} B_j z^j$$

with

$$B_j = \frac{2(2+\delta)(1-\alpha)\epsilon_j}{j^s(j+\delta)(j+1+\delta)\Psi(m,n,j,\alpha,\beta)}$$

belongs to the class $\mathcal{M}^s_{m,n}(\alpha,\beta)$ for $\delta > -2$, $0 \leqq \alpha < 1$, $\beta \geq 0$, $\epsilon_j \in \mathbb{C}$ and $|\epsilon_j| = 1$. Because, the function $f(z)$ gives us that

$$\sum_{j=2}^{\infty} j^s \Psi(m,n,j,\alpha,\beta)\,|B_j| \leqq \sum_{j=2}^{\infty} \frac{2(2+\delta)(1-\alpha)}{(j+\delta)(j+1+\delta)} = 2(1-\alpha).$$

3. RELATION FOR $\widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$ AND $\widetilde{\mathcal{M}}^s_{m,n}(\alpha,\beta)$

In view of Theorem 2.1 and Theorem 2.2, we now introduce the subclasses

$$\widetilde{\mathcal{N}}_{m,n}(\alpha,\beta) \subset \mathcal{N}_{m,n}(\alpha,\beta) \quad \text{and} \quad \widetilde{\mathcal{M}}^s_{m,n}(\alpha,\beta) \subset \mathcal{M}^s_{m,n}(\alpha,\beta)$$

which consist of functions

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad (a_j \geq 0) \tag{3.1}$$

whose Taylor-Maclaurin coefficients satisfy the inequalities (2.1) and (2.2), respectively. By the coefficient inequalities for the classes $\widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$ and $\widetilde{\mathcal{M}}^s_{m,n}(\alpha,\beta)$, we see:

Theorem 3.1.

$$\widetilde{\mathcal{N}}_{m,n}(\alpha,\beta_2) \subset \widetilde{\mathcal{N}}_{m,n}(\alpha,\beta_1)$$

for some β_1 and β_2, $0 \leqq \beta_1 \leqq \beta_2$.

Proof. For $0 \leqq \beta_1 \leqq \beta_2$ we obtain

$$\sum_{j=2}^{\infty} \Psi(m,n,j,\alpha,\beta_1) a_j \leqq \sum_{j=2}^{\infty} \Psi(m,n,j,\alpha,\beta_2) a_j.$$

Therefore, if $f(z) \in \widetilde{\mathcal{N}}_{m,n}(\alpha,\beta_2)$, then $f(z) \in \widetilde{\mathcal{N}}_{m,n}(\alpha,\beta_1)$. Hence we get the required result. □

By using Theorem 3.1, we also have

Corollary 3.2.

$$\widetilde{\mathcal{M}}^s_{m,n}(\alpha,\beta_2) \subset \widetilde{\mathcal{M}}^s_{m,n}(\alpha,\beta_1)$$

for some β_1 and β_2, $0 \leqq \beta_1 \leqq \beta_2$.

4. DISTORTION INEQUALITIES

Lemma 4.1. *If $f(z) \in \widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$, then we have*

$$\sum_{j=p+1}^{\infty} a_j \leqq \frac{2(1-\alpha) - \sum_{j=2}^{p} \Psi(m,n,j,\alpha,\beta)a_j}{\Psi(m,n,p+1,\alpha,\beta)}.$$

Proof. In view of Theorem 2.1, we can write

$$\sum_{j=p+1}^{\infty} \Psi(m,n,j,\alpha,\beta)a_j \leqq 2(1-\alpha) - \sum_{j=2}^{p} \Psi(m,n,j,\alpha,\beta)a_j. \tag{4.1}$$

Clearly $\Psi(m,n,j,\alpha,\beta)$ is an increasing function for j. Then from (2.2) and (4.1), we have

$$\Psi(m,n,p+1,\alpha,\beta) \sum_{j=p+1}^{\infty} a_j \leqq 2(1-\alpha) - \sum_{j=2}^{p} \Psi(m,n,j,\alpha,\beta)a_j.$$

Thus, we obtain

$$\sum_{j=p+1}^{\infty} a_j \leqq \frac{2(1-\alpha) - \sum_{j=2}^{p} \Psi(m,n,j,\alpha,\beta)a_j}{\Psi(m,n,p+1,\alpha,\beta)} = A_j.$$

□

Lemma 4.2. *If $f(z) \in \widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$, then*

$$\sum_{j=p+1}^{\infty} ja_j \leqq \frac{2(1-\alpha) - \sum_{j=2}^{p} \Psi(m,n,j,\alpha,\beta)a_j}{\Psi(m-1,n-1,p+1,\alpha,\beta)} = B_j.$$

Corollary 4.3. *If $f(z) \in \widetilde{M}^{s}_{m,n}(\alpha)$, then*

$$\sum_{j=p+1}^{\infty} a_j \leqq \frac{2(1-\alpha) - \sum_{j=2}^{p} j^s\Psi(m,n,j,\alpha,\beta)a_j}{(p+1)^s\Psi(m,n,p+1,\alpha,\beta)} = C_j$$

and

$$\sum_{j=p+1}^{\infty} ja_j \leqq \frac{2(1-\alpha) - \sum_{j=2}^{p} j^s\Psi(m,n,j,\alpha,\beta)a_j}{(p+1)^s\Psi(m-1,n-1,p+1,\alpha,\beta)} = D_j.$$

Theorem 4.4. *Let $f(z) \in \widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$. Then for $|z| = r < 1$*

$$r - \sum_{j=2}^{p} a_j |z|^j - A_j r^{p+1} \leqq |f(z)| \leqq r + \sum_{j=2}^{p} a_j |z|^j + A_j r^{p+1}$$

and

$$1 - \sum_{j=2}^{p} ja_j |z|^{j-1} - B_j r^p \leqq |f'(z)| \leqq 1 + \sum_{j=2}^{p} ja_j |z|^j + B_j r^p$$

where A_j and B_j are given by Lemma 4.1 and Lemma 4.2.

Proof. Let $f(z)$ given by (1.1). For $|z| = r < 1$,using Lemma 4.1, we have

$$\begin{aligned} |f(z)| &\leqq |z| + \sum_{j=2}^{p} a_j |z|^j + \sum_{j=p+1}^{\infty} a_j |z|^j \\ &\leqq |z| + \sum_{j=2}^{p} a_j |z|^j + |z|^{p+1} \sum_{j=p+1}^{\infty} a_j \\ &\leqq r + \sum_{j=2}^{p} a_j |z|^j + A_j r^{p+1} \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{j=2}^{p} a_j |z|^j - \sum_{j=p+1}^{\infty} a_j |z|^j \\ &\geq |z| - \sum_{j=2}^{p} a_j |z|^j - |z|^{p+1} \sum_{j=p+1}^{\infty} a_j \\ &\geq r - \sum_{j=2}^{p} a_j |z|^j - A_j r^{p+1}. \end{aligned}$$

Furthermore, for $|z| = r < 1$ using Lemma 4.2, we obtain

$$\begin{aligned} |f'(z)| &\leqq 1 + \sum_{j=2}^{p} j a_j |z|^{j-1} + \sum_{j=p+1}^{\infty} j a_j |z|^{j-1} \\ &\leqq 1 + \sum_{j=2}^{p} j a_j |z|^{j-1} + |z|^p \sum_{j=p+1}^{\infty} j a_j \\ &\leqq 1 + \sum_{j=2}^{p} j a_j |z|^{j-1} + B_j r^p \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq 1 - \sum_{j=2}^{p} j a_j |z|^{j-1} - \sum_{j=p+1}^{\infty} j a_j |z|^{j-1} \\ &\geq 1 - \sum_{j=2}^{p} j a_j |z|^{j-1} - |z|^p \sum_{j=p+1}^{\infty} j a_j \\ &\geq 1 - \sum_{j=2}^{p} j a_j |z|^{j-1} - B_j r^p. \end{aligned}$$

This completes the assertion of Theorem 4.4. □

Theorem 4.5. *Let* $f(z) \in \widetilde{\mathcal{M}}^s_{m,n}(\alpha, \beta)$. *Then*

$$r - \sum_{j=2}^{p} a_j |z|^j - C_j r^{p+1} \leqq |f(z)| \leqq r + \sum_{j=2}^{p} a_j |z|^j + C_j r^{p+1}$$

and

$$1-\sum_{j=2}^{p} ja_j\,|z|^{j-1}-D_j r^p \leqq |f'(z)| \leqq 1+\sum_{j=2}^{p} ja_j\,|z|^{j}+D_j r^p$$

where C_j and D_j are given by Corollary 4.3.

Proof. Using a similar method to that in the proof of Theorem 4.4 and making use Corollary 4.3, we get our result. □

Taking $p=1$ in Theorem 4.4 and Theorem 4.5, we have:

Corollary 4.6. *Let $f(z)\in\widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$. Then for $|z|=r<1$*

$$r-\frac{2(1-\alpha)}{\Psi(m,n,2,\alpha,\beta)}r^2 \leqq |f(z)| \leqq r+\frac{2(1-\alpha)}{\Psi(m,n,2,\alpha,\beta)}r^2$$

and

$$1-\frac{2(1-\alpha)}{\Psi(m-1,n-1,2,\alpha,\beta)}r \leqq |f'(z)| \leqq 1+\frac{2(1-\alpha)}{\Psi(m-1,n-1,2,\alpha,\beta)}r.$$

Corollary 4.7. *Let $f(z)\in\widetilde{\mathcal{M}}^s_{m,n}(\alpha,\beta)$. Then for $|z|=r<1$*

$$r-\frac{2(1-\alpha)}{2^s\Psi(m,n,2,\alpha,\beta)}r^2 \leqq |f(z)| \leqq r+\frac{2(1-\alpha)}{2^s\Psi(m,n,2,\alpha,\beta)}r^2$$

and

$$1-\frac{2(1-\alpha)}{2^s\Psi(m-1,n-1,2,\alpha,\beta)}r \leqq |f'(z)| \leqq 1+\frac{2(1-\alpha)}{2^s\Psi(m-1,n-1,2,\alpha,\beta)}r.$$

5. EXTREME POINTS

The determination of the extreme points of a family F of univalent functions enables us to solve many extremal problems for F. Now, let us determine extreme points of the classes $\widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$ and $\widetilde{\mathcal{M}}^s_{m,n}(\alpha,\beta)$.

Theorem 5.1. *Let $f_1(z)=z$ and*

$$f_j(z)=z+\frac{2(1-\alpha)}{\Psi(m,n,j,\alpha,\beta)}z^j \quad (j=2,3,...).$$

where $\Psi(m,n,j,\alpha,\beta)$ is defined by (2.2). Then $f\in\widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{j=1}^{\infty}\lambda_j f_j(z),$$

where $\lambda_j>0$ and $\sum_{j=1}^{\infty}\lambda_j=1$.

Proof. Suppose that

$$f(z)=\sum_{j=1}^{\infty}\lambda_j f_j(z)=z+\sum_{j=2}^{\infty}\lambda_j\frac{2(1-\alpha)}{\Psi(m,n,j,\alpha,\beta)}z^j.$$

Then

$$\begin{aligned}\sum_{j=2}^{\infty}\Psi(m,n,j,\alpha,\beta)\frac{2(1-\alpha)}{\Psi(m,n,j,\alpha,\beta)}\lambda_j &= \sum_{j=2}^{\infty}2(1-\alpha)\lambda_j \\ &= 2(1-\alpha)\sum_{j=2}^{\infty}\lambda_j \\ &= 2(1-\alpha)(1-\lambda_1) \\ &< 2(1-\alpha)\end{aligned}$$

Thus, $f(z) \in \widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$ from the definition of the class of $\widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$.

Conversely, suppose that $f \in \widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$. Since

$$a_j \leqq \frac{2(1-\alpha)}{\Psi(m,n,j,\alpha,\beta)} \qquad (j = 2, 3, ...),$$

we may set

$$\lambda_j = \frac{\Psi(m,n,j,\alpha,\beta)}{2(1-\alpha)}a_j$$

and

$$\lambda_1 = 1 - \sum_{j=2}^{\infty}\lambda_j.$$

Then,

$$f(z) = \sum_{j=1}^{\infty}\lambda_j f_j(z).$$

This completes the proof of the theorem. □

Corollary 5.2. *Let* $g_1(z) = z$ *and*

$$g_j(z) = z + \frac{2(1-\alpha)}{j^s\Psi(m,n,j,\alpha,\beta)}z^j \quad (j = 2, 3, ...).$$

Then $g \in \widetilde{\mathcal{M}}^s_{m,n}(\alpha,\beta)$ *if and only if it can be expressed in the form*

$$g(z) = \sum_{j=1}^{\infty}\lambda_j g_j(z),$$

where $\lambda_j > 0$ *and* $\sum_{j=1}^{\infty}\lambda_j = 1$.

Corollary 5.3. *The extreme points of* $\widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$ *are the functions* $f_1(z) = z$ *and*

$$f_j(z) = z + \frac{2(1-\alpha)}{\Psi(m,n,j,\alpha,\beta)}z^j \quad (j = 2, 3, ...).$$

Corollary 5.4. *The extreme points of* $\widetilde{\mathcal{M}}^s_{m,n}(\alpha,\beta)$ *are given by* $g_1(z) = z$ *and*

$$g_j(z) = z + \frac{2(1-\alpha)}{j^s\Psi(m,n,j,\alpha,\beta)}z^j \quad (j = 2, 3, ...).$$

6. INTEGRAL MEANS INEQUALITIES

We shall use the following definitions for fractional derivatives by Owa [6] (also Srivastava and Owa [7]).

Definition 6.1. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi \quad (0 \leqq \lambda < 1), \tag{6.1}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

Definition 6.2. Under the hypotheses of Definition 6.1, the fractional derivative of order $(p+\lambda)$ is defined, for a function $f(z)$, by

$$D_z^{p+\lambda} f(z) = \frac{d^p}{dz^p} D_z^{\lambda} f(z)$$

where $0 \leqq \lambda < 1$ and $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

It readily follows from (6.1) in Definition 6.1 that

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda} \quad (0 \leqq \lambda < 1). \tag{6.2}$$

Further, we need the concept of subordination between analytic functions and a subordination theorem by Littlewood [5] in our investigation.

Let us consider two functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U}$. The function $f(z)$ is said to be *subordinate* to $g(z)$ in $\mathbb{U}$ if there exists a function $w(z)$ analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

We denote this subordination by

$$f(z) \prec g(z).$$

Theorem 6.1 (Littlewood [5]). *If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$*

$$\int_0^{2\pi} |f(z)|^{\mu} \, d\theta \leqq \int_0^{2\pi} |g(z)|^{\mu} \, d\theta.$$

Theorem 6.2. *Let $f(z) \in \mathcal{A}$ given by (3.1) be in the class $\widetilde{\mathcal{N}}_{m,n}(\alpha, \beta)$ and suppose that*

$$\sum_{j=2}^{\infty} (j-p)_{p+1} a_j \leqq \frac{2(1-\alpha)\Gamma(k+1)\Gamma(3-\lambda-p)}{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)\Gamma(2-p)}$$

for some $0 \leqq p \leqq 2$, $0 \leqq \lambda < 1$ where $(j-p)_{p+1}$ denotes the Pochhammer symbol defined by $(j-p)_{p+1} = (j-p)(j-p+1)\cdots j$. Also given is the function $f_k(z)$ by

$$f_k(z) = z + \frac{2(1-\alpha)}{\Psi(m,n,k,\alpha,\beta)} z^k \quad (k \geq 2). \tag{6.3}$$

If there exists an analytic function $w(z)$ given by

$$\{w(z)\}^{k-1} = \frac{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)} \times \sum_{j=2}^{\infty}(j-p)_{p+1}\frac{\Gamma(j-p)}{\Gamma(j+1-\lambda-p)}a_j z^{j-1},$$

then for $z = re^{i\theta}$ $(0 < r < 1)$ and $\mu > 0$,

$$\int_0^{2\pi} \left|D_z^{p+\lambda} f(z)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|D_z^{p+\lambda} f_k(z)\right|^{\mu} d\theta.$$

Proof. By virtue of the fractional derivative formula (6.2) and Definition 6.2, we find from (1.1) that

$$\begin{aligned} D_z^{p+\lambda} f(z) &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left\{1+\sum_{j=2}^{\infty}\frac{\Gamma(2-\lambda-p)\Gamma(j+1)}{\Gamma(j+1-\lambda-p)}a_j z^{j-1}\right\} \\ &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left\{1+\sum_{j=2}^{\infty}\Gamma(2-\lambda-p)(j-p)_{p+1}\Phi(j)a_j z^{j-1}\right\} \end{aligned}$$

where

$$\Phi(j) = \frac{\Gamma(j-p)}{\Gamma(j+1-\lambda-p)}.$$

Since $\Phi(j)$ is a decreasing function of j, we have

$$0 < \Phi(j) \leqq \Phi(2) = \frac{\Gamma(2-p)}{\Gamma(3-\lambda-p)} \qquad (0 \leqq \lambda < 1 \quad ; 0 \leqq p \leqq 2 \leqq j).$$

Similarly, from (6.2), (6.3) and Definition 6.2, we obtain

$$D_z^{p+\lambda} f_k(z) = \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left\{1+\frac{2(1-\alpha)\Gamma(2-\lambda-p)\Gamma(k+1)}{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)}z^{k-1}\right\}.$$

For $z = re^{i\theta}, \quad 0 < r < 1$, we must show that

$$\int_0^{2\pi}\left|1+\sum_{j=2}^{\infty}\Gamma(2-\lambda-p)(j-p)_{p+1}\Phi(j)a_j z^{j-1}\right|^{\mu} d\theta \leqq \int_0^{2\pi}\left|1+\frac{2(1-\alpha)\Gamma(2-\lambda-p)\Gamma(k+1)}{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)}z^{k-1}\right|^{\mu} d\theta \quad (\mu > 0).$$

Thus by applying Littlewood's subordination theorem, it would suffice to show that

$$1+\sum_{j=2}^{\infty}\Gamma(2-\lambda-p)(j-p)_{p+1}\Phi(j)a_j z^{j-1} \prec 1+\frac{2(1-\alpha)\Gamma(2-\lambda-p)\Gamma(k+1)}{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)}z^{k-1}. \tag{6.4}$$

By setting

$$1+\sum_{j=2}^{\infty}\Gamma(2-\lambda-p)(j-p)_{p+1}\Phi(j)a_j z^{j-1} = 1+\frac{2(1-\alpha)\Gamma(2-\lambda-p)\Gamma(k+1)}{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)}\{w(z)\}^{k-1}$$

we find that

$$\{w(z)\}^{k-1}=\frac{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)}\sum_{j=2}^{\infty}(j-p)_{p+1}\Phi(j)a_j z^{j-1}$$

which readily yields $w(0)=0$.

Therefore, we have

$$\begin{aligned}
|w(z)|^{k-1} &= \left|\frac{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)}\sum_{j=2}^{\infty}(j-p)_{p+1}\Phi(j)a_j z^{j-1}\right| \\
&\leqq \frac{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)}\sum_{j=2}^{\infty}(j-p)_{p+1}\Phi(j)a_j\,|z|^{j-1} \\
&\leqq |z|\frac{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)}\Phi(2)\sum_{j=2}^{\infty}(j-p)_{p+1}a_j \\
&= |z|\frac{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)}\frac{\Gamma(2-p)}{\Gamma(3-\lambda-p)}\sum_{j=2}^{\infty}(j-p)_{p+1}a_j \\
&\leqq |z| < 1
\end{aligned}$$

by means of the hypothesis of Theorem 6.2. □

For the special case $p=0$, Theorem 6.2 readily yields the following result.

Corollary 6.3. *Let $f(z)\in\mathcal{A}$ given by (3.1) be in the class $\widetilde{\mathcal{N}}_{m,n}(\alpha,\beta)$ and suppose that*

$$\sum_{j=2}^{\infty} j a_j \leqq \frac{2(1-\alpha)\Gamma(k+1)\Gamma(3-\lambda)}{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda)}$$

for $0\leqq\lambda<1$. Also let the function $f_k(z)$ be given by (6.3). If there exists an analytic function $w(z)$ given by

$$\{w(z)\}^{k-1}=\frac{\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda)}{2(1-\alpha)\Gamma(k+1)}\sum_{j=2}^{\infty}\frac{\Gamma(j+1)}{\Gamma(j+1-\lambda)}a_j z^{j-1},$$

then for $z=re^{i\theta}$ and $0<r<1$,

$$\int_0^{2\pi}\left|D_z^{\lambda}f(z)\right|^{\mu}d\theta \leqq \int_0^{2\pi}\left|D_z^{\lambda}f_k(z)\right|^{\mu}d\theta \qquad (0\leqq\lambda<1,\ \ \mu>0).$$

Corollary 6.4. *Let $f(z)\in\mathcal{A}$ given by (3.1) be in the class $\widetilde{\mathcal{M}}^{s}_{m,n}(\alpha,\beta)$ and suppose that*

$$\sum_{j=2}^{\infty}(j-p)_{p+1}a_j \leqq \frac{2(1-\alpha)\Gamma(k+1)\Gamma(3-\lambda-p)}{k^s\Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)\Gamma(2-p)}$$

for some $0 \leqq p \leqq 2$, $0 \leqq \lambda < 1$. *Also let the function*

$$g_k(z) = z + \frac{2(1-\alpha)}{k^s \Psi(m,n,k,\alpha,\beta)} z^k, \quad (k \geq 2). \tag{6.5}$$

If there exists an analytic function $w(z)$ *given by*

$$\{w(z)\}^{k-1} = \frac{k^s \Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)} \times \sum_{j=2}^{\infty} (j-p)_{p+1} \frac{\Gamma(j-p)}{\Gamma(j+1-\lambda-p)} a_j z^{j-1},$$

then for $z = re^{i\theta}$ $(0 < r < 1)$ *and* $\mu > 0$,

$$\int_0^{2\pi} \left| D_z^{p+\lambda} f(z) \right|^\mu d\theta \leqq \int_0^{2\pi} \left| D_z^{p+\lambda} g_k(z) \right|^\mu d\theta.$$

For the special case $p = 0$, Corollary 6.4 readily yields,

Corollary 6.5. *Let* $f(z) \in \mathcal{A}$ *given by (3.1) be in the class* $\widetilde{\mathcal{M}}_{m,n}^s(\alpha,\beta)$ *and suppose that*

$$\sum_{j=2}^{\infty} j a_j \leqq \frac{2(1-\alpha)\Gamma(k+1)\Gamma(3-\lambda)}{k^s \Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda)}$$

for $0 \leqq \lambda < 1$. *Also let the function* $g_k(z)$ *be given by (6.5). If there exists an analytic function* $w(z)$ *given by*

$$\{w(z)\}^{k-1} = \frac{k^s \Psi(m,n,k,\alpha,\beta)\Gamma(k+1-\lambda)}{2(1-\alpha)\Gamma(k+1)} \sum_{j=2}^{\infty} \frac{\Gamma(j+1)}{\Gamma(j+1-\lambda)} a_j z^{j-1},$$

then for $z = re^{i\theta}$ $(0 < r < 1)$ *and* $\mu > 0$,

$$\int_0^{2\pi} \left| D_z^{\lambda} f(z) \right|^\mu d\theta \leqq \int_0^{2\pi} \left| D_z^{\lambda} g_k(z) \right|^\mu d\theta.$$

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