

A NOTE ON NEWTON'S INEQUALITY

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Abstract: We present a generalization of Newton's inequality, i.e., an inequality of mixed form connecting symmetric functions and weighted means. Two open problems are also stated.



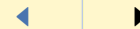
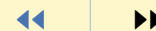
Newton's Inequality

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1. Introduction

A well-known theorem of Newton [1] states the following:

Theorem 1.1. *If all the zeros of a polynomial*

$$(1.1) \quad P_n(x) = e_0x^n + e_1x^{n-1} + \cdots + e_kx^{n-k} + \cdots + e_n, \quad e_0 = 1,$$

are real, then its coefficients satisfy

$$(1.2) \quad e_{k-1}e_{k+1} \leq A_k^{(n)}e_k^2, \quad k = 1, 2, \dots, n-1;$$

where $A_k^{(n)} := \frac{k}{k+1} \frac{n-k}{n+1-k}$.

For a sequence $\mathbf{a} = \{a_i\}_{i=1}^n$ of real numbers, by putting

$$(1.3) \quad P_n(x) = \prod_{i=1}^n (x + a_i) = \sum_{k=0}^n e_k x^{n-k},$$

we see that the coefficient $e_k = e_k(\mathbf{a})$ represents the k th *elementary symmetric function* of \mathbf{a} , i.e. the sum of all the products, k at a time, of different $a_i \in \mathbf{a}$.

There are several generalizations of Newton's inequality [2], [3]. In this article we give another one. For this purpose define the sequences $\mathbf{a}'_i := \mathbf{a}/\{a_i\}$, $i = 1, 2, \dots, n$, and by $e_k(\mathbf{a}'_i)$ denote the k -th elementary symmetric function over \mathbf{a}'_i . We have:

Theorem 1.2. *Let $\mathbf{c} = \{c_i\}_{i=1}^n$ be a weight sequence of non-negative numbers satisfying*

$$(1.4) \quad \sum_{i=1}^n c_i = 1,$$

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and, for an arbitrary sequence $\mathbf{a} = \{a_i\}_{i=1}^n$ of real numbers, define

$$(1.5) \quad E_k^{(c)} := \sum_{i=1}^n c_i e_k(\mathbf{a}'_i), \quad E_0^{(c)} = 1,$$

or equivalently,

$$(1.6) \quad E_k^{(c)} = e_k - e_{k-1}f_1 + e_{k-2}f_2^2 - \cdots + (-1)^r e_{k-r}f_r^r + \cdots + (-1)^k f_k^k,$$

where

$$(1.7) \quad f_s := \left(\sum_{i=1}^n c_i a_i^s \right)^{\frac{1}{s}}.$$

Then

$$(1.8) \quad E_{k-1}^{(c)} E_{k+1}^{(c)} \leq A_k^{(n-1)} \left(E_k^{(c)} \right)^2, \quad k = 1, 2, \dots, n-1.$$

Proof. We shall give an easy proof supposing that the sequence \mathbf{c} consists of arbitrary positive rational numbers. Since \mathbf{a} and \mathbf{c} are independent of each other, the truthfulness of the above theorem follows by the continuity principle.

Therefore, let $\mathbf{p} = \{p_k\}_{k=1}^n$ be an arbitrary sequence of positive integers and put

$$(1.9) \quad c_i = \frac{p_i}{\sum_{k=1}^n p_k}, \quad i = 1, 2, \dots, n; \quad \mathbf{p} \in \mathbb{N}.$$

Now, for a given real sequence \mathbf{a} , consider the polynomial $Q(x)$ defined by

$$(1.10) \quad Q(x) := \prod_{i=1}^n (x + a_i)^{p_i}.$$



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Since all its zeros are real, by the well-known Gauss theorem, the zeros of $Q'(x)$,

$$(1.11) \quad Q'(x) = Q(x) \sum_{i=1}^n \frac{p_i}{x + a_i},$$

are also real.

In particular, the same is valid for the polynomial $R(x)$ defined by

$$(1.12) \quad R(x) := \prod_{i=1}^n (x + a_i) \sum_{i=1}^n \frac{c_i}{(x + a_i)}.$$

Since

$$(1.13) \quad R(x) = x^{n-1} + E_1^{(c)} x^{n-2} + \dots + E_k^{(c)} x^{n-1-k} + \dots + E_{n-1}^{(c)},$$

the result follows by simple application of Theorem 1.1.

Remark 1. Since $\sum_{i=1}^n e_k(\mathbf{a}'_i) = (n - k)e_k(\mathbf{a})$, putting $c_i = 1/n$, $i = 1, 2, \dots, n$ in (1.5) and (1.8), we obtain the assertion from Theorem 1.1. Hence our result represents a generalization of Newton's theorem.

Also, denoting by $f_s^{(c)}(\mathbf{a}) = f_s := (\sum_{i=1}^n c_i a_i^s)^{1/s}$, $s > 0$, the classical weighted mean (with weights c) of order s , and using the identity

$$(1.14) \quad e_k(\mathbf{a}'_i) = e_k(\mathbf{a}) - a_i e_{k-1}(\mathbf{a}'_i),$$

an equivalent form of $E_k^{(c)}$ arises, i.e.,

$$(1.15) \quad E_k^{(c)} = e_k - e_{k-1} f_1 + e_{k-2} f_2^2 - \dots + (-1)^r e_{k-r} f_r^r + \dots + (-1)^k f_k^k.$$

Putting this in (1.8), we obtain a mixed inequality connecting elementary symmetric functions with weighted means of integer order. \square

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Problem 1.1. An interesting fact is that non-negativity of \mathbf{c} is *not* a necessary condition for (1.8) to hold. We shall illustrate this point by an example. For $k = 1$, $n = 3$, we have

$$\begin{aligned} & (E_1^{(c)})^2 - 4E_0^{(c)}E_2^{(c)} \\ &= (c_1(a_2 + a_3) + c_2(a_1 + a_3) + c_3(a_1 + a_2))^2 - 4(c_1a_2a_3 + c_2a_1a_3 + c_3a_1a_2) \\ &= (1 - c_2)^2(a_1 - a_2)^2 + 2(c_1 - c_2c_3)(a_1 - a_2)(a_3 - a_1) + (1 - c_3)^2(a_3 - a_1)^2, \end{aligned}$$

and this quadratic form is positive semi-definite whenever $c_1c_2c_3 \geq 0$.

Hence, in this case the inequality (1.8) is valid for all real sequences \mathbf{a} with \mathbf{c} satisfying

$$(1.16) \quad c_1 + c_2 + c_3 = 1, \quad c_1c_2c_3 \geq 0.$$

Therefore there remains the seemingly difficult problem of finding true bounds for the sequence \mathbf{c} satisfying (1.4), such that the inequality (1.8) holds for an arbitrary real sequence \mathbf{a} .

Problem 1.2. There is an interesting application of Theorem 1.2 to the well known Turan's problem. Under what conditions does the sequence of polynomials $\{Q_n(x)\}$ satisfy *Turan's inequality*

$$(1.17) \quad Q_{n-1}(x)Q_{n+1}(x) \leq (Q_n(x))^2,$$

for each $x \in [a, b]$ and $n \in [n_1, n_2]$?

This problem is solved for many classes of polynomials [4]. We shall consider here the following question [5].

An arbitrary sequence $\{d_i\}$, $i = 1, 2, \dots$ of real numbers generates a sequence of polynomials $\{P_n(x)\}$, $n = 0, 1, 2, \dots$ defined by

$$(1.18) \quad P_n(x) := x^n + d_1x^{n-1} + d_2x^{n-2} + \dots + d_{n-1}x + d_n, \quad P_0(x) := 1.$$



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Denote also by A_n the set of zeros of $P_n(x)$.

Now, if for some $m > 1$ the set A_m consists of real numbers only, then from Theorem 1.2, it follows that

$$(1.19) \quad P_{n-1}(a)P_{n+1}(a) \leq P_n^2(a),$$

for each $a \in A_m$ and $n \in [1, m - 1]$.

Is it possible to establish some simple conditions such that the Turan inequality

$$(1.20) \quad P_{n-1}(x)P_{n+1}(x) \leq P_n^2(x),$$

holds for each $x \in [\min a, \max a]_{a \in A_m}$ and $n \in [1, m - 1]$.

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