A NOTE ON NEWTON'S INEQUALITY

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Abstract: We present a generalization of Newton's inequality, i.e., an inequality of mixed form connecting symmetric functions and weighted means. Two open problems are also stated.

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1. Introduction

A well-known theorem of Newton [1] states the following:

Theorem 1.1. *If all the zeros of a polynomial*

$$P_n(x) = e_0x^n + e_1x^{n-1} + \cdots + e_kx^{n-k} + \cdots + e_n, \quad e_0 = 1, \tag{1.1}$$

are real, then its coefficients satisfy

$$e_{k-1}e_{k+1} \le A_k^{(n)} e_k^2, \quad k = 1, 2, \dots, n-1; \tag{1.2}$$

where $A_k^{(n)} := \frac{k}{k+1}\frac{n-k}{n+1-k}$.

For a sequence $\mathbf{a} = \{a_i\}_{i=1}^n$ of real numbers, by putting

$$P_n(x) = \prod_{i=1}^n (x + a_i) = \sum_{k=0}^n e_k x^{n-k}, \tag{1.3}$$

we see that the coefficient $e_k = e_k(\mathbf{a})$ represents the kth *elementary symmetric function* of $\mathbf{a}$, i.e. the sum of all the products, k at a time, of different $a_i \in \mathbf{a}$.

There are several generalizations of Newton's inequality [2], [3]. In this article we give another one. For this purpose define the sequences $\mathbf{a}'_i := \mathbf{a}/\{a_i\}, \quad i = 1, 2, \dots, n$, and by $e_k(\mathbf{a}'_i)$ denote the k-th elementary symmetric function over $\mathbf{a}'_i$. We have:

Theorem 1.2. *Let* $\mathbf{c} = \{c_i\}_{i=1}^n$ *be a weight sequence of non-negative numbers satisfying*

$$\sum_{i=1}^n c_i = 1, \tag{1.4}$$

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and, for an arbitrary sequence $\mathbf{a} = \{a_i\}_{i=1}^n$ *of real numbers, define*

$$E_k^{(c)} := \sum_{i=1}^{n} c_i e_k(\mathbf{a}_i'), \quad E_0^{(c)} = 1, \tag{1.5}$$

or equivalently,

$$E_k^{(c)} = e_k - e_{k-1} f_1 + e_{k-2} f_2^2 - \cdots + (-1)^r e_{k-r} f_r^r + \cdots + (-1)^k f_k^k, \tag{1.6}$$

where

$$f_s := \left(\sum_{i=1}^{n} c_i a_i^s \right)^{\frac{1}{s}}. \tag{1.7}$$

Then

$$E_{k-1}^{(c)} E_{k+1}^{(c)} \le A_k^{(n-1)} \left(E_k^{(c)} \right)^2, \qquad k = 1, 2, \ldots, n-1. \tag{1.8}$$

Proof. We shall give an easy proof supposing that the sequence $\mathbf{c}$ consists of arbitrary *positive rational numbers*. Since $\mathbf{a}$ and $\mathbf{c}$ are independent of each other, the truthfulness of the above theorem follows by the continuity principle.

Therefore, let $\mathbf{p} = \{p_k\}_{k=1}^n$ be an arbitrary sequence of positive integers and put

$$c_i = \frac{p_i}{\sum_1^n p_k}, \qquad i = 1, 2, \ldots, n; \quad \mathbf{p} \in \mathbb{N}. \tag{1.9}$$

Now, for a given real sequence $\mathbf{a}$, consider the polynomial $Q(x)$ defined by

$$Q(x) := \prod_{i=1}^{n} (x + a_i)^{p_i}. \tag{1.10}$$

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Since all its zeros are real, by the well-known Gauss theorem, the zeros of $Q'(x)$,

$$Q'(x) = Q(x) \sum_{i=1}^{n} \frac{p_i}{x + a_i}, \tag{1.11}$$

are also real.

In particular, the same is valid for the polynomial $R(x)$ defined by

$$R(x) := \prod_{i=1}^{n} (x + a_i) \sum_{i=1}^{n} \frac{c_i}{(x + a_i)}. \tag{1.12}$$

Since

$$R(x) = x^{n-1} + E_1^{(c)} x^{n-2} + \cdots + E_k^{(c)} x^{n-1-k} + \cdots + E_{n-1}^{(c)}, \tag{1.13}$$

the result follows by simple application of Theorem 1.1.

Remark 1. Since $\sum_{i=1}^{n} e_k(\mathbf{a}_i') = (n-k)e_k(\mathbf{a})$, putting $c_i = 1/n$, $i = 1, 2, \ldots, n$ in (1.5) and (1.8), we obtain the assertion from Theorem 1.1. Hence our result represents a generalization of Newton's theorem.

Also, denoting by $f_s^{(c)}(\mathbf{a}) = f_s := \left(\sum_{i=1}^{n} c_i a_i^s\right)^{1/s}$, $s > 0$, the classical weighted mean (with weights c) of order s, and using the identity

$$e_k(\mathbf{a}_i') = e_k(\mathbf{a}) - a_i e_{k-1}(\mathbf{a}_i'), \tag{1.14}$$

an equivalent form of $E_k^{(c)}$ arises, i.e.,

$$E_k^{(c)} = e_k - e_{k-1} f_1 + e_{k-2} f_2^2 - \cdots + (-1)^r e_{k-r} f_r^r + \cdots + (-1)^k f_k^k. \tag{1.15}$$

Putting this in (1.8), we obtain a mixed inequality connecting elementary symmetric functions with weighted means of integer order. □

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Problem 1.1. An interesting fact is that non-negativity of $\mathbf{c}$ *is not* a necessary condition for (1.8) to hold. We shall illustrate this point by an example. For $k = 1,\ n = 3$, we have

$$\begin{aligned}&(E_1^{(c)})^2 - 4E_0^{(c)}E_2^{(c)}\\&= (c_1(a_2+a_3)+c_2(a_1+a_3)+c_3(a_1+a_2))^2 - 4(c_1a_2a_3+c_2a_1a_3+c_3a_1a_2)\\&= (1-c_2)^2(a_1-a_2)^2 + 2(c_1-c_2c_3)(a_1-a_2)(a_3-a_1) + (1-c_3)^2(a_3-a_1)^2,\end{aligned}$$

and this quadratic form is positive semi-definite whenever $c_1c_2c_3 \geq 0$.

Hence, in this case the inequality (1.8) is valid for all real sequences $\mathbf{a}$ with $\mathbf{c}$ satisfying

$$c_1 + c_2 + c_3 = 1, \quad c_1c_2c_3 \geq 0. \tag{1.16}$$

Therefore there remains the seemingly difficult problem of finding true bounds for the sequence $\mathbf{c}$ satisfying (1.4), such that the inequality (1.8) holds for an arbitrary real sequence $\mathbf{a}$.

Problem 1.2. There is an interesting application of Theorem 1.2 to the well known Turan's problem. Under what conditions does the sequence of polynomials $\{Q_n(x)\}$ satisfy *Turan's inequality*

$$Q_{n-1}(x)Q_{n+1}(x) \leq (Q_n(x))^2, \tag{1.17}$$

for each $x \in [a,b]$ and $n \in [n_1, n_2]$?

This problem is solved for many classes of polynomials [4]. We shall consider here the following question [5].

An arbitrary sequence $\{d_i\}$, $i = 1, 2, \dots$ of real numbers generates a sequence of polynomials $\{P_n(x)\}$, $n = 0, 1, 2, \dots$ defined by

$$P_n(x) := x^n + d_1x^{n-1} + d_2x^{n-2} + \cdots + d_{n-1}x + d_n, \quad P_0(x) := 1. \tag{1.18}$$

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Denote also by A_n the set of zeros of $P_n(x)$.

Now, if for some $m > 1$ the set A_m consists of real numbers only, then from Theorem 1.2, it follows that

$$P_{n-1}(a)P_{n+1}(a) \leq P_n^2(a), \tag{1.19}$$

for each $a \in A_m$ and $n \in [1, m-1]$.

Is it possible to establish some simple conditions such that the Turan inequality

$$P_{n-1}(x)P_{n+1}(x) \leq P_n^2(x), \tag{1.20}$$

holds for each $x \in [\min a, \max a]_{a \in A_m}$ and $n \in [1, m-1]$.

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