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OUTER γ-CONVEX FUNCTIONS ON A NORMED SPACE

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ABSTRACT. For some given positive γ, a function f is called outer γ-convex if it satisfies the Jensen inequality $f(z_i) \le (1-\lambda_i)f(x_0) + \lambda_i f(x_1)$ for some $z_0 :\, = x_0, z_1, ..., z_k :\, = x_1 \in [x_0, x_1]$ satisfying $\|z_i - z_{i+1}\| \le \gamma$, where $\lambda_i :\, = \|x_0 - z_i\|/\|x_0 - x_1\|, i = 1, 2, ..., k-1$. Though the Jensen inequality is only required to hold true at some points (although the location of these points is uncertain) on the segment $[x_0, x_1]$, such a function has many interesting properties similar to those of classical convex functions. Among others it is shown that, if the infimum limit of an outer γ-convex function attains $-\infty$ at some point then this propagates to other points, and under some assumptions, a function is outer γ-convex iff its epigraph is an outer γ-convex set.

Key words and phrases: Convexity, Epigraph, Jensen inequality, Outer γ-convex set, Outer γ-convex function.

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1. INTRODUCTION

Convex functions belong to the most important objects investigated in mathematical programming. They have many interesting properties, for example, if a convex function attains $-\infty$ at some point then it attains $-\infty$ at every relative interior point of the domain, all lower level sets are convex and a function is convex iff its epigraph is convex; see [8]. It is worth mentioning that all of them follow from a single algebraic condition, namely the satisfaction of the Jensen inequality

$$f(x_\lambda) \le (1-\lambda)f(x_0) + \lambda f(x_1) \tag{1.1}$$
$$x_\lambda = (1-\lambda)x_0 + \lambda x_1, \quad \lambda \in [0,1]$$

everywhere on the segment connecting two arbitrary points of the domain. In a generalization of the classical convexity, for allowing small nonconvex blips, convexity is required to hold true

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between points, the distance between which is greater than some given positive real number, say, the roughness degree. Suppose D is a nonempty convex set in the normed linear space $(X, \|\cdot\|)$. According to Klötzler and Hartwig ([1]), a function $f : D \subset X \to \mathbb{R}$ is called roughly ρ-convex if the Jensen inequality (1.1) is satisfied *for all points* $x_\lambda \in [x_0, x_1] \subset D$ whenever $\|x_1 - x_0\| > \rho$, for some given $\rho > 0$. But the requirement of (1.1) *at all points* is sometimes too hard (see [7]). In the concept of Hu, Klee, and Larman [2], a function f is called δ-convex if (1.1) is fulfilled *at each point* $x_\lambda \in [x_0, x_1]$ with

$$\|x_\lambda - x_0\| \geq \frac{\delta}{2} \quad \text{and} \quad \|x_\lambda - x_1\| \geq \frac{\delta}{2}$$

for some given $\delta > 0$, which means that at least $\|x_1 - x_0\| \geq \delta$. According to H. X. Phu, for a fixed $\gamma > 0$, a function f is called γ-convex if

$$\|x_1 - x_0\| \geq \gamma \quad \text{implies} \quad f(x_0') + f(x_1') \leq f(x_0) + f(x_1)$$
$$\text{with} \quad x_i' \in [x_0, x_1], \|x_i - x_i'\| = \gamma, \quad i = 0, 1$$

([6]). It follows that f must fulfill the Jensen inequality (1.1) *at least at* x_0' *or* x_1'. In addition to this trend, γ-convexlikeness and outer γ-convexity were introduced respectively in [4] and [5] (and they are equivalent for lower semicontinuous functions). We recall that a function f is called outer γ-convex if (1.1) holds true *for some points*

$$z_0 := x_0, z_1, \ldots, z_k := x_1 \in [x_0, x_1] \quad \text{satisfying} \quad \|z_{i+1} - z_i\| \leq \gamma$$

(but the location of these points is *uncertain*). It was shown in [4] that, under some assumptions, a function is outer γ-convex (convex, respectively) iff the sum of this function and an arbitrary continuous linear functional always fulfills the property “each lower level set is outer γ-convex" (“each lower level set is convex", respectively) (see the definition of outer γ-convex sets in Section 2).

In this paper we show that although the demand “satisfying (1.1) at some points which are uncertain where" of outer γ-convexity is very weak it could conclude some more similar properties of classical convexity. In Section 2 some similar properties of classical convexity are given. Among others we get the nearest-point properties (Proposition 2.2). Some properties of outer γ-convex functions are given in Section 3. In particular, if the infimum limit of an outer γ-convex function attains $-\infty$ at some point then this propagates to other points (the so-called infection property) (Proposition 3.4). Finally, under some assumptions, Corollary 4.2 says that a function is outer γ-convex iff it its epigraph is outer γ-convex.

2. OUTER γ-CONVEX SETS

Let $(X, \|\cdot\|)$ be a normed linear space and γ be a fixed positive real number. For any x_0, $x_1 \in X$ and $\lambda \in [0, 1]$, we denote

$$x_\lambda := (1-\lambda)x_0 + \lambda x_1,$$
$$[x_0, x_1] := \{x_\lambda : 0 \leq \lambda \leq 1\},$$
$$[x_0, x_1[:= [x_0, x_1] \setminus \{x_1\},$$
$$]x_0, x_1[:= [x_0, x_1[\setminus \{x_0\}.$$

As usual, $B(x, r) := \{y \in X : \|x - y\| \leq r\}$ denotes the closed ball with centre x and radius $r > 0$. Let us recall the notion of outer γ-convex sets ([5]). A set $M \subset X$ is said to be *outer* γ*-convex* if for all $x_0, x_1 \in M$, there exist $z_0 := x_0, z_1, \ldots, z_k := x_1 \in [x_0, x_1] \cap M$ such that

$$\|z_{i+1} - z_i\| \leq \gamma \quad \text{for} \quad i = 0, 1, \ldots, k-1. \tag{2.1}$$

Obviously, every convex set is outer γ-convex for all $\gamma > 0$. Conversely, if a closed set M is outer γ-convex for all $\gamma > 0$ then M must be convex. It follows directly from the following.

Proposition 2.1 ([5]). *Let $M \subset X$ be outer γ-convex, and let x_0 and x_1 belong to M. Then*

$$[x'_0, x'_1[\subset [x_0, x_1] \setminus M \quad \textit{implies} \quad \|x'_0 - x'_1\| < \gamma.$$

By virtue of this proposition, such a set M is called outer γ-convex because a segment connecting two points of M may contain at most gaps (i.e., subsegments outside M) whose length is smaller than γ.

For each $x \in X$, set $Mx\colon = \{y^* \in M : \|x - y^*\| = \inf_{y \in M} \|x - y\|\}$.

Proposition 2.2. *Suppose that M is nonempty and outer γ-convex in X whose unit closed ball $B(0, 1)$ is strictly convex. Then $\operatorname{diam} Mx \leq \gamma$ for each $x \in X$.*

Proof. Assume the contrary that $\operatorname{diam} Mx > \gamma$. Then, there exist $x_0, x_1 \in Mx$ such that $\|x_0 - x_1\| > \gamma$. By the outer γ-convexity of M, there exists $z \in]x_0, x_1[\cap M$. The strict convexity of $B(0, 1)$ implies $\|x - z\| < \max\{\|x - x_0\|, \|x - x_1\|\} = \|x - x_0\|$, which conflicts with $x_0 \in Mx$. □

Note that the assumption of the strict convexity of $B(0, 1)$ is really needed. Moreover, the converse of Proposition 2.2 is false in case $\dim X \geq 2$. For example, the compact set

$$M\colon = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 1], y \in [-1, 1]\} \setminus \{(x, y) \in \mathbb{R}^2 : 0 < y < x\}$$

satisfies $\operatorname{diam} M(x, y) \leq \gamma$ for all $(x, y) \in \mathbb{R}^2$, where $\gamma\colon = 1$. But M is not outer γ-convex. As can be seen later, the converse of Proposition 2.2 holds true if $\dim X = 1$ and M is closed.

In view of Proposition 2.2, we get the following classical result which is a part of Motzkin's Theorem (see [9]).

Corollary 2.3. *Suppose that M is nonempty and convex in X whose unit closed ball $B(0, 1)$ is strictly convex. Then, for each $x \in X$, if the set Mx is nonempty, it is a singleton.*

Proof. Since M is convex, it is outer γ-convex for all $\gamma > 0$. By Proposition 2.2, $\operatorname{diam} Mx \leq \gamma$ for all $\gamma > 0$. It follows that $\operatorname{diam} Mx = 0$, i.e., Mx is a singleton. □

We recall that a set $M \subset X$ is γ*-convexlike* if $]x_0, x_1[\cap M \neq \emptyset$ holds true for all x_0, x_1 in M satisfying $\|x_0 - x_1\| > \gamma$ ([5]).

Clearly, each outer γ-convex set is γ-convexlike. In general the converse does not hold. The situation is quite different if M is closed.

Proposition 2.4 ([5]). *Suppose that M is closed. Then M is outer γ-convex iff it is γ-convexlike.*

Note that if $\dim X = 1$ and $\operatorname{diam} Mx \leq \gamma$ for each $x \in X$ then M is γ-convexlike (Indeed, if M were not γ-convexlike, i.e., there were $x_0, x_1 \in M$, $x_0 - x_1 > \gamma$ such that $]x_0, x_1[\cap M = \emptyset$, then $M\frac{x_0+x_1}{2} = \{x_0, x_1\}$ and therefore $\operatorname{diam} M\frac{x_0+x_1}{2} > \gamma$, a contradiction). Consequently, by Proposition 2.4, the converse of Proposition 2.2 holds true if $\dim X = 1$ and M is closed.

From Proposition 2.4 we have the following.

Proposition 2.5. *If M is outer γ-convex then $x_1, \ldots, x_m \in M$ and*

$$\inf_{x \in \mathit{conv}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m\}} \|x_i - x\| > \gamma \quad \textit{for all } i = 1, \ldots, m$$

and $m \geq 2$ imply that there exist $\lambda_i > 0$, $i = 1, \ldots, m$, such that $\sum_{i=1}^m \lambda_i = 1$ and $\sum_{i=1}^m \lambda_i x_i \in M$. If additionally M is closed, the converse is true.

Proof. Suppose that M is outer γ-convex. Then the above condition holds true for $m = 2$. It remains to prove that the above condition holds true for $m > 2$. The proof is by induction on m. Assume that the assertion holds for $m - 1$. Let $x_1, \dots, x_m \in M$ and

$$\inf_{x \in F_i} \|x_i - x\| > \gamma$$

for all $i = 1, \dots, m$, where $F_i := \text{conv}\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m\}$. It implies that

$$\inf_{x \in F_i} \|x_i - x\| > \gamma$$

for all $i = 1, \dots, m-1$. Therefore, by the induction assumption, we conclude that

$$y := \sum_{i=1}^{m-1} \lambda_i x_i \in M$$

with some $\lambda_i > 0$, $i = 1, \dots, m-1$ and $\sum_{i=1}^{m-1} \lambda_i = 1$. Since $\|y - x_m\| > \gamma$, there exists $\lambda_m \in]0, 1[$ such that $(1 - \lambda_m)y + \lambda_m x_m \in M$. Hence,

$$\sum_{i=1}^{m-1} \lambda_i (1 - \lambda_m) x_i + \lambda_m x_m \in M.$$

That is, the above condition always holds true.

Conversely, since the above condition holds true for $m = 2$, M is γ-convexlike. It follows from Proposition 2.4 that M is outer γ-convex. □

3. OUTER γ-CONVEX FUNCTIONS

Suppose D is a nonempty convex set in the normed linear space $(X, \|\cdot\|)$. We recall that $f : D \subset X \to \mathbb{R}$ is *outer γ-convex* if for all distinct points $x_0, x_1 \in D$, there exist $z_0 := x_0, z_1, \dots, z_k := x_1 \in [x_0, x_1]$ satisfying (2.1) and

$$f(z_i) \le (1 - \lambda_i) f(x_0) + \lambda_i f(x_1) \tag{3.1}$$

where $\lambda_i := \|x_0 - z_i\| / \|x_0 - x_1\|,\ \ i = 1, 2, \dots, k-1$ (see [5]).

Clearly, a convex function is outer γ-convex for all $\gamma > 0$. Conversely, if a lower semicontinuous function is outer γ-convex for all $\gamma > 0$ then it must be convex. Indeed, if a function is outer γ-convex for all $\gamma > 0$ then it is convexlike (see [1]) and therefore, by lower semicontinuouity, this function is convex.

In [4], a weaker notion of generalized convexity, namely γ-convexlikeness was introduced. We recall that a function f is *γ-convexlike* if for all x_0, x_1 in D, satisfying $\|x_0 - x_1\| > \gamma$, there exists $z \in]x_0, x_1[$ such that

$$f(z) \le (1 - \lambda) f(x_0) + \lambda f(x_1), \tag{3.2}$$

where $\lambda := \|x_0 - z\| / \|x_0 - x_1\|$.

Then, outer γ-convexity and γ-convexlikeness are equivalent for lower semicontinuous functions.

Proposition 3.1 ([5]). *Let f be lower semicontinuous. Then, f is outer γ-convex iff it is γ-convexlike.*

It is easy to see that a polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ of order 4 is not convex on $\mathbb{R}^1$ iff $0 < 3b^2 - 8ac$. But f is outer γ-convex for a suitable γ as the following shows.

Corollary 3.2. *Suppose that a polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ of order 4 is not convex on $\mathbb{R}^1$. Then f is outer γ-convex iff $a > 0$ and $\gamma \ge \frac{1}{2a}\sqrt{\frac{3(3b^2-8ac)}{2}}$.*

Proof. Proposition 3.1 allows us to conclude that outer γ-convexity of a polynomial is equivalent to γ-convexlikeness. Therefore, f is outer γ-convex iff for all $x_0, x_1 \in \mathbb{R}^1$ and $x_1 - x_0 > \gamma$, there exists $\lambda \in]0,1[$ such that

$$f\left(x_0 + \lambda(x_1 - x_0)\right) \leq (1-\lambda) f(x_0) + \lambda f(x_1).$$

This inequality is equivalent to

$$g(x_1) = 6ax_1^2 - \left(4a(2-\lambda)p - 3b\right) x_1 + a(3 - 3\lambda + \lambda^2)p^2 - b(2-\lambda)p + c \geq 0,$$

where $p: = x_1 - x_0$. Fix p and λ. Then, the polynomial $g(x_1)$ of order 2 is greater than 0 for all $x_1 \in \mathbb{R}^1$ iff $a > 0$ and

$$8a^2(1 - \lambda + \lambda^2)(x_1 - x_0)^2 \geq 9b^2 - 24ac \tag{3.3}$$

holds true for all $x_0, x_1 \in \mathbb{R}^1$ satisfying $x_1 - x_0 > \gamma$.

Now suppose that f is outer γ-convex. It follows from the above that $a > 0$ and (3.3) holds for all $x_0, x_1 \in \mathbb{R}^1$ satisfying $x_1 - x_0 > \gamma$. Since $\lambda \in [0,1]$, $0 < 1 - \lambda + \lambda^2 \leq 1$. Hence, by (3.2), $9b^2 - 24ac \leq 8a^2(x_1 - x_0)^2$ for all $x_0, x_1 \in \mathbb{R}^1$ satisfying $x_1 - x_0 > \gamma$. It follows that $0 < 3(3b^2 - 8ac) \leq 8a^2\gamma^2$.

Conversely, suppose that $a > 0$ and $0 < 3(3b^2 - 8ac) \leq 8a^2\gamma^2$. We prove that f is outer γ-convex. Assume the contrary that f is not outer γ-convex. Then, by (3.3), there exist $x_0, x_1 \in \mathbb{R}^1$ satisfying $x_1 - x_0 > \gamma$ and

$$8a^2(1 - \lambda + \lambda^2)(x_1 - x_0)^2 < 9b^2 - 24ac$$

for all $\lambda \in]0,1[$. It implies that $(x_1 - x_0)^2 \leq \gamma^2$, a contradiction. □

It is well known that f is convex iff the Jenssen inequality holds, namely $x_1, \ldots, x_m \in D$ imply that $f\left(\sum_{i=1}^m \lambda_i x_i\right) \leq \sum_{i=1}^m \lambda_i f(x_i)$ for all $\lambda_i \geq 0$, $i = 1, \ldots, m$ satisfying $\sum_{i=1}^m \lambda_i = 1$ (see, e.g. [8]).

Proposition 3.3. *If f is outer γ-convex then $x_1, \ldots, x_m \in D$ and*

$$\inf_{x \in \text{conv}\{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_m\}} \|x_i - x\| > \gamma \quad \textit{for all } i = 1, \ldots, m$$

and $m \geq 2$ imply that there exist $\lambda_i > 0$, $i = 1, \ldots, m$, $\sum_{i=1}^m \lambda_i = 1$, $f\left(\sum_{i=1}^m \lambda_i x_i\right) \leq \sum_{i=1}^m \lambda_i f(x_i)$. If additionally f is lower semicontinuous, the converse is true.

Proof. Suppose that f is outer γ-convex. We apply the argument given in the proof of Proposition 2.5 again, with "$\sum \lambda_i x_i \in M$" replaced by "$f(\sum \lambda_i x_i) \leq \sum \lambda_i f(x_i)$", to obtain the desired result.

Conversely, since the above condition holds true for $m = 2$, f is γ-convexlike. Hence, by Proposition 3.1, f is outer γ-convex. □

Note that the sufficiency of Proposition 3.3 fails to be true without the assumption on the lower semi continuity of f.

A property of generalized convex functions is called an infection property if this property transmits to other places after once appearing somewhere. Phu and Hai ([6]) showed that γ-convex functions on $\mathbb{R}$ possess some infection properties. Outer γ-convex functions also possess an infection property as the following proposition shows.

Proposition 3.4. *Let $f : D \subset X \to \mathbb{R}$ be outer γ-convex and $x_0 \in D$ satisfy $\liminf_{x \to x_0} f(x) = -\infty$. If there exists $y \in D$ satisfying*

$$\|y - x_0\| \geq 2\gamma \tag{3.4}$$

then there is some

$$z \in \left[x_0 + \gamma\frac{y - x_0}{\|y - x_0\|}, x_0 + 2\gamma\frac{y - x_0}{\|y - x_0\|}\right]$$

such that $\liminf_{x\to z} f(x) = -\infty$.

Proof. Assume that $x_0 = \lim_{m\to+\infty} x_m$ and $\lim_{m\to+\infty} f(x_m) = -\infty$ with some $\{x_m\} \subset D$. Since $\|y - x_0\| \geq 2\gamma$, we also assume that $\|y - x_m\| > \gamma$ for all m. Set $s_m \colon = (y - x_m)/\|y - x_m\|$. Because f is outer γ-convex, there exist $z_m^j = (1 - \lambda_m^j)x_m + \lambda_m^j y, j = 1, 2$ satisfying

$$\|z_m^2 - z_m^1\| \leq \gamma, \quad x_m + \gamma s_m \in [z_m^1, z_m^2], \tag{3.5}$$

and

$$f(z_m^j) \leq (1 - \lambda_m^j)f(x_m) + \lambda_m^j f(y), \tag{3.6}$$

where $\lambda_m^j \colon = \|x_m - z_m^j\|/\|y - x_m\|$. Since $\{\lambda_m^j\} \subset [0, 1]$, we can assume that $\lambda_m^j \to \lambda^j \in [0, 1]$ as $m \to +\infty$. It follows that $z_m^j \to z^j \colon = (1 - \lambda^j)x_0 + \lambda^j y$ as $m \to +\infty$, $j = 1, 2$. We now consider the following cases:

a) If $\lambda^2 \neq 1$, i.e., $z_m^2 \not\to y$ as $m \to +\infty$. This together with (3.6) yields

$$\begin{aligned}\liminf_{m\to+\infty} f(z_m^2) &\leq \liminf_{m\to+\infty}\{(1 - \lambda_m^2)f(x_m) + \lambda_m^2 f(y)\}\\ &\leq \liminf_{m\to+\infty}(1 - \lambda_m^2)f(x_m) + \limsup_{m\to+\infty} \lambda_m^2 f(y)\\ &= -\infty.\end{aligned}$$

Therefore

$$\liminf_{m\to+\infty} f(z_m^2) = -\infty.$$

That is,

$$\liminf_{x\to z^2} f(x) = -\infty.$$

Since $z_m^2 \in [x_m + \gamma s_m, x_m + 2\gamma s_m]$, we conclude that $z^2 \in [x_0 + \gamma s_0, x_0 + 2\gamma s_0]$.

b) If $\lambda^2 = 1$, i.e., $z_m^2 \to y$ as $m \to +\infty$. Then, by (3.5), we conclude that $\|x_0 - y\| = 2\gamma$, $z_m^1 \to z^1 = x_0 + \gamma s_0$ as $m \to +\infty$ and therefore $\lambda^1 \neq 1$. Applying the argument given in case **a**) again, with "z_m^2" replaced by "z_m^1", we get

$$\liminf_{x\to z^1} f(x) = -\infty.$$

This completes our proof. □

Note that the number 2γ in (3.4) is best possible. This is illustrated by

$$f(x) \colon = \begin{cases} 0 & \text{if } x \in \{0\} \cup [a, b] \\ \frac{1}{x(x-a)} & \text{if } x \in]0, a[\end{cases}$$

($1 < b < 2$ and $b - 1 < a < 1$). Obviously, f is outer γ-convex on $D \colon = [0, b]$ with $\gamma \colon = 1$. Choose $x_0 \colon = 0$ and $y \colon = b$ then $\liminf_{x\to x_0} f(x) = -\infty$ and $y - x_0 = b < 2\gamma$. In this case, $\lim_{x\to z} f(x) = 0$ for all $z \in [x_0 + \gamma, y]$ and the conclusion of Proposition 3.4 is false.

In the next section, a Lipschitz condition is assumed and therefore, the infection property above does not occur.

4. THE OUTER γ-CONVEXITY OF FUNCTIONS AND THEIR EPIGRAPHS

Similar to convex functions, outer γ-convex functions can be characterized by their epigraphs.

Theorem 4.1. *Suppose that* $\|(x,t)\|_1 := \max\{\|x\|, |t|\}$ *for all* $x \in X, t \in \mathbb{R}$. *If* $\operatorname{epi} f$ *is outer* γ*-convex then* f *is outer* γ*-convex. Conversely, if an outer* γ*-convex* f *is Lipschitz continuous with constant* $\alpha > 1$ *(*$\alpha \in [0,1]$*, respectively) then* $\operatorname{epi} f$ *is outer* $\alpha\gamma$*-convex (outer* γ*-convex, respectively).*

Proof. Suppose that $\operatorname{epi} f$ is outer γ-convex and $x_0, x_1 \in D$ such that $\|x_1 - x_0\| > \gamma$. Then

$$\| (x_1, f(x_1)) - (x_0, f(x_0)) \|_1 \geq \|x_1 - x_0\| > \gamma.$$

It follows that there exist

$$A_0 := (x_0, f(x_0)), A_1, \dots, A_k := (x_1, f(x_1)) \in [(x_0, f(x_0)), (x_1, f(x_1))] \cap \operatorname{epi} f$$

such that

$$\|A_{i+1} - A_i\|_1 \leq \gamma \quad \text{with} \quad i = 0, 1, \dots, k-1.$$

Suppose that $A_i = (z_i, t_i)$. Then

$$\|z_{i+1} - z_i\| \leq \|A_{i+1} - A_i\|_1 \leq \gamma \quad \text{with} \quad i = 0, 1, \dots, k-1.$$

On the other hand, since

$$A_i = (z_i, t_i) \in [(x_0, f(x_0)), (x_1, f(x_1))] \cap \operatorname{epi} f,$$

we get

$$f(z_i) \leq t_i = (1 - \lambda_i) f(x_0) + \lambda_i f(x_1)$$

where $\lambda_i := \|x_0 - z_i\| / \|x_0 - x_1\|,\ \ i = 1, 2, \dots, k-1$. That is, f is outer γ-convex.

Conversely, if an outer γ-convex function f is Lipschitz continuous with constant $\alpha > 1$ ($\alpha \in [0,1]$, respectively) then $\operatorname{epi} f$ is outer $\alpha\gamma$-convex (outer γ-convex, respectively). Indeed, let

$$Y_0 = (x_0, t_0), Y_1 = (x_1, t_1) \in \operatorname{epi} f.$$

Obviously, f is continuous on $[x_0, x_1]$.

Hence, $\{(x,t) \in \operatorname{epi} f : x \in [x_0, x_1]\}$ is closed. Assume without loss of generality, that

$$Y_0 = (x_0, f(x_0)), Y_1 = (x_1, f(x_1)).$$

Suppose

$$\|Y_1 - Y_0\|_1 > \alpha\gamma \quad \text{with} \quad \alpha > 1$$

($\|Y_1 - Y_0\|_1 > \gamma$ with $0 \leq \alpha \leq 1$, respectively). Then, by $\|(x,t)\|_1 := \max\{\|x\|, |t|\}$,

$$\alpha \|x_1 - x_0\| \geq |f(x_1) - f(x_0)|$$

implies

$$\|x_1 - x_0\| \geq \frac{\|Y_1 - Y_0\|_1}{\alpha} > \gamma \quad \text{with} \quad \alpha > 1$$

($\|x_1 - x_0\| = \|Y_1 - Y_0\|_1 > \gamma$ with $0 \leq \alpha \leq 1$, respectively). By the outer γ-convexity of f, there exist $z_0 := x_0, z_1, \dots, z_k := x_1 \in [x_0, x_1]$ satisfying (2.1) and (3.1). Set

$$A_i := (z_i, (1 - \lambda_i) f(x_0) + \lambda_i f(x_1)),$$

where $\lambda_i := \|x_0 - z_i\| / \|x_0 - x_1\|,\ \ i = 0, 1, \dots, k$. It follows that

$$A_0, A_1, \dots, A_k \in [Y_0, Y_1] \cap \operatorname{epi} f$$

and

$$\|A_{i+1} - A_i\|_1 \leq \alpha\gamma,\ i = 0, 1, \dots, k-1 \quad \text{with} \quad \alpha > 1$$

($\|A_{i+1} - A_i\|_1 \leq \gamma$, $i = 0, 1, \ldots, k-1$ with $0 \leq \alpha \leq 1$, respectively). Hence, $\operatorname{epi} f$ is outer $\alpha\gamma$-convex with $\alpha > 1$ ($\operatorname{epi} f$ is outer γ-convex with $0 \leq \alpha \leq 1$, respectively), and the proof is complete. □

Corollary 4.2. *Suppose that* $\|(x,t)\|_1 := \max\{\|x\|, |t|\}$ *for all* $x \in X$, $t \in \mathbb{R}$ *and* f *is Lipschitz continuous with constant* $\alpha \in [0,1]$. *Then,* f *is outer* γ*-convex iff* $\operatorname{epi} f$ *is outer* γ*-convex.*

Note that the assumptions of norm and Lipschitz condition in Theorem 4.1 and Corollary 4.2 are really needed.

5. CONCLUDING REMARKS

Some sufficient conditions for some kinds of outer γ-convex functions, namely strictly γ-convex functions and γ-convex functions, were given in [3] and [6]. Some sufficient conditions for outer γ-convex function will be a subject of another paper.

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