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A NOTE ON SUMMABILITY FACTORS

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Abstract

In this note we investigate the relation between two theorems proved by Bor [2, 3] on $|\bar{N}, p_n|_k$ summability of an infinite series.

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Contents

1. Introduction

Let $\sum a_n$ be a given infinite series with $\{s_n\}$ as the sequence of its n-th partial sums. Let $\{p_n\}$ be a sequence of positive constants such that $P_n = p_0 + p_1 + p_2 + \cdots + p_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

Let

$$t_n = \frac{1}{P_n} \sum_{\nu=1}^{n} p_\nu s_\nu .$$

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|$ if $\sum_1^\infty |t_n - t_{n-1}| < \infty$. It is said to be summable $|\bar{N}, p_n|_k,\ k \geq 1$ [1] if

$$\sum_1^\infty \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty, \tag{1.1}$$

and bounded $[\bar{N}, p_n]_k,\ \ k \geq 1$ if

$$\sum_1^n p_\nu |s_\nu|^k = O(P_n), \qquad n \longrightarrow \infty. \tag{1.2}$$

Concerning $|\bar{N}, p_n|$ summability factors of $\sum a_n$, T. Singh [6] proved the following theorem:

Theorem A. *If the sequences $\{p_n\}$ and $\{\lambda_n\}$ satisfy the conditions*

$$\sum_1^\infty p_n |\lambda_n| < \infty, \tag{1.3}$$

$$P_n|\Delta\lambda_n| \le Cp_n|\lambda_n|, \tag{1.4}$$

C is a constant, and if $\sum a_n$ is bounded $[\bar{N}, p_n]_1$, then $\sum a_n P_n \lambda_n$ is summable $|\bar{N}, p_n|$.

Earlier in 1968 N. Singh [5] had obtained the following theorem.

Theorem B. *If $\sum a_n$ is bounded $[\bar{N}, p_n]_1$ and $\{\lambda_n\}$ is a sequence satisfying the following conditions*

$$\sum_1^\infty \frac{p_n|\lambda_n|}{P_n} < \infty, \tag{1.5}$$

$$\frac{P_n}{p_n}\Delta\lambda_n = O(|\lambda_n|), \tag{1.6}$$

then $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|$.

In order to extend these theorems to the summability $|\bar{N}, p_n|_k,\ \ k \ge 1$, Bor [2, 3] proved the following theorems.

Theorem C. *Under the conditions (1.2), (1.3) and (1.4), the series $\sum a_n P_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \ge 1$.*

Theorem D. *If $\sum a_n$ is bounded $[\bar{N}, p_n]_k,\ \ k \ge 1$ and $\{\lambda_n\}$, is a sequence satisfying the conditions (1.4) and (1.5), then $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$.*

2. Results

In this note we propose to examine the relation between Theorem C and Theorem D.

We recall that recently Sarigol and Ozturk [4] constructed an example to demonstrate that the hypotheses of Theorem A are not sufficient for the summability $|\bar{N}, p_n|$ of $\sum a_n P_n \lambda_n$. They proved that Theorem A holds true if we assume the additional condition

$$p_{n+1} = O(p_n). \tag{2.1}$$

From (1.4) we find that

$$\left|\frac{\Delta\lambda_n}{\lambda_n}\right| = \left|1 - \frac{\lambda_{n+1}}{\lambda_n}\right| \leq \frac{Cp_n}{P_n},$$

Hence

$$\begin{aligned} \left|\frac{\lambda_{n+1}}{\lambda_n}\right| &= \left|\frac{\lambda_{n+1}}{\lambda_n} - 1 + 1\right| \\ &\leq \left|1 - \frac{\lambda_{n+1}}{\lambda_n}\right| + 1 \\ &\leq \frac{Cp_n}{P_n} + 1 \leq C. \end{aligned}$$

Thus $|\lambda_{n+1}| \leq C|\lambda_n|$, and combining this with (2.1) we get

$$p_{n+1}|\lambda_{n+1}| \leq Cp_n|\lambda_n|. \tag{2.2}$$

Clearly (2.1) and (1.4) imply (2.2). However (2.2) need not imply (2.1) or (1.4). In view of

$$\Delta(P_n\lambda_n) = P_n\Delta\lambda_n - p_{n+1}\lambda_{n+1}$$

it is clear that if (2.2) holds, then the condition (1.4) is equivalent to the condition

$$|\Delta(P_n\lambda_n)| \leq Cp_n|\lambda_n|. \tag{2.3}$$

It can be easily verified that a corrected version of Theorem A and Theorem C and also a slight generalization of the result of Sarigol and Ozturk for $k = 1$ can be stated as

Theorem 2.1. *Under the conditions (1.2), (1.3) (2.2) and (2.3) the series* $\sum a_nP_n\lambda_n$ *is summable* $|\bar{N}, p_n|_k,\ k \geq 1$

We now proceed to show that Theorem 2.1 holds good without condition (2.2).

Thus we have:

Theorem 2.2. *Under the conditions (1.2), (1.3) and (2.3) the series* $\sum a_nP_n\lambda_n$ *is summable* $|\bar{N}, p_n|_k,\ k \geq 1.$

To prove Theorem 2.2 we first prove the following lemma.

Lemma 2.3. *Under the conditions of Theorem 2.2*

$$\sum_1^m p_n|\lambda_n||s_n|^k = O(1) \ \text{ as } \ m \longrightarrow \infty. \tag{2.4}$$

3. Proofs

Proof of Lemma 2.3. In view of (1.3) and (2.3)

$$\sum_1^\infty |\Delta(\lambda_n P_n)| \leq C \sum_1^\infty p_n|\lambda_n| < \infty,$$

so it follows that $\{P_n\lambda_n\} \in BV$ and hence $P_n|\lambda_n| = O(1)$.

Now

$$\begin{aligned}
\sum_1^m p_n\lambda_n||s_n|^k &= \sum_1^{m-1} \Delta|\lambda_n| \sum_{\nu=1}^n p_\nu|s_\nu|^k + |\lambda_m| \sum_{\nu=1}^m p_\nu|s_\nu|^k \\
&= O(1) \sum_1^{m-1} |\Delta\lambda_n|P_n + O(|\lambda_m|P_m) \\
&= O(1) \left(\sum_1^{m-1} |\Delta(P_n\lambda_n)| + p_{n+1}|\lambda_{n+1}| \right) + O(1) \\
&= O(1) \sum_1^{m-1} p_n|\lambda_n| + O(1) \sum_1^m p_{n+1}|\lambda_{n+1}| + O(1) \\
&= O(1).
\end{aligned}$$

$\square$

Proof of Theorem 2.2. Let T_n denote the n^{th} $(\bar{N}, p_n)$ means of the series $\sum a_n P_n \lambda_n$.

Then

$$T_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \sum_{r=0}^{\nu} a_r P_r \lambda_r$$
$$= \frac{1}{P_n}\sum_{\nu=0}^{n} (P_n - P_{\nu-1})\, a_\nu P_\nu \lambda_\nu.$$

so that for $n \geq 1$

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu P_\nu \lambda_\nu$$
$$= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \Delta(P_{\nu-1} P_\nu \lambda_\nu) s_\nu + p_n \lambda_n s_n$$
$$= L_1 + L_2, \quad \text{say.}$$

Thus to prove the theorem it is sufficient to show that

$$\sum_1^\infty \left(\frac{P_n}{p_n}\right)^{k-1} |L_\nu|^k < \infty, \qquad \nu = 1, 2.$$

Now

$$|\Delta(P_{\nu-1} P_\nu \lambda_\nu)| \leq p_\nu P_\nu |\lambda_\nu| + P_\nu |\Delta(P_\nu \lambda_\nu)|$$
$$\leq C p_\nu P_\nu |\lambda_\nu|$$

in view of (2.3). So

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|L_1|^k = O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}p_\nu P_\nu|\lambda_\nu||s_\nu|\right)^k$$
$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{\nu=1}^{n-1}(P_\nu|\lambda_\nu|)^k|s_\nu|^k p_\nu\right)\left(\sum_{\nu=1}^{n-1}p_\nu\right)^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n-1}P_\nu|\lambda_\nu|p_\nu|s_\nu|^k$$
$$= O(1)\sum_{\nu=1}^{m}p_\nu|\lambda_\nu|s_\nu|^k = O(1)$$

in view of the lemma and $P_n|\lambda_n| = O(1)$.

Also

$$\sum_{1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|L_2|^k = O(1)\sum_{1}^{m+1}p_n|\lambda_n|^k|s_n|^kP_n^{k-1}$$
$$= O(1)\sum_{1}^{m+1}p_n|\lambda_n||s_n|^k$$
$$= O(1).$$

This proves Theorem 2.2. □

Thus a generalization of a corrected version of Theorem C is Theorem 2.2.

Writing $\lambda_n = \mu_n P_n$ the conditions (1.5) and (1.4) become

$$\sum_{1}^{\infty} p_n|\mu_n| < \infty, \tag{3.1}$$

$$|\Delta(P_n\mu_n)| \leq Cp_n|\mu_n|, \tag{3.2}$$

consequently Theorem D can be stated as:

If $\sum a_n$ *is bounded* $[\bar{N}, P_n]_k,\ k \geq 1$ *and* $\{\mu_n\}$ *is a sequence satisfying (3.1) and (3.2) then* $\sum a_n P_n \mu_n$ *is summable* $|\bar{N}, p_n|_k,\ k \geq 1.$

Thus Theorem D is the same as Theorem 2.2 which is a generalization of the corrected version of Theorem C.

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