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DIRICHLET GREEN FUNCTIONS FOR PARABOLIC OPERATORS WITH SINGULAR LOWER-ORDER TERMS

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ABSTRACT. We prove the existence and uniqueness of a continuous Green function for the parabolic operator $L = \partial/\partial t - \operatorname{div}(A(x,t)\nabla_x) + \nu \cdot \nabla_x + \mu$ with the initial Dirichlet boundary condition on a $C^{1,1}$-cylindrical domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$, $n \geq 1$, satisfying lower and upper estimates, where $\nu = (\nu_1, \dots, \nu_n)$, ν_i and μ are in general classes of signed Radon measures covering the well known parabolic Kato classes.

Key words and phrases: Green function, Parabolic operator, Initial-Dirichlet problem, Boundary behavior, Singular potential, Singular drift term, Radon measure, Schrödinger heat kernel, Parabolic Kato class.

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1. INTRODUCTION

In this paper we are interested in the parabolic operator

$$L = L_0 + \nu \cdot \nabla_x + \mu,$$

where $L_0 = \partial/\partial t - \operatorname{div}(A(x,t)\nabla_x)$ on $\Omega = D \times]0,T[$, D is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 1$ and $0 < T < \infty$. The matrix A is assumed to be real, symmetric, uniformly elliptic with Lipschitz continuous coefficients, $\nu = (\nu_1, \dots, \nu_n)$, ν_i and μ are signed Radon measures on Ω. Recall that Zhang studied the perturbations $L_0 + B(x,t) \cdot \nabla_x$ [37, 40] and $L_0 + V(x,t)$ [38, 39] of L_0 with B and V in some parabolic Kato classes. Using the well known results by Aronson [1] for parabolic operators with coefficients in $L^{p,q}$-spaces and an approximation argument, he proved, in both cases, the existence and uniqueness of a Green function G for the initial-Dirichlet problem on Ω. The existence of the Green function allowed him to solve some initial boundary value problems. In [28] and [31], we have established two-sided pointwise estimates for the Green functions describing, completely, their behavior near the boundary. These estimates are used to prove some potential-theoretic results, namely, the equivalence of

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harmonic measures [31], the coincidence of the Martin boundary and the parabolic boundary [27]; and they simplify proofs of certain known results such as the Harnack inequality, the boundary Harnack principles [28], etc. In the elliptic setting, similar estimates are well known (see [3, 8, 11, 12, 43]) and have played a major role in potential analysis; for instance they were used to prove the well known $3G$-Theorems and the comparability of perturbed Green functions (see [10, 13, 26, 29, 30, 32, 43]).

Our aim in this paper is to introduce general conditions on the measures ν and μ which guarantee the existence and uniqueness of a continuous L-Green function G for the initial-Dirichlet problem on Ω satisfying two-sided estimates like the ones in the unperturbed case. In fact, we establish the existence of G when ν and μ are in general classes covering the parabolic Kato classes used by Zhang [37] – [40]. Some partial counterpart results in the elliptic setting have recently been proved in [13, 30] and are based on new $3G$-Theorems which cover the classical ones due to Chung and Zhao [3], Cranston and Zhao [4] and Zhao [43]. In the parabolic setting it is not clear whether versions of these theorems hold. Here we establish basic inequalities (Lemmas 3.1 – 3.3 below) which imply the elliptic new $3G$-Theorems for all dimensions $n \geq 1$, and which are a key in proving the existence result. The paper is organized as follows.

In Section 2, we give some notations and state some known results. In Section 3, we prove some useful inequalities that will be used in the next sections. Parabolic versions of the elliptic $3G$-Theorems [13, 26, 29, 30, 32] are proved. In Section 4, we introduce general classes of drift terms ν and potentials μ denoted by $\mathcal{K}_c^{\mathrm{loc}}(\Omega)$ and $\mathcal{P}_c^{\mathrm{loc}}(\Omega)$, respectively, and we study some of their properties. In Section 5, we prove the existence and uniqueness of a continuous L-Green function G for the initial-Dirichlet problem on Ω satisfying lower and upper estimates as in the unperturbed case, when ν and μ are in the classes $\mathcal{K}_c^{\mathrm{loc}}(\Omega)$ and $\mathcal{P}_c^{\mathrm{loc}}(\Omega)$, with small norms $M^c(\nu)$ and $N^c(\mu^-)$, respectively (see Theorem 5.6 and Corollary 5.7). In particular, these results extend the ones proved in [14, 28, 31, 37, 38] to a more general class of parabolic operators. In Section 6, we consider the time-independent case $A = A(x)$, $\nu = 0$, $\mu = V(x)dx$ and we establish global-time estimates for Schrödinger heat kernels.

Throughout the paper the letters $C, C' \ldots$ denote positive constants which may vary in value from line to line.

2. NOTATIONS AND KNOWN RESULTS

We consider the parabolic operator

$$L = \frac{\partial}{\partial t} - \operatorname{div}(A(x,t)\nabla_x) + \nu \cdot \nabla_x + \mu$$

on $\Omega = D\times]0,T[$, where D is a $C^{1,1}$-bounded domain in $\mathbb{R}^n$, $n \geq 1$ and $0 < T < \infty$. By a domain we mean an open connected set. For $n = 1$, $D =]a,b[$ with $a, b \in \mathbb{R}, a < b$. We assume that the matrix A is real, symmetric, uniformly elliptic, i.e. there is $\lambda \geq 1$ such that $\lambda^{-1}\|\xi\|^2 \leq \langle A(x,t)\xi, \xi\rangle \leq \lambda\|\xi\|^2$, for all $(x,t) \in \Omega$ and all $\xi \in \mathbb{R}^n$ with λ-Lipschitz continuous coefficients on Ω, $\nu = (\nu_1, \ldots, \nu_n)$, ν_i and μ are signed Radon measures. When $\nu = 0$ and $\mu = 0$, we denote L by L_0. We denote by G_0 the L_0-Green function for the initial-Dirichlet problem on Ω. In the time-independent case, we denote by g_0 (resp. $g_{-\Delta}$) the Green function of $\mathcal{L}_0 = -\operatorname{div}(A(x)\nabla_x)$ (resp. $-\Delta$) with the Dirichlet boundary condition on D. By [12], there exists a constant $C = C(n, \lambda, D) > 0$ such that $C^{-1}g_{-\Delta} \leq g_0 \leq Cg_{-\Delta}$. Using this comparison and the estimates on $g_{-\Delta}$ proved in [8, 11, 43] for $n \geq 3$, in [3] for $n = 2$ and the formula

$$g_{-\Delta}(x,y) = \frac{(b - x \vee y)(x \wedge y - a)}{b - a} \quad \text{for} \quad n = 1,$$

we have the following.

Theorem 2.1. *There exists a constant* $C = C(n, \lambda, D) > 0$ *such that, for all* $x,\ y \in D$*,*

$$C^{-1}\Psi(x, y) \leq g_0(x, y) \leq C\Psi(x, y),$$

where

$$\Psi(x,y) = \begin{cases} \frac{d(x)d(y)|x-y|^{2-n}}{d(x)d(y)+|x-y|^2} & \text{if } n \geq 3; \\ \operatorname{Log}\left(1 + \frac{d(x)d(y)}{|x-y|^2}\right) & \text{if } n = 2; \\ \frac{d(x)d(y)}{|x-y|+\sqrt{d(x)d(y)}} & \text{if } n = 1, \end{cases}$$

with $d(x) = d(x, \partial D)$*, the distance from* x *to the boundary of* D*.*

For $a > 0,\ x,\ y \in D$ and $s < t$, let

$$\Gamma_a(x, t; y, s) = \frac{1}{(t - s)^{n/2}} \exp\left(-a\frac{|x - y|^2}{t - s}\right),$$

$$\gamma_a(x, t; y, s) = \min\left(1, \frac{d(x)}{\sqrt{t - s}}\right) \min\left(1, \frac{d(y)}{\sqrt{t - s}}\right) \Gamma_a(x, t; y, s),$$

and

$$\psi_a(x, t; y, s) = \psi_a^*(y, t; x, s) = \min\left(1, \frac{d(y)}{\sqrt{t - s}}\right) \frac{\Gamma_a(x, t; y, s)}{\sqrt{t - s}}.$$

The following estimates on the L_0-Green function G_0 were recently proved in [31].

Theorem 2.2. *There exist constants* $k_0,\ c_1,\ c_2 > 0$ *depending only on* $n,\ \lambda,\ D$ *and* T *such that for all* $x,\ y \in D$ *and* $0 \leq s < t \leq T$*,*

(i) $k_0^{-1}\gamma_{c_2}(x, t; y, s) \leq G_0(x, t; y, s) \leq k_0\, \gamma_{c_1}(x, t; y, s)$,
(ii) $|\nabla_x G_0|(x, t; y, s) \leq k_0\psi_{c_1}(x, t; y, s)$ *and*
(iii) $|\nabla_y G_0|(x, t; y, s) \leq k_0\psi^*_{c_1}(x, t; y, s)$.

3. BASIC INEQUALITIES

In this section we prove some basic inequalities which are a key in obtaining the existence results.

Lemma 3.1 (3γ-Inequality)**.** *Let* $0 < a < b$*. Then for any* $0 < c < \min(a, b - a)$*, there exists a constant* $C_0 = C_0(a, b, c) > 0$ *such that, for all* $x,\ y,\ z \in D,\ s < \tau < t$*,*

$$\frac{\gamma_a(x, t; z, \tau)\gamma_b(z, \tau; y, s)}{\gamma_a(x, t; y, s)} \leq C_0 \left[\frac{d(z)}{d(x)}\gamma_c(x, t; z, \tau) + \frac{d(z)}{d(y)}\gamma_c(z, \tau; y, s)\right].$$

Proof. We may assume $s = 0$. Let $x,\ y,\ z \in D,\ 0 < \tau < t$. We have

$$\gamma_a(x, t; z, \tau)\gamma_b(z, \tau; y, 0) = w\Gamma_a(x, t; z, \tau)\Gamma_b(z, \tau; y, 0), \tag{3.1}$$

where

$$w = \min\left(1, \frac{d(x)}{\sqrt{t - \tau}}\right) \min\left(1, \frac{d(z)}{\sqrt{t - \tau}}\right) \min\left(1, \frac{d(z)}{\sqrt{\tau}}\right) \min\left(1, \frac{d(y)}{\sqrt{\tau}}\right).$$

Let $\rho \in]0, 1[$ which will be fixed later.

Case 1. $\tau \in]0, \rho t]$. We have

$$\frac{1}{(t - \tau)^{n/2}} \leq \frac{1}{((1 - \rho)t)^{n/2}}.$$

Combining with the inequality

$$\frac{|x-z|^2}{t-\tau}+\frac{|z-y|^2}{\tau}\geq\frac{|x-y|^2}{t},\quad \text{for all}\quad \tau\in]0,t[,$$

we obtain

$$\Gamma_a(x,t;z,\tau)\Gamma_b(z,\tau;y,0)\leq\frac{1}{(1-\rho)^{n/2}}\Gamma_{b-a}(z,\tau;y,0)\Gamma_a(x,t;y,0). \tag{3.2}$$

Moreover, using the inequalities

$$\frac{\alpha\beta}{\alpha+\beta}\leq\min(\alpha,\beta)\leq 2\frac{\alpha\beta}{\alpha+\beta},$$

for $\alpha,\beta>0$, and $|d(z)-d(y)|\leq|z-y|$, we have

$$\begin{aligned}\min\left(1,\frac{d(z)}{\sqrt{t-\tau}}\right)&\leq 2\frac{d(z)}{d(y)}\min\left(1,\frac{d(y)}{\sqrt{t-\tau}}\right)\left(1+\frac{|z-y|}{\sqrt{t-\tau}}\right)\\ &\leq\frac{2}{1-\rho}\frac{d(z)}{d(y)}\min\left(1,\frac{d(y)}{\sqrt{t}}\right)\left(1+\frac{|z-y|}{\sqrt{\tau}}\right)\end{aligned} \tag{3.3}$$

Combining (3.1) – (3.3), we obtain, for all $\tau\in]0,\rho t]$,

$$\begin{aligned}\gamma_a(x,t;z,\tau)\gamma_b(z,\tau;y,0)\leq\frac{2}{(1-\rho)^{\frac{n+3}{2}}}\frac{d(z)}{d(y)}\gamma_c(z,\tau;y,0)\gamma_a(x,t;y,0)\\ \times\left(1+\frac{|z-y|}{\sqrt{\tau}}\right)\exp\left(-(b-a-c)\frac{|z-y|^2}{\tau}\right).\end{aligned}$$

Using the inequality $(1+\theta)\exp(-\alpha\theta^2)\leq 1+\alpha^{-1/2}$, for all $\alpha,\theta\geq 0$, it follows that

$$\gamma_a(x,t;z,\tau)\gamma_b(z,\tau;y,0)\leq C_0\frac{d(z)}{d(y)}\gamma_c(z,\tau;y,0)\gamma_a(x,t;y,0), \tag{3.4}$$

where $C_0=C_0(a,b,c,\rho)>0$.

Case 2. $\tau\in[\rho t,t[$. If $|z-y|\geq(\frac{a}{b})^{1/2}|x-y|$, then

$$\exp\left(-b\frac{|z-y|^2}{\tau}\right)\leq\exp\left(-a\frac{|x-y|^2}{t}\right). \tag{3.5}$$

If $|z-y|\leq(\frac{a}{b})^{1/2}|x-y|$, then

$$|x-z|\geq|x-y|-|z-y|\geq\left(1-\left(\frac{a}{b}\right)^{\frac{1}{2}}\right)|x-y|,$$

which yields

$$\begin{aligned}&\exp\left(-a\frac{|x-z|^2}{t-\tau}\right)\\ &\leq\exp\left(-\left(\frac{a+c}{2}\right)\frac{|x-z|^2}{t-\tau}\right)\exp\left(-\left(\frac{a-c}{2}\right)\frac{|x-y|^2}{t-\tau}\left(1-\left(\frac{a}{b}\right)^{\frac{1}{2}}\right)^2\right)\\ &\leq\exp\left(-\left(\frac{a+c}{2}\right)\frac{|x-z|^2}{t-\tau}\right)\exp\left(-\left(\frac{a-c}{2}\right)\frac{|x-y|^2}{(1-\rho)t}\left(1-\left(\frac{a}{b}\right)^{\frac{1}{2}}\right)^2\right).\end{aligned}$$

Now taking ρ so that

$$\frac{(a-c)\left(1-\left(\frac{a}{b}\right)^{\frac{1}{2}}\right)^2}{2a(1-\rho)}=1,$$

we obtain

$$\exp\left(-a\frac{|x-z|^2}{t-\tau}\right)\le\exp\left(-\left(\frac{a+c}{2}\right)\frac{|x-z|^2}{t-\tau}\right)\exp\left(-a\frac{|x-y|^2}{t}\right). \tag{3.6}$$

From (3.5) and (3.6), we have

$$\Gamma_a(x,t;z,\tau)\Gamma_b(z,\tau;y,0)\le\frac{1}{\rho^{n/2}}\Gamma_{\frac{a+c}{2}}(x,t;z,\tau)\Gamma_a(x,t;y,0). \tag{3.7}$$

Note that (3.7) is similar to the inequality (3.2). Then by the same method used to prove (3.4), we obtain

$$\gamma_a(x,t;z,\tau)\gamma_b(z,\tau;y,0)\le C_0\frac{d(z)}{d(x)}\gamma_c(x,t;z,\tau)\gamma_a(x,t;y,0). \tag{3.8}$$

Combining (3.4), (3.8) and using the fact that

$$\frac{(a-c)\left(1-\left(\frac{a}{b}\right)^{\frac{1}{2}}\right)^2}{2a(1-\rho)}=1,$$

we get the inequality of Lemma 3.1 with $C_0=C_0(a,b,c)>0$. □

Lemma 3.2. *Let* $0<a<b$*. Then for any* $0<c<\min(a,b-a)$*, there exists a constant* $C_1=C_1(a,b,c)>0$ *such that, for all* $x,\ y,\ z\in D,\ s<\tau<t$*,*

$$\frac{\gamma_a(x,t;z,\tau)\psi_b(z,\tau;y,s)}{\gamma_a(x,t;y,s)}\le C_1\left[\psi_c(x,t;z,\tau)+\psi_c^*(z,\tau;y,s)\right].$$

Proof. We may assume that $s=0$. Letting $x,\ y,\ z\in D,\ 0<\tau<t$, we have

$$\gamma_a(x,t;z,\tau)\psi_b(z,\tau;y,0)=w\Gamma_a(x,t;z,\tau)\Gamma_b(z,\tau;y,0), \tag{3.9}$$

where

$$w=\min\left(1,\frac{d(x)}{\sqrt{t-\tau}}\right)\min\left(1,\frac{d(z)}{\sqrt{t-\tau}}\right)\min\left(1,\frac{d(y)}{\sqrt{\tau}}\right)\frac{1}{\sqrt{\tau}}.$$

Let $\rho\in]0,1[$ that will be fixed later.

Case 1. $\tau\in]0,\rho t]$. As in (3.2), we have

$$\begin{aligned}\Gamma_a(x,t;z,\tau)\Gamma_b(z,\tau;y,0)&\le\frac{1}{(1-\rho)^{n/2}}\Gamma_{b-a}(z,\tau;y,0)\Gamma_a(x,t;y,0)\\&\le\frac{1}{(1-\rho)^{n/2}}\Gamma_c(z,\tau;y,0)\Gamma_a(x,t;y,0)\end{aligned} \tag{3.10}$$

Moreover, by using the same inequalities as in (3.3), we obtain

$$w\le\frac{4}{(1-\rho)^{3/2}}\min\left(1,\frac{d(x)}{\sqrt{t}}\right)\min\left(1,\frac{d(y)}{\sqrt{t}}\right)\min\left(1,\frac{d(z)}{\sqrt{\tau}}\right)\left(1+\frac{|z-y|}{\sqrt{\tau}}\right)^2\frac{1}{\sqrt{\tau}}. \tag{3.11}$$

Combining (3.9) – (3.11) and using the inequality

$$(1+\theta)^2\exp(-\alpha\theta^2)\le 2\left(1+\frac{1}{\sqrt{\alpha}}\right)^2,$$

for all $\alpha,\theta\ge 0$, it follows that

$$\gamma_a(x,t;z,\tau)\psi_b(z,\tau;y,0)\le C_1\psi_c^*(z,\tau;y,0)\gamma_a(x,t;y,0),$$

with

$$C_1 = 8\left(1+\frac{1}{\sqrt{b-a-c}}\right)(1-\rho)^{-\frac{n+3}{2}}.$$

Case 2. $\tau \in [\rho t, t[$. If $|z-y| \geq (\frac{a}{b})^{1/2}|x-y|$, then

$$\exp\left(-b\frac{|z-y|^2}{\tau}\right) \leq \exp\left(-a\frac{|x-y|^2}{t}\right). \tag{3.12}$$

If $|z-y| \leq (\frac{a}{b})^{1/2}|x-y|$, then $|x-z| \geq (1-(\frac{a}{b})^{1/2})|x-y|$, which yields

$$\exp\left(-a\frac{|x-z|^2}{t-\tau}\right) \leq \exp\left(-c\frac{|x-z|^2}{t-\tau}\right)\exp\left(-(a-c)\frac{|x-y|^2}{(1-\rho)t}\left(1-\left(\frac{a}{b}\right)^{1/2}\right)^2\right).$$

Now taking ρ so that

$$\frac{(a-c)\left(1-\left(\frac{a}{b}\right)^{1/2}\right)^2}{a(1-\rho)} = 1,$$

we obtain

$$\exp\left(-a\frac{|x-z|^2}{t-\tau}\right) \leq \exp\left(-c\frac{|x-z|^2}{t-\tau}\right)\exp\left(-a\frac{|x-y|^2}{t}\right). \tag{3.13}$$

Combining (3.12) and (3.13), we have

$$\Gamma_a(x,t;z,\tau)\Gamma_b(z,\tau;y,0) \leq \frac{1}{\rho^{n/2}}\Gamma_c(x,t;z,\tau)\Gamma_a(x,t;y,0). \tag{3.14}$$

Moreover,

$$\min\left(1,\frac{d(x)}{\sqrt{t-\tau}}\right)\frac{1}{\sqrt{\tau}} \leq \frac{1}{\sqrt{\rho}}\min\left(1,\frac{d(x)}{\sqrt{t}}\right)\frac{1}{\sqrt{t-\tau}}$$

and so

$$w \leq \frac{1}{\rho}\min\left(1,\frac{d(x)}{\sqrt{t}}\right)\min\left(1,\frac{d(y)}{\sqrt{t}}\right)\min\left(1,\frac{d(z)}{\sqrt{t-\tau}}\right)\frac{1}{\sqrt{t-\tau}}. \tag{3.15}$$

Combining (3.9), (3.14) and (3.15), we obtain

$$\gamma_a(x,t;z,\tau)\psi_b(z,\tau;y,0) \leq \frac{1}{\rho^{n/2+1}}\psi_c(x,t;z,\tau)\gamma_a(x,t;y,0),$$

which ends the proof. □

Replacing γ_a by ψ_a in Lemma 3.2 and following the same manner of proof, we also obtain

Lemma 3.3. *Let $0 < a < b$. Then for any $0 < c < \min(a, b-a)$, there exists a constant $C_2 = C_2(a,b,c) > 0$ such that for all x, y, $z \in D$, $s < \tau < t$,*

$$\frac{\psi_a(x,t;z,\tau)\psi_b(z,\tau;y,s)}{\psi_a(x,t;y,s)} \leq C_2\Big[\psi_c(x,t;z,\tau)+\psi_c^*(z,\tau;y,s)\Big].$$

By simple computations we also have the following inequalities.

Lemma 3.4. *For $0 < a < b < c$, there exists a constant $C_3 = C_3(a,b,c) > 0$ such that, for all $x, y \in D$ and $s < t$,*

$$C_3^{-1}\min\left(1,\frac{d^2(y)}{t-s}\right)\Gamma_c(x,t;y,s) \leq \frac{d(y)}{d(x)}\gamma_b(x,t;y,s) \leq C_3\min\left(1,\frac{d^2(y)}{t-s}\right)\Gamma_a(x,t;y,s).$$

4. THE CLASSES $\mathcal{K}_c^{\text{loc}}(\Omega)$ AND $\mathcal{P}_c^{\text{loc}}(\Omega)$

In this section we introduce general classes of drift terms $\nu = (\nu_1, \dots, \nu_n)$ and potentials μ which guarantee the existence and uniqueness of a continuous L-Green function G for the initial-Dirichlet problem on Ω satisfying two-sided estimates like the ones in the unperturbed case (Theorem 2.2).

Definition 4.1 (see [37, 40])**.** Let B be a locally integrable $\mathbb{R}^n$-valued function on Ω. We say that B is in the parabolic Kato class if it satisfies, for some $c > 0$,

$$\lim_{r\to 0}\left\{\sup_{(x,t)\in\Omega}\int_{t-r}^{t}\int_{D\cap\{|x-z|\le\sqrt{r}\}}\frac{\Gamma_c(x,t;z,\tau)}{\sqrt{t-\tau}}|B(z,\tau)|dzd\tau \right.$$
$$\left. + \sup_{(y,s)\in\Omega}\int_{s}^{s+r}\int_{D\cap\{|z-y|\le\sqrt{r}\}}\frac{\Gamma_c(z,\tau;y,s)}{\sqrt{\tau-s}}|B(z,\tau)|dzd\tau\right\} = 0.$$

Remark 4.1.

(1) Clearly, by the compactness of $\overline{\Omega}$, if B is in the parabolic Kato class then

$$\sup_{(x,t)\in\Omega}\int_{0}^{t}\int_{D}\frac{\Gamma_c(x,t;z,\tau)}{\sqrt{t-\tau}}|B(z,\tau)|dzd\tau + \sup_{(y,s)\in\Omega}\int_{s}^{T}\int_{D}\frac{\Gamma_c(z,\tau;y,s)}{\sqrt{\tau-s}}|B(z,\tau)|dzd\tau < \infty.$$

(2) In the time-independent case, the parabolic Kato class is identified to the elliptic Kato class K_{n+1} (see [4], for $n \ge 3$), i.e. the class of locally integrable $\mathbb{R}^n$-valued functions $B = B(x)$ on D satisfying

$$\lim_{r\to 0}\sup_{x\in D}\int_{D\cap\{|x-z|<\sqrt{r}\}}\varphi(x,z)|B(z)|dz = 0,$$

where

$$\varphi(x,z) = \begin{cases} \frac{1}{|x-z|^{n-1}} & \text{if } n \ge 2 \\ 1 \vee \operatorname{Log}\frac{1}{|x-z|} & \text{if } n = 1. \end{cases}$$

Note that if $B \in K_{n+1}$, then

$$\sup_{x\in\overline{D}}\int_{D}\varphi(x,z)|B(z)|dz < \infty.$$

Definition 4.2. Let $c > 0$ and $\nu = (\nu_1, \dots, \nu_n)$ with ν_i a signed Radon measure on Ω. We say that ν is in the class $\mathcal{K}_c^{\text{loc}}(\Omega)$ if it satisfies

$$M^c(\nu) := \sup_{(x,t)\in\Omega}\int_0^t\int_D \psi_c(x,t;z,\tau)|\nu|(dzd\tau) + \sup_{(y,s)\in\Omega}\int_s^T\int_D \psi_c^*(z,\tau;y,s)|\nu|(dzd\tau) < \infty, \tag{4.1}$$

and, for any compact subset $E \subset \Omega$,

$$\lim_{r\to 0}\left\{\sup_{(x,t)\in E}\int_{t-r}^{t}\int_{D\cap\{|x-z|\leq\sqrt{r}\}}\psi_c(x,t;z,\tau)|\nu|(dzd\tau) + \sup_{(y,s)\in E}\int_{s}^{s+r}\int_{D\cap\{|z-y|\leq\sqrt{r}\}}\psi_c^*(z,\tau;y,s)|\nu|(dzd\tau)\right\} = 0. \tag{4.2}$$

Remark 4.2.

(1) From Definitions 4.1, 4.2 and Remark 4.1.1, the class $\mathcal{K}_c^{\mathrm{loc}}(\Omega)$ contains the parabolic Kato class.

(2) In the time-independent case, $\mathcal{K}_c^{\mathrm{loc}}(\Omega)$ is identified to the class $\mathcal{K}^{\mathrm{loc}}(D)$ of signed Radon measures $\nu = (\nu_1, \dots, \nu_n)$ on D satisfying

$$\sup_{x\in D}\int_D \psi(x,z)|\nu|(dz) < \infty, \tag{4.3}$$

and, for any compact subset $E \subset D$,

$$\lim_{r\to 0}\sup_{x\in E}\int_{D\cap\{|x-z|<\sqrt{r}\}}\psi(x,z)|\nu|(dz) = 0, \tag{4.4}$$

where

$$\psi(x,z) = \begin{cases} \min\left(1, \frac{d(z)}{|x-z|}\right)\frac{1}{|x-z|^{n-1}} & \text{if } n \geq 2, \\ \mathrm{Log}\left(1 + \frac{d(z)}{|x-z|}\right) & \text{if } n = 1. \end{cases}$$

For $n \geq 3$, the class $\mathcal{K}^{\mathrm{loc}}(D)$ was recently introduced in [13] to study the existence and uniqueness of a continuous Green function for the elliptic operator $\Delta + B(x)\cdot\nabla_x$ with the Dirichlet boundary condition on D.

Proposition 4.3. *For all $\alpha \in]1,2]$, the drift term*

$$|B_\alpha(z)| = \frac{1}{d(z)\left(\mathrm{Log}\left(\frac{d(D)}{d(z)}\right)\right)^{\alpha}} \in \mathcal{K}^{\mathrm{loc}}(D)\setminus K_{n+1},$$

where $d(D)$ is the diameter of D.

Proof. Case 1: $n = 1$. We will prove that B_α is in the class $\mathcal{K}^{\mathrm{loc}}(D)$. Clearly $|B_\alpha| \in L^\infty_{\mathrm{loc}}(D)$ and so it satisfies (4.4). We will show that B_α satisfies (4.3). We have

$$\begin{aligned}\int_D \psi(x,z)|B_\alpha(z)|dz &= \int_D \mathrm{Log}\left(1 + \frac{d(z)}{|x-z|}\right)\frac{dz}{d(z)\left(\mathrm{Log}\left(\frac{d(D)}{d(z)}\right)\right)^{\alpha}} \\ &= \int_{D\cap(|x-z|\leq d(z)/2)}\dots dz + \int_{D\cap(|x-z|\geq d(z)/2)}\dots dz \\ &:= I_1 + I_2. \end{aligned} \tag{4.5}$$

In the case $|x-z| \leq d(z)/2$, we have $\frac{2}{3}d(x) \leq d(z) \leq 2d(x)$, and so

$$\begin{aligned} I_1 &\leq \frac{1}{(\text{Log } 2)^\alpha} \cdot \frac{3}{2d(x)} \int_{|x-z|\leq d(x)} \text{Log}\left(1+\frac{2d(x)}{|x-z|}\right) dz \\ &\leq \frac{C}{d(x)} \int_{|r|\leq d(x)} \text{Log}\left(1+\frac{2d(x)}{|r|}\right) dr \\ &= 2C \int_0^1 \text{Log}\left(1+\frac{2}{t}\right) dt = C'. \end{aligned} \tag{4.6}$$

Moreover, by using the inequality $\text{Log}(1+t) \leq t$, for all $t \geq 0$, we have

$$\begin{aligned} I_2 &\leq \int_D \frac{dz}{|x-z|\left(\text{Log}\left(\frac{d(D)}{|x-z|}\right)\right)^\alpha} \\ &\leq C \int_0^{d(D)} \frac{dr}{r\left(\text{Log}\left(\frac{d(D)}{r}\right)\right)^\alpha} = C'. \end{aligned} \tag{4.7}$$

Combining $(4.5)-(4.7)$, we obtain that B_α satisfies (4.3).

Now we prove that B_α does not belong to the class K_{n+1}. Without loss of generality, we may assume that $D =]0,1[$. We have

$$\begin{aligned} \sup_{x\in\overline{D}} \int_D \varphi(x,z)|B_\alpha(z)|dz &= \sup_{x\in[0,1]} \int_0^1 \left(\text{Log}\,\frac{1}{|x-z|}\right) \frac{\left(\text{Log}\left(\frac{1}{d(z)}\right)\right)^{-\alpha}}{d(z)} dz \\ &\geq \int_0^{1/2} \frac{1}{z}\left(\text{Log}\left(\frac{1}{z}\right)\right)^{1-\alpha} dz = \infty. \end{aligned}$$

Case 2: $n \geq 2$. We will prove that B_α is in the class $\mathcal{K}^{\text{loc}}(D)$. Clearly $|B_\alpha| \in L^\infty_{\text{loc}}(D)$ and so it satisfies (4.4). We will show that B_α satisfies (4.3). We have

$$\begin{aligned} \int_D \psi(x,z)|B_\alpha(z)|dz &= \int_D \min\left(1, \frac{d(z)}{|x-z|}\right) \frac{1}{|x-z|^{n-1}} \frac{dz}{d(z)\left(\text{Log}\left(\frac{d(D)}{d(z)}\right)\right)^\alpha} \\ &= \int_{D\cap(|x-z|\leq d(z)/2)} \dots dz + \int_{D\cap(|x-z|\geq d(z)/2)} \dots dz \\ &:= J_1 + J_2. \end{aligned} \tag{4.8}$$

In the case $|x-z| \leq d(z)/2$, we have $\frac{2}{3}d(x) \leq d(z) \leq 2d(x)$, and so

$$\begin{aligned} J_1 &\leq \frac{1}{(\text{Log } 2)^\alpha} \frac{3}{2d(x)} \int_{|x-z|\leq d(x)} \frac{dz}{|x-z|^{n-1}} \\ &\leq \frac{C}{d(x)} \int_0^{d(x)} dr = C. \end{aligned} \tag{4.9}$$

Moreover,

$$\begin{aligned} J_2 &\leq \int_D \frac{dz}{|x-z|^n\left(\text{Log}\left(\frac{d(D)}{|x-z|}\right)\right)^\alpha} \\ &\leq C \int_0^{d(D)} \frac{dr}{r\left(\text{Log}\left(\frac{d(D)}{r}\right)\right)^\alpha} = C'. \end{aligned} \tag{4.10}$$

Combining $(4.8)-(4.10)$, we obtain that B_α satisfies (4.3).

Now we prove that B_α does not belong to the class K_{n+1}. Without loss of generality, we may assume that $0\in\partial D$. D is a $C^{1,1}$-domain and so there exists $r_0>0$ such that

$$D\cap B(0,r_0)=B(0,r_0)\cap\{x=(x',x_n):x'\in\mathbb{R}^{n-1},\,x_n>f(x')\},$$

and

$$\partial D\cap B(0,r_0)=B(0,r_0)\cap\{x=(x',f(x')):x'\in\mathbb{R}^{n-1}\},$$

where f is a $C^{1,1}$-function. For some $\rho_0>0$ small (see [30, p. 220]) the set

$$V_0=\{z=(z',z_n):|z'|<\rho_0,\ \text{and } 0<z_n-f(z')<r_0/4\}$$

satisfies

$$D\cap B(0,\rho_0)\subset V_0\subset D\cap B(0,r_0/2)$$

and for all $z\in V_0,\ d(z)\le z_n-f(z')\le Cd(z)$ and $|f(z')|\le C'|z'|$, where C and C' depend only on the $C^{1,1}$-constant. From these observations, we have

$$\begin{aligned}
&\sup_{x\in\overline{D}}\int_D\varphi(x,z)|B_\alpha(z)|dz\\
&\ge\int_{V_0}\varphi(0,z)|B_\alpha(z)|dz\\
&=\int_{V_0}|z|^{1-n}\frac{\left(\operatorname{Log}\left(\frac{1}{d(z)}\right)\right)^{-\alpha}}{d(z)}dz\\
&\ge\frac{1}{C}\int_{|z'|<\rho_0}\int_{0<z_n-f(z')<r_0/4}(|z'|^2+|z_n|^2)^{\frac{1-n}{2}}\frac{\left(\operatorname{Log}\left(\frac{1}{z_n-f(z')}\right)\right)^{-\alpha}}{z_n-f(z')}dz_ndz'\\
&\ge\frac{1}{C'}\int_{|z'|<\rho_0}\int_{0<z_n-f(z')<r_0/4}(|z'|^2+|z_n-f'(z)|^2)^{\frac{1-n}{2}}\frac{\left(\operatorname{Log}\left(\frac{1}{z_n-f(z')}\right)\right)^{-\alpha}}{z_n-f(z')}dz_ndz'\\
&=\frac{1}{C'}\int_{|z'|<\rho_0}\int_0^{r_0/4}(|z'|^2+r^2)^{\frac{1-n}{2}}\frac{(\operatorname{Log}(\frac{1}{r}))^{-\alpha}}{r}drdz'\\
&=\frac{1}{C''}\int_0^{r_0/4}\frac{1}{r}\left(\operatorname{Log}\left(\frac{1}{r}\right)\right)^{-\alpha}\int_0^{\rho_0}\frac{t^{n-2}}{(t^2+r^2)^{\frac{n-1}{2}}}dtdr\\
&=\frac{1}{C''}\int_0^{r_0/4}\frac{1}{r}\left(\operatorname{Log}\left(\frac{1}{r}\right)\right)^{-\alpha}\int_0^{\rho_0/r}\frac{s^{n-2}}{(s^2+1)^{\frac{n-1}{2}}}dsdr\\
&\ge\frac{1}{C''}\int_0^{r_0/4}\frac{1}{r}\left(\operatorname{Log}\left(\frac{1}{r}\right)\right)^{1-\alpha}dr=\infty.
\end{aligned}$$

□

Definition 4.3 (see [38, 39])**.** Let V be a potential in $L^1_{\text{loc}}(\Omega)$. We say that V is in the parabolic Kato class if it satisfies, for some $c>0$,

$$\lim_{r\to0}\left\{\sup_{(x,t)\in\Omega}\int_{t-r}^t\int_{D\cap\{|x-z|<\sqrt{r}\}}\Gamma_c(x,t;z,\tau)|V(z,\tau)|dzd\tau\right.$$
$$\left.+\sup_{(y,s)\in\Omega}\int_s^{s+r}\int_{D\cap\{|x-z|<\sqrt{r}\}}\Gamma_c(z,\tau;y,s)|V(z,\tau)|dzd\tau\right\}=0.$$

Remark 4.4.

(1) If V is in the parabolic Kato class, then, by the compactness of $\overline{\Omega}$, we have

$$\sup_{(x,t)\in\Omega}\int_0^t\int_D \Gamma_c(x,t;z,\tau)|V(z,\tau)|dzd\tau + \sup_{(y,s)\in\Omega}\int_s^T\int_D \Gamma_c(z,\tau;y,s)|V(z,\tau)|dzd\tau < \infty.$$

(2) In the time-independent case the parabolic Kato class is identified to the elliptic Kato class K_n, i.e. the class of functions $V = V(x) \in L^1_{\text{loc}}(D)$ satisfying

$$\lim_{r\to 0}\sup_{x\in D}\int_{D\cap(|x-z|<\sqrt{r})}\Phi(x,z)|V(z)|dz = 0,$$

where

$$\Phi(x,z) = \begin{cases} \frac{1}{|x-z|^{n-2}} & \text{if } n\geq 3;\\ 1\vee \operatorname{Log}\frac{1}{|x-z|} & \text{if } n=2;\\ 1 & \text{if } n=1.\end{cases}$$

Note that, if $V \in K_n$, then

$$\sup_{x\in\overline{D}}\int_D \Phi(x,z)|V(z)|dz < \infty.$$

In particular $K_n \subset L^1(D)$.

Definition 4.4. Let $c > 0$ and μ a signed Radon measure on Ω. We say that μ is in the class $\mathcal{P}_c^{\text{loc}}(\Omega)$ if it satisfies

$$N^c(\mu) := \sup_{(x,t)\in\Omega}\int_0^t\int_D \frac{d(z)}{d(x)}\gamma_c(x,t;z,\tau)|\mu|(dzd\tau) + \sup_{(y,s)\in\Omega}\int_s^T\int_D \frac{d(z)}{d(y)}\gamma_c(z,\tau;y,s)|\mu|(dzd\tau) < \infty, \tag{4.11}$$

and, for any compact subset $E \subset \Omega$,

$$\lim_{r\to 0}\left\{\sup_{(x,t)\in E}\int_{t-r}^t\int_{D\cap\{|x-z|\leq\sqrt{r}\}}\Gamma_c(x,t;z,\tau)|\mu|(dzd\tau) + \sup_{(y,s)\in E}\int_s^{s+r}\int_{D\cap\{|z-y|\leq\sqrt{r}\}}\Gamma_c(z,\tau;y,s)|\mu|(dzd\tau)\right\} = 0. \tag{4.12}$$

Remark 4.5.

(1) From Definitions 4.3, 4.4, Remark 4.4.1 and Lemma 3.4, the class $\mathcal{P}_c^{\text{loc}}(\Omega)$ contains the parabolic Kato class.
(2) In the time-independent case, $\mathcal{P}_c^{\text{loc}}(\Omega)$ is identified to the class $\mathcal{P}^{\text{loc}}(D)$ of signed Radon measures μ on D satisfying

$$\|\mu\| := \sup_{x\in D}\int_D \frac{d(z)}{d(x)}g_0(x,z)|\mu|(dz) < \infty, \tag{4.13}$$

and, for any compact subset $E \subset D$,

$$\lim_{r\to 0}\sup_{x\in E}\int_{D\cap\{|x-z|<\sqrt{r}\}} g_0(x,z)|\mu|(dz) = 0. \tag{4.14}$$

This is clear by integrating with respect to time and using Theorem 2.1. For $n \geq 3$, the class $\mathcal{P}^{\text{loc}}(D)$ is introduced in [30] to study the existence and uniqueness of a continuous Green function with the Dirichlet boundary condition for the Schrödinger equation $\Delta - \mu = 0$ on bounded Lipschitz domains. For $n = 2$, the same results hold on regular bounded Jordan domains (see [29]).

Proposition 4.6. *For $\alpha \in [1,2[$, the potential*

$$V_\alpha(z) = d(z)^{-\alpha} \in \mathcal{P}^{\text{loc}}(D) \setminus K_n.$$

Proof. For $n \geq 3$, this is done in [30, Corollary 4.8]. We will give the proof for $n \in \{1,2\}$. Note that for $\alpha \geq 1, V_\alpha \notin L^1(D)$ (see [30, Proposition 4.7]) and so $V_\alpha \notin K_n$. We will prove that $V_\alpha \in \mathcal{P}^{\text{loc}}(D)$.

Case 1: $n = 1$. $V_\alpha \in L^\infty_{\text{loc}}(D)$ and so it satisfies (4.14). We show that V_α satisfies (4.13). By Theorem 2.1, we have

$$\begin{aligned}\int_D \frac{d(z)}{d(x)} g_0(x,z)|V_\alpha(z)|dz &\leq C\int_D \frac{d^{2-\alpha}(z)}{|x-z| + \sqrt{d(x)d(z)}}dz \\ &= C\left(\int_{D\cap(|x-z|\leq d(z)/2)} \dots dz + \int_{D\cap(|x-z|\geq d(z)/2)} \dots dz\right) \\ &:= C(I_1 + I_2). \end{aligned}\tag{4.15}$$

In the case $|x-z| \leq d(z)/2$, we have $\frac{2}{3}d(x) \leq d(z) \leq 2d(x)$, and so

$$\begin{aligned} I_1 &\leq Cd^{1-\alpha}(x)\int_{|x-z|\leq d(x)} dz \\ &\leq 2Cd^{2-\alpha}(D) < \infty. \end{aligned}\tag{4.16}$$

Moreover,

$$\begin{aligned} I_2 &\leq C\int_{D\cap(|x-z|\geq d(z)/2)} \frac{|x-z|^{2-\alpha}}{|x-z| + \sqrt{d(x)d(z)}}dz \\ &\leq C\int_D |x-z|^{1-\alpha}dz \\ &\leq C'd^{2-\alpha}(D) < \infty. \end{aligned}\tag{4.17}$$

Combining (4.15) – (4.17), we obtain $\|V_\alpha\| < \infty$.

Case 2: $n = 2$. $V_\alpha \in L^\infty_{\text{loc}}(D)$ and so it satisfies (4.14). We show that V_α satisfies (4.13). By Theorem 2.1, we have

$$\begin{aligned}\int_D \frac{d(z)}{d(x)} g_0(x,z)|V_\alpha(z)|dz &\leq C\int_D \frac{d^{1-\alpha}(z)}{d(x)} \operatorname{Log}\left(1 + \frac{d(x)d(z)}{|x-z|^2}\right) dz \\ &= C\left(\int_{D\cap(|x-z|\leq d(z)/2)} \dots dz + \int_{D\cap(|x-z|\geq d(z)/2)} \dots dz\right) \\ &:= C(J_1 + J_2). \end{aligned}\tag{4.18}$$

Recalling that in the case $|x - z| \leq d(z)/2$, we have $\frac{2}{3}d(x) \leq d(z) \leq 2d(x)$, and using the inequality $\mathrm{Log}(1+t) \leq t$, for all $t \geq 0$, we have

$$\begin{aligned} J_1 &\leq C d^{-\alpha}(x) \int_{|x-z|\leq d(x)} \mathrm{Log}\left(1 + \frac{2d(x)}{|x-z|}\right)^2 dz \\ &\leq 4C d^{1-\alpha}(x) \int_{|x-z|\leq d(x)} \frac{dz}{|x-z|} \\ &= C' d^{2-\alpha}(x) \\ &\leq C' d^{2-\alpha}(D) < \infty. \end{aligned} \tag{4.19}$$

Moreover, by using the inequality $\mathrm{Log}(1+t) \leq t$, for all $t \geq 0$, we also have

$$\begin{aligned} J_2 &\leq C \int_{D\cap(|x-z|\geq d(z)/2)} \frac{d^{2-\alpha}(z)}{|x-z|^2} dz \\ &\leq C \int_D |x-z|^{-\alpha} dz \\ &\leq C' \int_0^{d(D)} r^{1-\alpha} dr \\ &= C'' d^{2-\alpha}(D) < \infty. \end{aligned} \tag{4.20}$$

Combining (4.18) – (4.20), we obtain $\|V_\alpha\| < \infty$. □

5. THE L-GREEN FUNCTION FOR THE INITIAL DIRICHLET PROBLEM

In this section we fix a positive constant $c < c_1/8$, where c_1 is the constant in Theorem 2.2, and we study the existence and uniqueness of a continuous L-Green function for the initial-Dirichlet problem on Ω when ν and μ are in the classes $\mathcal{K}_c^{\mathrm{loc}}(\Omega)$ and $\mathcal{P}_c^{\mathrm{loc}}(\Omega)$, respectively. A Borel measurable function $G : \Omega \times \Omega \rightarrow]0, \infty]$ is called an L-Green function for the initial-Dirichlet problem if, for all $(y, s) \in \Omega$, $G(\cdot, \cdot; y, s) \in L^1_{\mathrm{loc}}(\Omega)$ and satisfies

$$(*) \qquad \begin{cases} LG(\cdot, \cdot; y, s) = \varepsilon_{(y,s)} \\ G(\cdot, \cdot; y, s) = 0 \quad \text{on} \quad \partial D \times [s, T[\\ \lim_{t\to s^+} G(x, t; y, s) = \varepsilon_y, \end{cases}$$

in the distributional sense, where $\varepsilon_{(y,s)}$ and ε_y are the Dirac measures at (y, s) and y, respectively. In particular, for all $f \in L^1(D \times [s, T[)$ and $u_0 \in C_0(\overline{D})$, the initial Dirichlet problem

$$\begin{cases} Lu = f \quad \text{on } D \times [s, T[\\ u = 0 \quad \text{on} \quad \partial D \times [s, T[\\ u(x, s) = u_0(x), \ x \in D \end{cases}$$

admits a unique weak solution (see [37] – [40]) given by

$$u(x, t) = \int_D G(x, t; y, s) u_0(y) dy + \int_s^t \int_D G(x, t; z, \tau) f(z, \tau) dz d\tau.$$

We say that the Green function G is *continuous* if it is continuous outside the diagonal. Our first result is the following.

Theorem 5.1. *Let ν be in the class $\mathcal{K}_c^{\mathrm{loc}}(\Omega)$ with $M^c(\nu) \leq c_0$ for some suitable constant c_0. Then, there exists a unique continuous $(L_0 + \nu \cdot \nabla_x)$-Green function $\mathcal{G}$ for the initial-Dirichlet problem on Ω satisfying the estimates:*

$$C^{-1}\gamma_{c_3}(x,t;y,s) \leq \mathcal{G}(x,t;y,s) \leq C\,\gamma_{\frac{c_1}{2}}(x,t;y,s),$$

for all $x,\ y \in D$ and $0 \leq s < t \leq T$, where $C,\ c_3$ are positive constants depending on $n,\ \lambda, D$ and T.

To prove the theorem we need the following lemma.

Lemma 5.2. *Let $\Theta = \{(x,t;y,s) \in \Omega \times \Omega : t > s\}$, $f : \Theta \to \mathbb{R}$ continuous, satisfying $|f| \leq C\gamma_{\frac{c_1}{2}}$, for some positive constant C and ν be in the class $\mathcal{K}_c^{\mathrm{loc}}(\Omega)$. Then, the function*

$$p(x,t;y,s) = \int_s^t \int_D f(x,t;z,\tau)\nabla_z G_0(z,\tau;y,s) \cdot \nu(dzd\tau)$$

is continuous on Θ.

Proof of Lemma 5.2. For simplicity we use the notation $X = (x,t),\ Y = (y,s),\ Z = (z,\tau)$ and $dZ = dzd\tau$. By Lemma 3.2, we have, for all $(X;Y) \in \Theta$,

$$\begin{aligned}|p|(X;Y) &\leq C\int_s^t \int_D \gamma_{\frac{c_1}{2}}(X;Z)\psi_{c_1}(Z;Y)|\nu|(dZ)\\ &\leq C\gamma_{\frac{c_1}{2}}(X;Y)\int_s^t \int_D \left[\psi_c(X;Z) + \psi_c^*(Z;Y)\right]|\nu|(dZ)\\ &\leq CM^c(\nu)\gamma_{\frac{c_1}{2}}(X;Y),\end{aligned}$$

and so p is a real finite valued function. Let $(X_0;Y_0) := (x_0,t_0;y_0,s_0) \in \Theta$ be fixed and let

$$r_0 := \delta(X_0,\partial\Omega) \wedge \delta(Y_0,\partial\Omega) \wedge \delta(X_0;Y_0) > 0,$$

where

$$\delta(X_0,Y_0) = |x_0 - y_0| \vee |t_0 - s_0|^{\frac{1}{2}}$$

is the parabolic distance between X_0 and Y_0. Consider the compact subsets $E_1 = \overline{B}_\delta\left(X_0,\frac{r_0}{2}\right)$ and $E_2 = \overline{B}_\delta\left(Y_0,\frac{r_0}{2}\right)$. Since $\nu \in \mathcal{K}_c^{\mathrm{loc}}(\Omega)$, for $\varepsilon > 0$, there is $r \in \left]0,\frac{r_0}{2}\right[$ such that

$$\sup_{X\in E_1}\int\int_{B_\delta(X,r)} \psi_c(X;Z)|\nu|(dZ) < \varepsilon,$$

and

$$\sup_{Y\in E_2}\int\int_{B_\delta(Y,r)} \psi_c^*(Z;Y)|\nu|(dZ) < \varepsilon.$$

For $X \in B_\delta\left(X_0,\frac{r}{4}\right),\ Y \in B_\delta\left(Y_0,\frac{r}{4}\right)$, we have

$$\begin{aligned}p(X;Y) &= \int_s^t \int_D f(X;Z)\nabla_z G_0(Z;Y).\nu(dZ)\\ &= \int\int_{B_\delta(X_0,\frac{r}{2})} + \int\int_{B_\delta(Y_0,\frac{r}{2})} + \int\int_{B_\delta^c(X_0,\frac{r}{2})\cap B_\delta^c(Y_0,\frac{r}{2})}\\ &:= p_1(X;Y) + p_2(X;Y) + p_3(X;Y).\end{aligned}$$

Clearly, for $Z \in B^c_\delta\left(X_0, \frac{r}{2}\right) \cap B^c_\delta\left(Y_0, \frac{r}{2}\right)$, the function $(X;Y) \to f(X;Z)\nabla_z G_0(Z;Y)$ is continuous on $B_\delta\left(X_0, \frac{r}{4}\right) \times B_\delta\left(Y_0, \frac{r}{4}\right)$ and satisfies

$$\begin{aligned} |f|(X;Z)|\nabla_z G_0|(Z;Y) &\le C\gamma_{\frac{c_1}{4}}(X_0 + (0, r^2/8); Z) \\ &\le Cd(D)\psi_{\frac{c_1}{4}}(X_0 + (0, r^2/8); Z), \end{aligned}$$

for some $C = C(k_0, c_1, r, Y_0) > 0$ with

$$\int_0^{t_0 + r^2/8} \int_D \psi_{\frac{c_1}{4}}(X_0 + (0, r^2/8); Z)|\nu|(dZ) \le M^c(\nu) < \infty.$$

It then follows from the dominated convergence theorem that p_3 is continuous on $B_\delta\left(X_0, \frac{r}{4}\right) \times B_\delta\left(Y_0, \frac{r}{4}\right)$. Moreover, for $X \in B_\delta\left(X_0, \frac{r}{4}\right)$, $Z \in B_\delta\left(X_0, \frac{r}{2}\right)$ and $Y \in B_\delta\left(Y_0, \frac{r}{4}\right)$, we have

$$|f|(X;Z)|\nabla_z G_0|(Z;Y) \le C\gamma_{\frac{c_1}{2}}(X;Z),$$

for some $C = C(k_0, c_1, r_0) > 0$. So, for all $X \in B_\delta\left(X_0, \frac{r}{4}\right)$ and $Y \in B_\delta\left(Y_0, \frac{r}{4}\right)$,

$$\begin{aligned} |p_1|(X;Y) &\le C \int\int_{B_\delta(X_0, \frac{r}{2})} \gamma_{\frac{c_1}{2}}(X;Z)|\nu|(dZ) \\ &\le Cd(D) \int\int_{B_\delta(X,r)} \psi_{\frac{c_1}{2}}(X;Z)|\nu|(dZ) \\ &\le Cd(D)\varepsilon. \end{aligned}$$

In the same way, for $X \in B_\delta(X_0, \frac{r}{4}), Z \in B_\delta(Y_0, \frac{r}{2})$ and $Y \in B_\delta(Y_0, \frac{r}{4})$, we have

$$|f|(X;Z)|\nabla_z G_0|(Z;Y) \le C\psi_{c_1}(Z;Y),$$

for some $C = C(k_0, c_1, r_0) > 0$. So, for all $X \in B_\delta(X_0, \frac{r}{4})$ and $Y \in B_\delta(Y_0, \frac{r}{4})$,

$$\begin{aligned} |p_2|(X;Y) &\le C \int\int_{B_\delta(Y_0, \frac{r}{2})} \psi_{c_1}(Z;Y)|\nu|(dZ) \\ &\le C' \int\int_{B_\delta(Y,r)} \psi^*_{c_1}(Z;Y)|\nu|(dZ) \\ &\le C'\varepsilon. \end{aligned}$$

Thus p is continuous at $(X_0;Y_0)$. □

Proof of Theorem 5.1. For $\alpha > 0$ let

$$\mathcal{B}_\alpha = \{f : \Theta \to \mathbb{R},\ \text{continuous} : |f| \le C\,\gamma_\alpha, \text{for some } C \in \mathbb{R}\}.$$

For $f \in \mathcal{B}_\alpha$ we put

$$\|f\| = \sup_\Theta \frac{|f|}{\gamma_\alpha}.$$

Clearly, $(\mathcal{B}_\alpha, \|\cdot\|)$ is a Banach space. Let us define the operator Λ on $\mathcal{B}_{\frac{c_1}{2}}$ by

$$\Lambda f(x,t;y,s) = \int_s^t \int_D f(x,t;z,\tau)\nabla_z G_0(z,\tau;y,s) \cdot \nu(dzd\tau),$$

for all $f \in \mathcal{B}_{\frac{c_1}{2}}$. By the estimate (ii) of Theorem 2.2, Lemma 3.2 and Lemma 5.2, Λ is a bounded linear operator from $\mathcal{B}_{\frac{c_1}{2}}$ into $\mathcal{B}_{\frac{c_1}{2}}$ with $\|\Lambda\| \le k_0 C_1 M^c(\nu)$. Assume that $k_0 C_1 M^c(\nu) < 1$ and define $\mathcal{G}$ by

$$\mathcal{G}(x,t;y,s)=\begin{cases}(I-\Lambda)^{-1}G_0(x,t;y,s)=\sum_{m\geq 0}\Lambda^m G_0(x,t;y,s) & \text{for } (x,t;y,s)\in\Theta\\ G_0(x,t;y,s) \quad \text{for } (x,t),(y,s)\in\Omega, t\leq s.\end{cases}$$

Thus $\mathcal{G}$ satisfies the integral equation:

$$\mathcal{G}(x,t;y,s)=G_0(x,t;y,s)-\int_s^t\int_D \mathcal{G}(x,t;z,\tau)\nabla_z G_0(z,\tau;y,s)\cdot\nu(dzd\tau),$$

for all $(x,t),(y,s)\in\Omega$, and it is continuous outside the diagonal. This integral equation implies that $\mathcal{G}$ is a solution of the problem $(*)$. Moreover by Theorem 2.2 and Lemma 3.2, we have, for all $(x,t;y,s)\in\Theta$,

$$\begin{aligned}|\mathcal{G}(x,t;y,s)-G_0(x,t;y,s)| &\leq k_0\sum_{m\geq 1}(k_0C_1M^c(\nu))^m\gamma_{\frac{c_1}{2}}(x,t;y,s)\\ &=\frac{k_0^2C_1M^c(\nu)}{1-k_0C_1M^c(\nu)}\gamma_{\frac{c_1}{2}}(x,t;y,s).\end{aligned}\tag{5.1}$$

By taking

$$k_0C_1M^c(\nu)\leq\frac{1}{2k_0^2e^{c_2}+1}\leq\frac{1}{2}$$

and recalling that

$$k_0^{-1}\gamma_{c_2}\leq G_0\leq k_0\gamma_{c_1},$$

we get from (5.1),

$$\mathcal{G}(x,t;y,s)\leq 2k_0\gamma_{\frac{c_1}{2}}(x,t;y,s),$$

for all $(x,t;y,s)\in\Theta$, and

$$\mathcal{G}(x,t;y,s)\geq\frac{e^{-c_2}}{2k_0}\min\left(1,\frac{d(x)}{\sqrt{t-s}}\right)\min\left(1,\frac{d(y)}{\sqrt{t-s}}\right)\frac{1}{(t-s)^{\frac{n}{2}}},\tag{5.2}$$

for all $(x,t;y,s)\in\Theta$ with $\frac{|x-y|^2}{t-s}\leq 1$. Using (5.2) and the reproducing property of the Green function $\mathcal{G}$ (which follows from the reproducing property of G_0) we obtain, as in [31], the existence of constants $C,\ c_3>0$ such that

$$\mathcal{G}(x,t;y,s)\geq\frac{1}{C}\gamma_{c_3}(x,t;y,s),$$

for all $(x,t;y,s)\in\Theta$. □

Corollary 5.3. *Let $\nu\in\mathcal{K}_c^{\mathrm{loc}}(\Omega)$ with $M^c(\nu)\leq c_0$ and $\mathcal{G}$ be the $(L_0+\nu\cdot\nabla_x)$-Green function for the initial-Dirichlet problem on Ω. Then,*

$$|\nabla_x\mathcal{G}|(x,t;y,s)\leq 2k_0\,\psi_{\frac{c_1}{2}}(x,t;y,s)$$

for all $x,\ y\in D$ and $0\leq s<t\leq T$.

Proof. By using the inequality (ii) of Theorem 2.2 and Lemma 3.3, we obtain by induction,

$$|\Lambda^m(\nabla_x G_0)|(x,t;y,s)\leq k_0(k_0C_1M^c(\nu))^m\psi_{\frac{c_1}{2}}(x,t;y,s),$$

for all $x,y\in D,\ 0\leq s<t\leq T$ and $m\in\mathbb{N}$. Assume $k_0C_1M^c(\nu)\leq 1/2$, the derivative with respect to x of the Green function $\mathcal{G}=\sum_{m\geq 0}\Lambda^m G_0$ is given by

$$\nabla_x\mathcal{G}=\sum_{m\geq 0}\Lambda^m(\nabla_x G_0)$$

and satisfies

$$|\nabla_x \mathcal{G}|(x,t;y,s) \le 2k_0 \psi_{\frac{c_1}{2}}(x,t;y,s),$$

for all $x, y \in D$, $0 \le s < t \le T$. □

Theorem 5.4. *Let ν be in the class $\mathcal{K}_c^{\mathrm{loc}}(\Omega)$ with $M^c(\nu) \le c_0$, $\mathcal{G}$ be the $(L_0 + \nu.\nabla_x)$-Green function for the initial-Dirichlet problem on Ω and μ be a nonnegative measure in the class $\mathcal{P}_c^{\mathrm{loc}}(\Omega)$. Then, there exists a unique continuous L-Green function G for the initial-Dirichlet problem on Ω satisfying the estimates $C^{-1}\gamma_{c_4} \le G \le C\gamma_{\frac{c_1}{4}}$ on Θ, for some positive constants C and c_4.*

To prove the theorem we need the following lemma.

Lemma 5.5. *Let $f : \Theta \to \mathbb{R}$ be a continuous function satisfying $|f| \le C\gamma_{\frac{c_1}{4}}$ for some positive constant C and μ be a nonnegative measure in the class $\mathcal{P}_c^{\mathrm{loc}}(\Omega)$. Then, the function*

$$q(x,t;y,s) = \int_s^t \int_D \mathcal{G}(x,t;z,\tau) f(z,\tau;y,s)\mu(dzd\tau)$$

is continuous on Θ.

Proof of Lemma 5.5. For simplicity we use the notation $X = (x,t)$, $Y = (y,s)$, $Z = (z,\tau)$ and $dZ = dzd\tau$. By Lemma 3.1, we have, for all $(X;Y) \in \Theta$,

$$\begin{aligned}
|q|(X;Y) &\le C \int_s^t \int_D \gamma_{\frac{c_1}{2}}(X;Z)\gamma_{\frac{c_1}{4}}(Z;Y)\mu(dZ) \\
&\le C\gamma_{\frac{c_1}{4}}(X;Y) \int_s^t \int_D \left[\frac{d(z)}{d(x)}\gamma_c(X;Z) + \frac{d(z)}{d(y)}\gamma_c(Z;Y) \right] \mu(dZ) \\
&\le CN^c(\mu)\gamma_{\frac{c_1}{4}}(X;Y),
\end{aligned}$$

and so q is a real finite valued function. Let $(X_0;Y_0) := (x_0,t_0;y_0,s_0) \in \Theta$ be fixed and let

$$r_0 := \delta(X_0, \partial\Omega) \wedge \delta(Y_0, \partial\Omega) \wedge \delta(X_0, Y_0) > 0.$$

Consider the compact subsets $E_1 = \overline{B}_\delta\left(X_0, \frac{r_0}{2}\right)$ and $E_2 = \overline{B}_\delta\left(Y_0, \frac{r_0}{2}\right)$. Since $\mu \in \mathcal{P}_c^{\mathrm{loc}}(\Omega)$, for $\varepsilon > 0$, there is $r \in \left]0, \frac{r_0}{2}\right[$ such that

$$\sup_{X \in E_1} \int\int_{B_\delta(X,r)} \Gamma_c(X;Z)\mu(dZ) < \varepsilon,$$

and

$$\sup_{Y \in E_2} \int\int_{B_\delta(Y,r)} \Gamma_c(Z;Y)\mu(dZ) < \varepsilon.$$

For $X \in B_\delta\left(X_0, \frac{r}{4}\right)$, we have

$$\begin{aligned}
q(X;Y) &= \int_s^t \int_D \mathcal{G}(X;Z) f(Z;Y)\mu(dZ) \\
&= \int\int_{B_\delta(X_0,\frac{r}{2})} + \int\int_{B_\delta(Y_0,\frac{r}{2})} + \int\int_{B_\delta^c(X_0,\frac{r}{2}) \cap B_\delta^c(Y_0,\frac{r}{2})} \\
&:= q_1(X;Y) + q_2(X;Y) + q_3(X;Y).
\end{aligned}$$

For $Z \in B_\delta^c\left(X_0, \frac{r}{2}\right) \cap B_\delta^c\left(Y_0, \frac{r}{2}\right)$, the function $(X;Y) \to \mathcal{G}(X;Z)f(Z;Y)$ is continuous on $B_\delta\left(X_0, \frac{r}{4}\right) \times B_\delta\left(Y_0, \frac{r}{4}\right)$ with

$$\mathcal{G}(X;Z)|f|(Z;Y) \le C\gamma_{\frac{c_1}{4}}(X_0 + (0, r^2/8); Z)\gamma_{\frac{c_1}{8}}(Z; Y_0 - (0, r^2/8)),$$

for some $C = C(k_0, c_1, r, X_0, Y_0) > 0$ and by Lemma 3.1,

$$\int_{s_0-\frac{r^2}{8}}^{t_0+\frac{r^2}{8}} \int_D \gamma_{\frac{c_1}{4}}(X_0 + (0, r^2/8); Z)\gamma_{\frac{c_1}{8}}(Z; Y_0 - (0, r^2/8))\mu(dZ) \leq C_0 N^c(\mu)\gamma_{\frac{c_1}{8}}(X_0 + (0, r^2/8); Y_0 - (0, r^2/8)) < \infty.$$

It then follows, from the dominated convergence theorem, that q_3 is continuous on $B_\delta\left(X_0, \frac{r}{4}\right) \times B_\delta\left(Y_0, \frac{r}{4}\right)$. Moreover, for $Z \in B_\delta\left(X_0, \frac{r}{2}\right)$, $X \in B_\delta\left(X_0, \frac{r}{4}\right)$, $Y \in B_\delta\left(Y_0, \frac{r}{4}\right)$, we have

$$\mathcal{G}(X; Z)|f|(Z; Y) \leq C\Gamma_c(X; Z),$$

for some $C = C(k_0, c_1, r_0) > 0$ and so

$$q_1(X; Y) \leq C \int\int_{B_\delta(X,r)} \Gamma_c(X; Z)\mu(dZ) \leq C\varepsilon.$$

In the same way,

$$q_2(X; Y) \leq C \int\int_{B_\delta(Y,r)} \Gamma_c(Z, Y)\mu(dZ) \leq C\varepsilon.$$

Thus q is continuous at $(X_0; Y_0)$. □

Proof of Theorem 5.4. Let μ be a nonnegative measure in the class $\mathcal{P}_c^{\text{loc}}(\Omega)$ and define the operator T^μ on $\mathcal{B}_{\frac{c_1}{4}}$ by

$$T^\mu f(x, t; y, s) = \int_s^t \int_D \mathcal{G}(x, t; z, \tau)f(z, \tau; y, s)\mu(dzd\tau),$$

for all $f \in \mathcal{B}_{\frac{c_1}{4}}$. By Lemma 3.1 and Lemma 5.5, T^μ is a bounded linear operator from $\mathcal{B}_{\frac{c_1}{4}}$ into $\mathcal{B}_{\frac{c_1}{4}}$ with

$$\|T^\mu\| = \left\|T^\mu\gamma_{\frac{c_1}{4}}\right\| \leq 2C_0k_0N^c(\mu).$$

Its spectral radius is given by

$$r_{\mathcal{B}_{\frac{c_1}{4}}}(T^\mu) = \lim_{m\to\infty} \|(T^\mu)^m\|^{\frac{1}{m}} = \inf_m \|(T^\mu)^m\|^{\frac{1}{m}} = \inf_m \left\|(T^\mu)^m\gamma_{\frac{c_1}{4}}\right\|^{\frac{1}{m}}.$$

Note that if $N^c(\mu) < \frac{1}{2C_0k_0}$, then $\|T^\mu\| < 1$ and so $I + T^\mu$ is invertible on $\mathcal{B}_{\frac{c_1}{4}}$ with $\|(I + T^\mu)^{-1}\| \leq 1$. Thus, for a nonnegative measure σ in the class $\mathcal{P}_c^{\text{loc}}(\Omega)$ with $N^c(\sigma) < \frac{1}{2C_0k_0}$, we have

$$I + T^{\mu+\sigma} = I + T^\mu + T^\sigma = (I + T^\mu)[I + (I + T^\mu)^{-1}T^\sigma]$$

with $\|(I+T^\mu)^{-1}T^\sigma\| \leq \|T^\sigma\| < 1$ and so $I+T^{\mu+\sigma}$ is invertible on $\mathcal{B}_{\frac{c_1}{4}}$. From this observation we deduce that for any nonnegative measure μ in $\mathcal{P}_c^{\text{loc}}(\Omega)$, the operator $I + T^\mu$ is invertible on $\mathcal{B}_{\frac{c_1}{4}}$. Let us then define the function G by

$$G(x, t; y, s) = \begin{cases} (I + T^\mu)^{-1}\mathcal{G}(x, t; y, s) & \text{for } (x, t; y, s) \in \Theta \\ \mathcal{G}(x, t; y, s) & \text{for } (x, t), (y, s) \in \Omega, t \leq s. \end{cases}$$

Then $G \in \mathcal{B}_{\frac{c_1}{4}}$ and satisfies the integral equation:

$$G(x, t; y, s) = \mathcal{G}(x, t; y, s) - \int_s^t \int_D \mathcal{G}(x, t; z, \tau)G(z, \tau; y, s)\mu(dzd\tau),$$

for all $(x, t), (y, s) \in \Omega$. In particular, G is continuous outside the diagonal, a solution of the problem $(*)$ and satisfies $G \leq C\gamma_{\frac{c_1}{4}}$ on Θ. Moreover, by using this upper estimate, the integral equation and the arguments as in the proof of Theorem 5.1, we obtain a positive constant $c_4 > 0$ such that $G \geq C^{-1}\gamma_{c_4}$ on Θ. □

Theorem 5.6. *Let ν be in the class $\mathcal{K}_c^{\mathrm{loc}}(\Omega)$ with $M^c(\nu) \leq c_0$, $\mathcal{G}$ be the $(L_0 + \nu \cdot \nabla_x)$-Green function for the initial-Dirichlet problem on Ω and μ be in the class $\mathcal{P}_c^{\mathrm{loc}}(\Omega)$.*

Assume that $r_{\mathcal{B}_{\frac{c_1}{4}}}[(I + T^{\mu^+})^{-1}T^{\mu^-}] < 1$, then there exists a unique continuous L-Green function G for the initial-Dirichlet problem on Ω satisfying the estimates $C^{-1}\gamma_{c_4} \leq G \leq C\gamma_{\frac{c_1}{4}}$ on Θ.

Conversely, assume that there exists a unique continuous L-Green function G for the initial-Dirichlet problem on Ω satisfying the estimates $C^{-1}\gamma_{c_4} \leq G \leq C\gamma_{\frac{c_1}{4}}$ on Θ, then $r_{\mathcal{B}_{c_4}}[(I + T^{\mu^+})^{-1}T^{\mu^-}] < 1$.

Proof. For simplicity let $S = (I+T^{\mu^+})^{-1}T^{\mu^-}$. Since $r_{\mathcal{B}_{\frac{c_1}{4}}}(S) < 1$, for all $f \in \mathcal{B}_{\frac{c_1}{4}}$, $\sum_{m\geq 0} S^m f \in \mathcal{B}_{\frac{c_1}{4}}$. Let us then define G by

$$G(x,t;y,s) = \begin{cases} \sum_{m\geq 0} S^m[(I + T^{\mu^+})^{-1}\mathcal{G}](x,t;y,s) & \text{for } (x,t;y,s) \in \Theta \\ \mathcal{G}(x,t;y,s) & \text{for } (x,t),(y,s) \in \Omega, t \leq s. \end{cases}$$

Thus

$$G = (I + T^{\mu^+})^{-1}\mathcal{G} + SG \quad \text{on } \Theta,$$

which yields

$$(I + T^{\mu^+})G = \mathcal{G} + T^{\mu^-}G \quad \text{on } \Theta$$

and so

$$G(x,t;y,s) = \mathcal{G}(x,t;y,s) - \int_s^t \int_D \mathcal{G}(x,t;z,\tau)G(z,\tau;y,s)\mu(dzd\tau),$$

for all $(x,t),(y,s) \in \Omega$. Using this integral equation and the same arguments as in the proof of Theorem 5.4, G is a solution of the problem $(*)$, continuous outside the diagonal and satisfies the estimates $C^{-1}\gamma_{c_4} \leq G \leq C\gamma_{\frac{c_1}{4}}$ on Θ.

Conversely, assume that there exists a unique continuous L-Green function G for the initial-Dirichlet problem on Ω satisfying the estimates $C^{-1}\gamma_{c_4} \leq G \leq C\gamma_{\frac{c_1}{4}}$ on Θ, then we have

$$G = (I + T^{\mu^+})^{-1}\mathcal{G} + SG \quad \text{on } \Theta,$$

which implies that

$$G = \sum_{m\geq 0} S^m[(I + T^{\mu^+})^{-1}\mathcal{G}] \quad \text{on } \Theta.$$

By recalling that $(I+T^{\mu^+})^{-1}\mathcal{G}$ is the $(L_0+\nu\cdot\nabla_x+\mu^+)$-Green function for the initial-Dirichlet problem on Ω which satisfies the lower bound $(I + T^{\mu^+})^{-1}\mathcal{G} \geq C^{-1}\gamma_{c_4}$ on Θ, it follows that $r_{\mathcal{B}_{c_4}}(S) < 1$. □

Corollary 5.7. *Let ν and μ be in the classes $\mathcal{K}_c^{\mathrm{loc}}(\Omega)$ and $\mathcal{P}_c^{\mathrm{loc}}(\Omega)$, respectively, with $M^c(\nu) \leq c_0$ and $N^c(\mu^-) \leq c_0'$ for some suitable constants c_0 and c_0'. Then, there exists a unique continuous L-Green function G for the initial-Dirichlet problem on Ω satisfying the estimates $C^{-1}\gamma_{c_4} \leq G \leq C\gamma_{\frac{c_1}{4}}$ on Θ.*

Proof. It suffices to note that for $c_0' \leq \frac{1}{2k_0C_0}$, we have $\|T^{\mu^-}\| < 1$ which yields

$$\|(I + T^{\mu^+})^{-1}T^{\mu^-}\| \leq \|T^{\mu^-}\| < 1,$$

and so $r_{\mathcal{B}_{\frac{c_1}{4}}}[(I + T^{\mu^+})^{-1}T^{\mu^-}] < 1$. □

Remark 5.8.

(1) Note that the condition $\|T^{\mu^-}\| < 1$ is sufficient for the existence of the Green function and not necessary. More precisely, we may find a negative measure $\mu \in \mathcal{P}_c^{\mathrm{loc}}(\Omega)$ with $\|T^{-\mu}\|$ as large as we wish, however its spectral radius $r(T^{-\mu}) < 1$ (see [10]).

(2) As in [31], from the estimates $C^{-1}\gamma_{c_4} \le G \le C\gamma_{\frac{c_1}{4}}$ on Θ, we may deduce two-sided estimates for the L-Poisson kernel on Ω which imply the equivalence of the L-harmonic measure and the surface measure on the lateral boundary $\partial D \times]0, T[$ of Ω.

6. Global Estimates for Dirichlet Schrödinger Heat Kernels

Despite the wide study of the behavior of Schrödinger semigroups over the last three decades (see for example [2], [5] – [7], [14] – [17], [20] – [25], [33, 34, 36, 41, 42]), global pointwise estimates for certain Schrödinger heat kernels on bounded smooth domains remain unknown. In this section, we are concerned ourselves with this problem and obtained global-time estimates for heat kernels of certain subcritical Schrödinger operators on bounded $C^{1,1}$-domains. In particular, we rectify the heat kernel estimates given by Zhang for the Dirichlet Laplacian [42, Theorem 1.1 (b)] with an incomplete proof. We will use the notation $f \sim h$ to mean that $C^{-1}h \le f \le Ch$ for some positive constant C.

Let $A = A(x)$ be a real, symmetric, uniformly elliptic matrix with λ-Lipschitz continuous coefficients on D. Let $\mathcal{L}_0 = -\operatorname{div}(A(x)\nabla_x)$ and g_0 be the Green function with the Dirichlet boundary condition on D. By integrating the inequality in Lemma 3.1 with respect to τ and next with respect to t and using the fact that

$$\int_0^\infty \gamma_c(x,t;y,0)dt \sim \Psi(x,y) \sim g_0(x,y),$$

we obtain the following $3g_0$-Theorem valid for all dimensions $n \ge 1$ (see [29] for $n = 2$, [9, 26, 30] and [32] for $n \ge 3$).

Lemma 6.1 ($3g_0$-Theorem)**.** *There exists $C_4 = C_4(n, \lambda, D) > 0$ such that for all $x, y, z \in D$,*

$$\frac{g_0(x,z)g_0(z,y)}{g_0(x,y)} \le C_4 \left[\frac{d(z)}{d(x)} g_0(x,z) + \frac{d(z)}{d(y)} g_0(z,y)\right].$$

Let $V = V(x)$ be a function in the class $\mathcal{P}^{\mathrm{loc}}(D)$ defined in Remark 4.5.2 and put $\mathcal{L} = \mathcal{L}_0 + V$ with the Dirichlet boundary condition on D. By Lemma 6.1 and Theorem 9.1 in [10], we know that when $\|V^-\| \le 1/4C_4$, the Schrödinger operator $\mathcal{L}$ admits a continuous Green function g on D comparable to g_0. In particular, $\mathcal{L}$ is subcritical in the sense of [18, 19, 44]. Let σ_0 be the first eigenvalue of $\mathcal{L}$ on D which is strictly positive and G be the Dirichlet heat kernel of $\mathcal{L}$ on D (the existence of G follows from Corollary 5.7 and the reproducing property). We have the following global-time estimates on G.

Theorem 6.2. *Let V be in the class $\mathcal{P}^{\mathrm{loc}}(D)$ with $\|V^-\| \le c_0'$ for some suitable constant c_0'. Then the Dirichlet heat kernel G for the Schrödinger operator $\mathcal{L} = \mathcal{L}_0 + V$ satisfies the following estimates: there exist constants C, c_5, $c_6 > 0$ depending only on n, λ, D and on V only in terms of the quantity $\|V\|$, such that for all x, $y \in D$ and $t > 0$,*

$$C^{-1}e^{-\sigma_0 t}\varphi_{c_6}(x,t;y,0) \le G(x,t;y,0) \le C\, e^{-\sigma_0 t}\varphi_{c_5}(x,t;y,0),$$

where

$$\varphi_a(x,t;y,0) = \min\left(1, \frac{d(x)}{1 \wedge \sqrt{t}}\right) \min\left(1, \frac{d(y)}{1 \wedge \sqrt{t}}\right) \frac{\exp\left(-a\frac{|x-y|^2}{t}\right)}{1 \wedge t^{n/2}}, \quad a > 0.$$

Proof. Let h_0 be the first eigenfunction normalized by $\|h_0\|_2 = 1$. Clearly by the comparability $g \sim g_0$ and Theorem 2.1, it follows that $h_0(x) \sim d(x)$. From the reproducing property of G and the estimates

$$C^{-1}\gamma_{c_4}(x,t;y,0) \le G(x,t;y,0) \le C\gamma_{\frac{c_1}{4}}(x,t;y,0),$$

for $x, y \in D,\ t \in]0,1[$ (Corollary 5.7), we have

$$C^{-t}d(x)d(y) \le G(x,t;y,0) \le C^t d(x)d(y),$$

for all $t > 0$ and all $x, y \in D$; and so the semigroup $e^{-t\mathcal{L}}$ of $\mathcal{L}$ is intrinsically ultracontractive in the sense of [2, 5, 6, 7]. Thus, for any $C > 1$, there exists $T > 1$ such that

$$C^{-1}d(x)d(y)e^{-\sigma_0 t} \le G(x,t;y,0) \le Cd(x)d(y)e^{-\sigma_0 t},$$

for all $x, y \in D$ and $t \ge T$. Combining these estimates with the finite-time estimates

$$C^{-1}\gamma_{c_4}(x,t;y,0) \le G(x,t;y,0) \le C\gamma_{\frac{c_1}{4}}(x,t;y,0),$$

for $x, y \in D,\ t \in]0,T[$, we clearly obtain the global-time estimates stated in Theorem 6.2. □

Corollary 6.3. *Let λ_0 be the bottom eigenvalue of $\mathcal{L}_0$ on D. Then, the Dirichlet heat kernel G_0 of $\mathcal{L}_0$ on D satisfies the following estimates: there exist constants $C,\ c_5,\ c_6 > 0$ depending only on $n,\ \lambda$ and D, such that for all $x,\ y \in D$ and $t > 0$,*

$$C^{-1}e^{-\lambda_0 t}\varphi_{c_6}(x,t;y,0) \le G_0(x,t;y,0) \le C\,e^{-\lambda_0 t}\varphi_{c_5}(x,t;y,0), \tag{6.1}$$

and

$$|\nabla_x G_0|(x,t;y,0) \le C\,e^{-\lambda_0 t}\Phi_{c_5}(x,t;y,0), \tag{6.2}$$

where

$$\Phi_a(x,t;y,0) = \min\left(1, \frac{d(y)}{1 \wedge \sqrt{t}}\right)\frac{\exp\left(-a\frac{|x-y|^2}{t}\right)}{1 \wedge t^{(n+1)/2}}, \quad a > 0.$$

Proof. The estimates (6.1) are given by Theorem 6.2. We will prove (6.2). From the reproducing property of G_0, the finite-time inequality (ii) in Theorem 2.2 and the inequality $G_0 \le Ce^{-\lambda_0 t}\varphi_{c_5},\ c_5 < c_1$, we have, for all $t > 2$,

$$\nabla_x G_0(x,t;y,0) = \int_D \nabla_x G_0(x,t;z,t-1)G_0(z,t-1;y,0)dz,$$

and so

$$\begin{aligned}
|\nabla_x G_0|(x,t;y,0) &\le \int_D |\nabla_x G_0|(x,1;z,0)G_0(z,t-1;y,0)dz \\
&\le k^2 e^{-\lambda_0(t-1)}\int_D \psi_{c_1}(x,1;z,0)\varphi_{c_5}(z,t-1;y,0)dz \\
&\le k^2 e^{-\lambda_0(t-1)}\min(1,d(y))\int_D e^{-c_1|x-z|^2}e^{-c_5\frac{|z-y|^2}{t-1}}dz \\
&\le Ce^{-\lambda_0 t}\min(1,d(y))\int_D e^{-c_5(|x-z|^2+\frac{|z-y|^2}{t-1})}dz \\
&\le Ce^{-\lambda_0 t}\min(1,d(y))\exp\left(-c_5\frac{|x-y|^2}{t}\right) \\
&= Ce^{-\lambda_0 t}\Phi_{c_5}(x,t;y,0).
\end{aligned}$$

This inequality combined with the finite-time inequality (ii) of Theorem 2.2 yields the estimate (6.2). □

The following inequalities extend the ones, proved in [13] for $n \geq 3$, to all dimensions $n \geq 1$.

Corollary 6.4. *There exists a constant $C = C(n, \lambda, D) > 0$ such that, for all $x, y, z \in D$,*

$$|\nabla_x g_0|(x,y) \leq C\psi(x,y), \tag{6.3}$$

$$\frac{g_0(x,z)|\nabla_z g_0|(z,y)}{g_0(x,y)} \leq C[\psi(x,z) + \psi^*(z,y)] \tag{6.4}$$

and

$$\frac{|\nabla_x g_0|(x,z)|\nabla_z g_0|(z,y)}{\psi(x,y)} \leq C[\psi(x,z) + \psi^*(z,y)], \tag{6.5}$$

where

$$\psi(x,z) = \psi^*(z,x) = \begin{cases} \min\left(1, \frac{d(z)}{|x-z|}\right)\frac{1}{|x-z|^{n-1}} & \text{if } n \geq 2 \\ \operatorname{Log}\left(1 + \frac{d(z)}{|x-z|}\right) & \text{if } n = 1. \end{cases}$$

Proof. Inequality (6.3) holds by integrating (6.2) of Corollary 6.3 with respect to time and using the fact that

$$\int_0^\infty \Phi_{c_5}(x,t;y,0)dt \sim \psi(x,y).$$

Inequality (6.4) (resp. (6.5)) holds by integrating the inequality of Lemma 3.2 (resp. Lemma 3.3) with respect to τ and next with respect to t, using the facts that

$$\int_0^\infty \psi_c(x,t;y,0)dt \sim \psi(x,y)$$

and

$$\int_0^\infty \gamma_c(x,t;y,0)dt \sim \Psi(x,y) \sim g_0(x,y).$$

□

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