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GENERAL SYSTEM OF STRONGLY PSEUDOMONOTONE NONLINEAR VARIATIONAL INEQUALITIES BASED ON PROJECTION SYSTEMS

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ABSTRACT. Let K_1 and K_2, respectively, be non empty closed convex subsets of real Hilbert spaces H_1 and H_2. The $Approximation - solvability$ of a generalized system of nonlinear variational inequality $(SNVI)$ problems based on the convergence of projection methods is discussed. The SNVI problem is stated as follows: find an element $(x^*, y^*) \in K_1 \times K_2$ such that

$$\langle \rho S(x^*, y^*), x - x^* \rangle \geq 0, \ \forall x \in K_1 \text{ and for } \rho > 0,$$
$$\langle \eta T(x^*, y^*), y - y^* \rangle \geq 0, \ \forall y \in K_2 \text{ and for } \eta > 0,$$

where $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ are nonlinear mappings.

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1. INTRODUCTION

Projection-like methods in general have been one of the most fundamental techniques for establishing the convergence analysis for solutions of problems arising from several fields, such as complementarity theory, convex quadratic programming, and variational problems. There exists a vast literature on approximation-solvability of several classes of variational/hemivariational inequalities in different space settings. The author [6, 7] introduced and studied a new system of nonlinear variational inequalities in Hilbert space settings. This class encompasses several classes of nonlinear variational inequality problems. In this paper we intend to explore, based on a general system of projection-like methods, the approximation-solvability of a system of nonlinear strongly pseudomonotone variational inequalities in Hilbert spaces. The obtained results extend/generalize the results in [1], [5] – [7] to the case of strongly pseudomonotone system of nonlinear variational inequalities. Approximation solvability of this system can also be established using the resolvent operator technique but in the more relaxed setting of Hilbert spaces. For more details, we refer the reader to [1] – [10].

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Let H_1 and H_2 be two real Hilbert spaces with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ be any mappings on $K_1 \times K_2$, where K_1 and K_2 are nonempty closed convex subsets of H_1 and H_2, respectively. We consider a system of nonlinear variational inequality (abbreviated as SNVI) problems: determine an element $(x^*, y^*) \in K_1 \times K_2$ such that

$$\langle \rho S(x^*, y^*), x - x^*\rangle \geq 0 \; \forall x \in K_1 \tag{1.1}$$

$$\langle \eta T(x^*, y^*), y - y^*\rangle \geq 0 \; \forall y \in K_2, \tag{1.2}$$

where $\rho, \eta > 0$.

The SNVI $(1.1) - (1.2)$ problem is equivalent to the following projection formulas

$$x^* = P_k[x^* - \rho S(x^*, y^*)] \quad \text{for } \rho > 0$$
$$y^* = Q_k[y^* - \eta T(x^*, y^*)] \quad \text{for } \eta > 0,$$

where P_k is the projection of H_1 onto K_1 and Q_K is the projection of H_2 onto K_2.

We note that the SNVI $(1.1)-(1.2)$ problem extends the NVI problem: determine an element $x^* \in K_1$ such that

$$\langle S(x^*), x - x^*\rangle \geq 0, \; \forall x \in K_1. \tag{1.3}$$

Also, we note that the SNVI $(1.1)-(1.2)$ problem is equivalent to a system of nonlinear complementarities (abbreviated as SNC): find an element $(x^*, y^*) \in K_1 \times K_2$ such that $S(x^*, y^*) \in K_1^*$, $T(x^*, y^*) \in K_2^*$, and

$$\langle \rho S(x^*, y^*), x^*\rangle = 0 \quad \text{for} \quad \rho > 0, \tag{1.4}$$

$$\langle \eta T(x^*, y^*), y^*\rangle = 0 \quad \text{for} \quad \eta > 0, \tag{1.5}$$

where K_1^* and K_2^*, respectively, are polar cones to K_1 and K_2 defined by

$$K_1^* = \{f \in H_1 : \langle f, x\rangle \geq 0, \; \forall x \in K_1\}.$$
$$K_2^* = \{g \in H_2 : \langle g, y\rangle \geq 0, \; \forall g \in K_2\}.$$

Now, we recall some auxiliary results and notions crucial to the problem on hand.

Lemma 1.1. *For an element $z \in H$, we have*

$$x \in K \quad \textit{and} \quad \langle x - z, y - x\rangle \geq 0, \quad \forall y \in K \quad \textit{if and only if} \quad x = P_k(z).$$

Lemma 1.2 ([3]). *Let $\{\alpha^k\}, \{\beta^k\}$, and $\{\gamma^k\}$ be three nonnegative sequences such that*

$$\alpha^{k+1} \leq (1 - t^k)\alpha^k + \beta^k + \gamma^k \quad \textit{for} \quad k = 0, 1, 2, ...,$$

where $t^k \in [0, 1]$, $\sum_{k=0}^{\infty} t^k = \infty$, $\beta^k = o(t^k)$, and $\sum_{k=0}^{\infty} \gamma^k < \infty$. Then $\alpha^k \to 0$ as $k \to \infty$.

A mapping $T : H \to H$ from a Hilbert space H into H is called monotone if $\langle T(x) - T(y), x - y\rangle \geq 0$ for all $x, y \in H$. The mapping T is $(r)-$*strongly* monotone if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y\rangle \geq r||x - y||^2 \quad \text{for a constant } r > 0.$$

This implies that $\|T(x) - T(y)\| \geq r\|x - y\|$, that is, T is (r)-expansive, and when $r = 1$, it is expansive. The mapping T is called (s)-Lipschitz continuous (or Lipschitzian) if there exists a constant $s \geq 0$ such that $\|T(x) - T(y)\| \leq s\|x - y\|$, $\forall\, x, y \in H$. T is called (μ)-cocoercive if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y\rangle \geq \mu||T(x) - T(y)||^2 \quad \text{for a constant } \mu > 0.$$

Clearly, every (μ)-cocoercive mapping T is $(\frac{1}{\mu})$-Lipschitz continuous. We can easily see that the following implications on monotonicity, strong monotonicity and expansiveness hold:

$$\begin{array}{ccc} \text{strong monotonicity} & \Rightarrow & \text{expansiveness} \\ \Downarrow & & \\ \text{monotonicity} & & \end{array}$$

T is called relaxed (γ)-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq (-\gamma)\|T(x) - T(y)\|^2, \quad \forall x, y \in H.$$

T is said to be (r)-strongly pseudomonotone if there exists a positive constant r such that

$$\langle T(y), x - y\rangle \geq 0 \Rightarrow \langle T(x), x - y\rangle \geq r\left\|x - y\right\|^2, \quad \forall x, y \in H.$$

T is said to be relaxed (γ, r)-cocoercive if there exist constants γ,$r > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq (-\gamma)\left\|T(x) - T(y)\right\|^2 + r\left\|x - y\right\|^2.$$

Clearly, it implies that

$$\langle T(x) - T(y), x - y\rangle \geq (-\gamma)\left\|T(x) - T(y)\right\|^2,$$

that is, T is relaxed (γ)-cocoercive.

T is said to be relaxed (γ, r)-pseudococoercive if there exist positive constants γ and r such that

$$\langle T(y), x - y\rangle \geq 0 \Rightarrow \langle T(x), x - y\rangle \geq (-\gamma)\left\|T(x) - T(y)\right\|^2 + r\left\|x - y\right\|^2, \quad \forall x, y \in H.$$

Thus, we have following implications:

$$\begin{array}{ccc} (r)\text{-strong monotonicity} & \Rightarrow & \text{strong } (r)\text{-pseudomonotonicity} \\ \Downarrow & & \\ \text{relaxed } (\gamma, r)\text{-cocoercivity} & & \\ \Downarrow & & \\ \text{relaxed } (\gamma, r)\text{-pseudococoercivity} & & \end{array}$$

2. GENERAL PROJECTION METHODS

This section deals with the convergence of projection methods in the context of the approximation-solvability of the SNVI $(1.1) - (1.2)$ problem.

Algorithm 2.1. For an arbitrarily chosen initial point $(x^0, y^0) \in K_1 \times K_2$, compute the sequences $\{x^k\}$ and $\{y^k\}$ such that

$$x^{k+1} = (1 - a^k - b^k)x^k + a^k P_k[x^k - \rho S(x^k, y^k)] + b^k u^k$$

$$y^{k+1} = (1 - \alpha^k - \beta^k)y^k + \alpha^k Q_K[y^k - \eta T(x^k, y^k)] + \beta^k v^k,$$

where P_K is the projection of H_1 onto K_1, Q_K is the projection of H_2 onto K_2, ρ, $\eta > 0$ are constants, $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ are any two mappings, and u^k and v^k, respectively, are bounded sequences in K_1 and K_2. The sequences $\{a^k\}, \{b^k\}, \{\alpha^k\}$, and $\{\beta^k\}$ are in $[0, 1]$ with $(k \geq 0)$

$$0 \leq a^k + b^k \leq 1, \qquad 0 \leq \alpha^k + \beta^k \leq 1.$$

Algorithm 2.2. For an arbitrarily chosen initial point $(x^0, y^0) \in K_1 \times K_2$, compute the sequences $\{x^k\}$ and $\{y^k\}$ such that

$$x^{k+1} = (1 - a^k - b^k)x^k + a^k P_k[x^k - \rho S(x^k, y^k)] + b^k u^k$$

$$y^{k+1} = (1 - a^k - b^k)y^k + a^k Q_K[y^k - \eta T(x^k, y^k)] + b^k v^k,$$

where P_K is the projection of H_1 onto K_1, Q_K is the projection of H_2 onto K_2, ρ, $\eta > 0$ are constants, $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ are any two mappings, and u^k and v^k, respectively, are bounded sequences in K_1 and K_2. The sequences $\{a^k\}$ and $\{b^k\}$, are in $[0, 1]$ with $(k \geq 0)$

$$0 \leq a^k + b^k \leq 1.$$

Algorithm 2.3. For an arbitrarily chosen initial point $(x^0, y^0) \in K_1 \times K_2$, compute the sequences $\{x^k\}$ and $\{y^k\}$ such that

$$x^{k+1} = (1 - a^k)x^k + a^k P_k[x^k - \rho S(x^k, y^k)]$$

$$y^{k+1} = (1 - a^k)y^k + a^k Q_K[y^k - \eta T(x^k, y^k)],$$

where P_K is the projection of H_1 onto K_1, Q_K is the projection of H_2 onto K_2, ρ, $\eta > 0$ are constants, $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ are any two mappings. The sequence $\{a^k\} \in [0, 1]$ for $k \geq 0$.

We consider, based on Algorithm 2.2, the approximation solvability of the SNVI $(1.1)-(1.2)$ problem involving strongly pseudomonotone and Lipschitz continuous mappings in Hilbert space settings.

Theorem 2.1. *Let H_1 and H_2 be two real Hilbert spaces and, K_1 and K_2, respectively, be nonempty closed convex subsets of H_1 and H_2. Let $S : K_1 \times K_2 \to H_1$ be strongly $(r)-$pseudomonotone and $(\mu)-$Lipschitz continuous in the first variable and let S be $(\nu)-$Lipschitz continuous in the second variable. Let $T : K_1 \times K_2 \to H_2$ be strongly $(s)-$pseudomonotone and $(\beta)-$Lipschitz continuous in the second variable and let T be $(\tau)-$Lipschitz continuous in the first variable. Let $\| \cdot \|^*$ denote the norm on $H_1 \times H_2$ defined by*

$$\|(x, y)\|^* = (\|x\| + \|y\|) \, \forall (x, y) \in H_1 \times H_2.$$

In addition, let

$$\theta = \sqrt{1 - 2\rho r + \rho + \left(\frac{\rho\mu^2}{2}\right) + \rho^2\mu^2} + \eta\tau < 1$$

$$\sigma = \sqrt{1 - 2\eta r + \eta + \left(\frac{\eta\beta^2}{2}\right) + \eta^2\beta^2} + \rho\nu < 1,$$

let $(x^*, y^*) \in K_1 \times K_2$ form a solution to the SNVI $(1.1) - (1.2)$ problem, and let sequences $\{x^k\}$, and $\{y^k\}$ be generated by Algorithm 2.2. Furthermore, let

(i) $\langle S(x^*, y^k), x^k - x^* \rangle \geq 0;$
(ii) $\langle T(x^k, y^*), y^k - y^* \rangle \geq 0;$
(iii) $0 \leq a^k + b^k \leq 1;$
(iv) $\sum_{k=0}^{\infty} a^k = \infty$, *and* $\sum_{k=0}^{\infty} b^k < \infty;$
(v) $0 < \rho < \frac{2r}{\mu^2}$ *and* $0 < \eta < \frac{2s}{\beta^2}$.

Then the sequence $\{x^k, y^k\}$ converges to (x^*, y^*).

Proof. Since $(x^*, y^*) \in K_1 \times K_2$ forms a solution to the SNVI $(1.1) - (1.2)$ problem, it follows that

$$x^* = P_K[x^* - \rho S(x^*, y^*)] \quad \text{and} \quad y^* = Q_K[x^* - \eta T(x^*, y^*)].$$

Applying Algorithm 2.2, we have

$$\begin{aligned}
(2.1)\qquad \|x^{k+1} - x^*\| &= \|(1 - a^k - b^k)x^k + a^k P_K[x^k - \rho S(x^k, y^k)] + b^k u^k \\
&\qquad - (1 - a^k - b^k)x^* - a^k P_K[x^* - \rho S(x^*, y^*)] - b^k x^*\| \\
&\le (1 - a^k - b^k)\|x^k - x^*\| \\
&\qquad + a^k\|P_K[x^k - \rho S(x^k, y^k)] - P_K[x^* - \rho S(x^*, y^*)]\| + Mb^k \\
&\le (1 - a^k)\|x^k - x^*\| + a^k\|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k) \\
&\qquad + S(x^*, y^k) - S(x^*, y^*)]\| + Mb^k \\
&\le (1 - a^k)\|x^k - x^*\| + a^k\|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k)]\| \\
&\qquad + \rho\|[S(x^*, y^k) - S(x^*, y^*)]\| + Mb^k,
\end{aligned}$$

where

$$M = \max\{\sup\|u^k - x^*\|, \sup\|v^k - y^*\|\} < \infty.$$

Since S is strongly $(r)-$ pseudomonotone and $(\mu)-$Lipschitz continuous in the first variable, and S is $(\nu)-$*Lipschitz* continuous in the second variable, we have in light of (i) that

$$\begin{aligned}
(2.2)\qquad & \|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k)]\|^2 \\
&= \|x^k - x^*\|^2 - 2\rho\langle S(x^k, y^k) - S(x^*, y^k), x^k - x^*\rangle + \rho^2\|S(x^k, y^k) - S(x^*, y^k)\|^2 \\
&= \|x^k - x^*\|^2 - 2\rho\langle S(x^k, y^k), x^k - x^*\rangle + 2\rho\langle S(x^*, y^k), x^k - x^*\rangle \\
&\qquad + \rho^2\|S(x^k, y^k) - S(x^*, y^k)\|^2 \\
&\le \|x^k - x^*\|^2 - 2\rho r\|x^k - x^*\|^2 + \rho^2\mu^2\|x^k - x^*\|^2 + 2\rho\langle S(x^*, y^k), x^k - x^*\rangle \\
&\le \|x^k - x^*\|^2 - 2\rho r\|x^k - x^*\|^2 + \rho^2\mu^2\|x^k - x^*\|^2 + 2\rho\langle S(x^*, y^k), x^k - x^*\rangle \\
&= [1 - 2\rho r + \rho^2\mu^2]\|x^k - x^*\|^2 + 2\rho\langle S(x^*, y^k), x^k - x^*\rangle.
\end{aligned}$$

On the other hand, we have

$$2\rho\langle S(x^*, y^k), x^k - x^*\rangle \le \rho[\|S(x^*, y^k)\|^2 + \|x^k - x^*\|^2]$$

and

$$\begin{aligned}
(2.3)\qquad & \|S(x^*, y^k)\|^2 \\
&= \frac{1}{2}\{\|S(x^*, y^k) - S(x^k, y^k)\|^2 - 2[\|S(x^k, y^k)\|^2 - \frac{1}{2}\|S(x^k, y^k) + S(x^*, y^k)\|^2\} \\
&\le \frac{1}{2}\{\|S(x^*, y^k) - S(x^k, y^k)\|^2 - 2[\|S(x^k, y^k)\|^2 - \frac{1}{2}\mid \|S(x^k, y^k) - S(x^*, y^k)\| \mid^2]\} \\
&\le \frac{1}{2}\{\|S(x^*, y^k) - S(x^k, y^k)\|^2 \\
&\le \frac{\mu^2}{2}\|x^k - x^*\|^2,
\end{aligned}$$

where

$$\|S(x^k, y^k)\|^2 - \frac{1}{2}\mid \|S(x^k, y^k) - S(x^*, y^k)\| \mid^2 > 0.$$

Therefore, we get

$$2\rho\langle S(x^*, y^k), x^k - x^*\rangle \le \left[\rho + \left(\frac{\rho\mu^2}{2}\right)\right] \|x^k - x^*\|^2.$$

It follows that

$$\|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k)]\|^2 \le \left[1 - 2\rho r + \rho + \left(\frac{\rho\mu^2}{2}\right) + (\rho\mu)^2\right] \|x^k - x^*\|^2.$$

As a result, we have

$$\|x^{k+1} - x^*\| \le (1 - a^k)\|x^k - x^*\| + a^k\theta\|x^k - x^*\| + a^k\rho\nu\|y^k - y^*\| + Mb^k, \tag{2.4}$$

where $\theta = \sqrt{1 - 2\rho r + \rho + \left(\frac{\rho\mu^2}{2}\right) + \rho^2\mu^2}$.

Similarly, we have

$$\left\|y^{k+1} - y^*\right\| \le (1 - a^k)\|y^k - y^*\| + a^k\sigma\|y^k - y^*\| + a^k\eta\tau\|x^k - x^*\| + Mb^k, \tag{2.5}$$

where $\sigma = \sqrt{1 - 2\eta r + \eta + \left(\frac{\eta\beta^2}{2}\right) + \eta^2\beta^2}$.

It follows from (2.4) and (2.5) that

$$\begin{aligned} &\|x^{k+1} - x^*\| + \|y^{k+1} - y^*\| \\ &\le (1 - a^k)\|x^k - x^*\| + a^k\theta\|x^k - x^*\| + a^k\eta\tau\|x^k - x^*\| + Mb^k \\ &\qquad + (1 - a^k)\|y^k - y^*\| + a^k\sigma\|y^k - y^*\| + a^k\rho\nu\|y^k - y^*\| + Mb^k \\ &= [1 - (1 - \delta)a^k](\|x^k - x^*\| + \|y^k - y^*\|) + 2Mb^k, \end{aligned} \tag{2.6}$$

where $\delta = \max\{\theta + \eta\tau, \sigma + \rho\nu\}$ and $H_1 \times H_2$ is a Banach space under the norm $\|\cdot\|^*$.

If we set

$$\alpha^k = \|x^k - x^*\| + \|y^k - y^*\|, \qquad t^k = (1 - \delta)a^k,$$
$$\beta^k = 2Mb^k \qquad \text{for } k = 0, 1, 2, ...,$$

in Lemma 1.2, and apply (iii) and (iv), we conclude that

$$\|x^k - x^*\| + \|y^k - y^*\| \to 0$$

as $k \to \infty$.

Hence,

$$\|x^{k+1} - x^*\| + \|y^{k+1} - y^*\| \to 0.$$

Consequently, the sequence $\{(x^k, y^k)\}$ converges strongly to (x^*, y^*), a solution to the SNVI $(1.1) - (1.2)$ problem. This completes the proof. □

Note that the proof of the following theorem follows rather directly without using Lemma 1.2.

Theorem 2.2. *Let H_1 and H_2 be two real Hilbert spaces and, K_1 and K_2, respectively, be nonempty closed convex subsets of H_1 and H_2. Let $S : K_1 \times K_2 \to H_1$ be strongly $(r)-$pseudomonotone and $(\mu)-$Lipschitz continuous in the first variable and let S be $(\nu)-$Lipschitz continuous in the second variable. Let $T : K_1 \times K_2 \to H_2$ be strongly $(s)-$pseudomonotone and $(\beta)-$Lipschitz continuous in the second variable and let T be $(\tau)-$Lipschitz continuous in the first variable. Let $\|\cdot\|^*$ denote the norm on $H_1 \times H_2$ defined by*

$$\|(x, y)\|^* = (\|x\| + \|y\|) \quad \forall (x, y) \in H_1 \times H_2.$$

In addition, let

$$\theta = \sqrt{1 - 2\rho r + \rho + \left(\frac{\rho\mu^2}{2}\right) + \rho^2\mu^2 + \eta\tau} < 1,$$

$$\sigma = \sqrt{1 - 2\eta r + \eta + \left(\frac{\eta\beta^2}{2}\right) + \eta^2\beta^2 + \rho\nu} < 1,$$

let $(x^*, y^*) \in K_1 \times K_2$ *form a solution to the SNVI* $(1.1) - (1.2)$ *problem, and let sequences* $\{x^k\}$, *and* $\{y^k\}$ *be generated by Algorithm 2.3. Furthermore, let*

(i) $\langle S(x^*, y^k), x^k - x^*\rangle \geq 0$
(ii) $\langle T(x^k, y^*), y^k - y^*\rangle \geq 0$
(iii) $0 \leq a^k \leq 1$
(iv) $\sum_{k=0}^{\infty} a^k = \infty$
(v) $0 < \rho < \frac{2r}{\mu^2}$ *and* $0 < \eta < \frac{2s}{\beta^2}$.

Then the sequence $\{x^k, y^k)\}$ *converges strongly to* (x^*, y^*).

Proof. Since $(x^*, y^*) \in K_1 \times K_2$ forms a solution to the SNVI $(1.1) - (1.2)$ problem, it follows that

$$x^* = P_K[x^* - \rho S(x^*, y^*)] \qquad \text{and} \qquad y^* = Q_K[x^* - \eta T(x^*, y^*)].$$

Applying Algorithm 2.3, we have

$$\begin{aligned} \|x^{k+1} - x^*\| &\\ &= \|(1 - a^k)x^k + a^k P_K[x^k - \rho S(x^k, y^k)] - (1 - a^k)x^* - a^k P_K[x^* - \rho S(x^*, y^*)]\| \\ &\leq (1 - a^k)\|x^k - x^*\| + a^k\|P_K[x^k - \rho S(x^k, y^k)] - P_K[x^* - \rho S(x^*, y^*)]\| \\ &\leq (1 - a^k)\|x^k - x^*\| \\ &\quad + a^k\|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k) + S(x^*, y^k) - S(x^*, y^*)]\| \\ &\leq (1 - a^k)\|x^k - x^*\| + a^k\|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k)]\| \\ &\quad + \rho\|[S(x^*, y^k) - S(x^*, y^*)]\|. \end{aligned} \tag{2.7}$$

Since S is strongly $(r)-$ pseudomonotone and $(\mu)-$Lipschitz continuous in the first variable, and S is $(\nu)-$*Lipschitz* continuous in the second variable, we have in light of (i) that

$$\begin{aligned} \|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k)]\|^2 &\\ &= \|x - x^*\|^2 - 2\rho\langle S(x^k, y^k) - S(x^*, y^k), x^k - x^*\rangle + \rho^2\|S(x^k, y^k) - S(x^*, y^k)\|^2 \\ &= \|x - x^*\|^2 - 2\rho\langle S(x^k, y^k), x^k - x^*\rangle + 2\rho\langle S(x^*, y^k), x^k - x^*\rangle \\ &\quad + \rho^2\|S(x^k, y^k) - S(x^*, y^k)\|^2 \\ &\leq \|x^k - x^*\|^2 - 2\rho r\|x^k - x^*\|^2 + \rho^2\mu^2\|x^k - x^*\|^2 + 2\rho\langle S(x^*, y^k), x^k - x^*\rangle \\ &\leq \|x^k - x^*\|^2 - 2\rho r\|x^k - x^*\|^2 + \rho^2\mu^2\|x^k - x^*\|^2 + 2\rho\langle S(x^*, y^k), x^k - x^*\rangle \\ &= [1 - 2\rho r + \rho^2\mu^2]\|x^k - x^*\|^2 + 2\rho\langle S(x^*, y^k), x^k - x^*\rangle. \end{aligned} \tag{2.8}$$

On the other hand, we have

$$2\rho\langle S(x^*, y^k), x^k - x^*\rangle \leq \rho[\|S(x^*, y^k)\|^2 + \|x^k - x^*\|^2]$$

and

$$\|S(x^*, y^k)\|^2 \tag{2.9}$$
$$= \frac{1}{2}\Bigg\{\|S(x^*, y^k) - S(x^k, y^k)\|^2$$
$$- 2\left[\|S(x^k, y^k)\|^2 - \frac{1}{2}\|S(x^k, y^k) + S(x^*, y^k)\|^2\right]\Bigg\}$$
$$\leq \frac{1}{2}\Bigg\{\|S(x^*, y^k) - S(x^k, y^k)\|^2$$
$$- 2\left[\|S(x^k, y^k)\|^2 - \frac{1}{2}\mid \|S(x^k, y^k) - S(x^*, y^k)\| \mid^2\right]\Bigg\}$$
$$\leq \frac{1}{2}\|S(x^*, y^k) - S(x^k, y^k)\|^2$$
$$\leq \frac{\mu^2}{2}\|x^k - x^*\|^2,$$

where

$$\|S(x^k, y^k)\|^2 - \frac{1}{2}\left|\|S(x^k, y^k) - S(x^*, y^k)\|\right|^2 > 0.$$

Therefore, we get

$$2\rho\langle S(x^*, y^k), x^k - x^*\rangle \leq \left[\rho + \left(\frac{\rho\mu^2}{2}\right)\right]\|x^k - x^*\|^2.$$

It follows that

$$\|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k)]\|^2 \leq \left[1 - 2\rho r + \rho + \left(\frac{\rho\mu^2}{2}\right) + (\rho\mu)^2\right]\|x^k - x^*\|^2.$$

As a result, we have

$$\|x^{k+1} - x^*\| \leq (1 - a^k)\|x^k - x^*\| + a^k\theta\|x^k - x^*\| + a^k\rho\nu\|y^k - y^*\|, \tag{2.10}$$

where $\theta = \sqrt{1 - 2\rho r + \rho + \left(\frac{\rho\mu^2}{2}\right) + \rho^2\mu^2}$.

Similarly, we have

$$\|y^{k+1} - y^*\| \leq (1 - a^k)\|y^k - y^*\| + a^k\sigma\|y^k - y^*\| + a^k\eta\tau\|x^k - x^*\|, \tag{2.11}$$

where $\sigma = \sqrt{1 - 2\eta r + \eta + \left(\frac{\eta\beta^2}{2}\right) + \eta^2\beta^2}$.

It follows from (2.9) and (2.10) that

$$\|x^{k+1} - x^*\| + \|y^{k+1} - y^*\| \tag{2.12}$$
$$\leq (1 - a^k)\|x^k - x^*\| + a^k\theta\|x^k - x^*\| + a^k\eta\tau\|x^k - x^*\|$$
$$+ (1 - a^k)\|y^k - y^*\| + a^k\sigma\|y^k - y^*\| + a^k\rho\nu\|y^k - y^*\|$$
$$= [1 - (1 - \delta)a^k](\|x^k - x^*\| + \|y^k - y^*\|)$$
$$\leq \prod_{j=0}^{k}[1 - (1 - \delta)a^j](\|x^0 - x^*\| + \|y^0 - y^*\|),$$

where $\delta = \max\{\theta + \eta\tau, \sigma + \rho\nu\}$ and $H_1 \times H_2$ is a Banach space under the norm $\|\cdot\|^*$.

Since $\delta < 1$ and $\sum_{k=0}^{\infty} a^k$ is divergent, it follows that

$$\lim_{k\to\infty} \prod_{j=0}^{k} [1 - (1-\delta)a^j] = 0 \quad \text{as } k \to \infty.$$

Therefore,

$$\|x^{k+1} - x^*\| + \|y^{k+1} - y^*\| \to 0,$$

and consequently, the sequence $\{(x^k, y^k)\}$ converges strongly to (x^*, y^*), a solution to the $SNVI\,(1.1) - (1.2)$ problem. This completes the proof. □

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