

# ON SOME NEW MEAN VALUE INEQUALITIES



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*Abstract:* In this paper, using the arithmetic-geometric mean inequality, we obtain some new mean value inequalities. Finally, some applications are given, they are extension of Hölder's inequalities.

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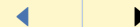
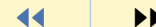
Some New Mean Value  
Inequalities

Liang-Cheng Wang and Cai-Liang Li  
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---

[Title Page](#)

[Contents](#)



Page 1 of 17

[Go Back](#)

[Full Screen](#)

[Close](#)

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# Contents

<b>1</b>	<b>Introduction and Main Results</b>	<b>3</b>
<b>2</b>	<b>Proof of Theorem and Corollary</b>	<b>7</b>
<b>3</b>	<b>Applications</b>	<b>11</b>



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**Some New Mean Value  
Inequalities**

Liang-Cheng Wang and Cai-Liang Li

vol. 8, iss. 3, art. 87, 2007

---

Title Page

Contents



Page 2 of 17

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



[Title Page](#)

[Contents](#)



Page 3 of 17

[Go Back](#)

[Full Screen](#)

[Close](#)

## 1. Introduction and Main Results

Let  $a > 0$ ,  $b > 0$  and  $t \in (0, 1)$ . It is well-known that the following arithmetic-geometric mean inequality holds

$$(1.1) \quad a^t b^{1-t} \leq ta + (1-t)b.$$

The arithmetic-geometric mean inequality is a classical inequality with many applications. Also, there exist extensive works devoted to generalizing or improving the arithmetic-geometric mean inequality. In this respect, we refer the reader to [1] – [7] and the references cited therein for updated results.

In this paper, by (1.1), we obtain some new mean value inequalities. Finally, some applications are given.

In this paper, we agree

$$\sum_{i=q+1}^q b_i = 0, \quad (b_i \in \mathbb{R}, \quad q \in \mathbb{N}).$$

**Theorem 1.1.** Let  $x_i > 0$  ( $i = 1, 2, \dots, n$ ;  $n \geq 2$ ) and  $t \in (0, 1)$ .

1. For the following

$$B(k) \triangleq \frac{1}{n^2} \left[ k \sum_{i=1}^k x_i + \left( \sum_{i=1}^n x_i^t \right) \left( \sum_{i=k+1}^n x_i^{1-t} \right) + \left( \sum_{i=k+1}^n x_i^t \right) \left( \sum_{i=1}^k x_i^{1-t} \right) \right], \quad (k = 1, 2, \dots, n)$$

and

$$C(j) \triangleq \frac{1}{n^2} \left[ (n-j+1) \sum_{i=j}^n x_i + \left( \sum_{i=1}^n x_i^t \right) \left( \sum_{i=1}^{j-1} x_i^{1-t} \right) + \left( \sum_{i=1}^{j-1} x_i^t \right) \left( \sum_{i=j}^n x_i^{1-t} \right) \right], \quad (j = 1, 2, \dots, n),$$

we have

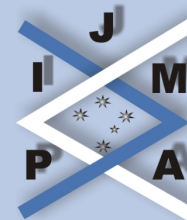
$$(1.2) \quad \left( \frac{1}{n} \sum_{i=1}^n x_i^t \right) \left( \frac{1}{n} \sum_{i=1}^n x_i^{1-t} \right) = B(1) \leq B(2) \leq \dots \leq B(k) \leq B(k+1) \leq \dots \leq B(n) = \frac{1}{n} \sum_{i=1}^n x_i$$

and

$$(1.3) \quad \left( \frac{1}{n} \sum_{i=1}^n x_i^t \right) \left( \frac{1}{n} \sum_{i=1}^n x_i^{1-t} \right) = C(n) \leq C(n-1) \leq \dots \leq C(j) \leq C(j-1) \leq \dots \leq C(1) = \frac{1}{n} \sum_{i=1}^n x_i.$$

2. For  $1 \leq j < k < l \leq n$  ( $n \geq 3$ ), we have

$$(1.4) \quad (k-j+1) \sum_{i=j}^k x_i + (l-k+1) \sum_{i=k}^l x_i + \left( \sum_{i=j}^l x_i^t \right) \left( \sum_{i=j}^l x_i^{1-t} \right)$$



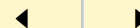
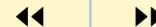
Some New Mean Value  
Inequalities

Liang-Cheng Wang and Cai-Liang Li

vol. 8, iss. 3, art. 87, 2007

Title Page

Contents



Page 4 of 17

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

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$$\leq (l-j+1) \sum_{i=j}^l x_i + \left( \sum_{i=j}^k x_i^t \right) \left( \sum_{i=j}^k x_i^{1-t} \right) + \left( \sum_{i=k}^l x_i^t \right) \left( \sum_{i=k}^l x_i^{1-t} \right).$$

**Corollary 1.2.** Let  $x_i > 0$  ( $i = 1, 2, \dots, n$ ,  $n \geq 2$ ) and  $p, q$  be any two positive numbers.

1. For

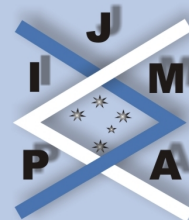
$$D(k) \triangleq \frac{1}{n^2} \left[ k \sum_{i=1}^k x_i^{p+q} + \left( \sum_{i=1}^n x_i^p \right) \left( \sum_{i=k+1}^n x_i^q \right) + \left( \sum_{i=k+1}^n x_i^p \right) \left( \sum_{i=1}^k x_i^q \right) \right], \quad (k = 1, 2, \dots, n)$$

and

$$E(j) \triangleq \frac{1}{n^2} \left[ (n-j+1) \sum_{i=j}^n x_i^{p+q} + \left( \sum_{i=1}^n x_i^p \right) \left( \sum_{i=1}^{j-1} x_i^q \right) + \left( \sum_{i=1}^{j-1} x_i^p \right) \left( \sum_{i=j}^n x_i^q \right) \right], \quad (j = 1, 2, \dots, n),$$

we have

$$(1.5) \quad \left( \frac{1}{n} \sum_{i=1}^n x_i^p \right) \left( \frac{1}{n} \sum_{i=1}^n x_i^q \right) = D(1) \leq D(2) \leq \dots \leq D(k) \leq D(k+1) \leq \dots \leq D(n) = \frac{1}{n} \sum_{i=1}^n x_i^{p+q}$$



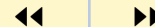
Some New Mean Value  
Inequalities

Liang-Cheng Wang and Cai-Liang Li

vol. 8, iss. 3, art. 87, 2007

Title Page

Contents



Page 5 of 17

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

and

$$(1.6) \quad \left( \frac{1}{n} \sum_{i=1}^n x_i^p \right) \left( \frac{1}{n} \sum_{i=1}^n x_i^q \right) \\ = E(n) \leq E(n-1) \leq \dots \leq E(j) \leq E(j-1) \leq \dots \leq E(1) = \frac{1}{n} \sum_{i=1}^n x_i^{p+q}.$$

2. For  $1 \leq j < k < l \leq n$  ( $n \geq 3$ ), we have

$$(1.7) \quad (k-j+1) \sum_{i=j}^k x_i^{p+q} + (l-k+1) \sum_{i=k}^l x_i^{p+q} + \left( \sum_{i=j}^l x_i^p \right) \left( \sum_{i=j}^l x_i^q \right) \\ \leq (l-j+1) \sum_{i=j}^l x_i^{p+q} + \left( \sum_{i=j}^k x_i^p \right) \left( \sum_{i=j}^k x_i^q \right) + \left( \sum_{i=k}^l x_i^p \right) \left( \sum_{i=k}^l x_i^q \right).$$



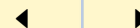
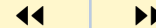
Some New Mean Value  
Inequalities

Liang-Cheng Wang and Cai-Liang Li

vol. 8, iss. 3, art. 87, 2007

Title Page

Contents



Page 6 of 17

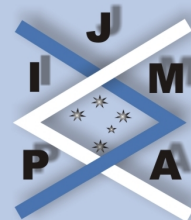
Go Back

Full Screen

Close

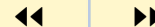
journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



Title Page

Contents



Page 7 of 17

Go Back

Full Screen

Close

## 2. Proof of Theorem and Corollary

*Proof of Theorem 1.1.* (1) Two equalities are clear in (1.2). To complete the proof of (1.2), we only need to prove that  $B(k) \leq B(k+1)$  ( $1 \leq k \leq n-1$ ). Indeed, from (1.1) we have

$$(2.1) \quad x_{k+1}^t \sum_{i=1}^k x_i^{1-t} = \sum_{i=1}^k x_{k+1}^t x_i^{1-t} \leq \sum_{i=1}^k (tx_{k+1} + (1-t)x_i),$$

and

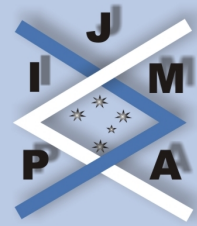
$$(2.2) \quad x_{k+1}^{1-t} \sum_{i=1}^k x_i^t = \sum_{i=1}^k x_{k+1}^{1-t} x_i^t \leq \sum_{i=1}^k ((1-t)x_{k+1} + tx_i).$$

Using (2.1) and (2.2), after a simple manipulation we get

$$(2.3) \quad x_{k+1}^t \sum_{i=1}^k x_i^{1-t} + x_{k+1}^{1-t} \sum_{i=1}^k x_i^t \leq kx_{k+1} + \sum_{i=1}^k x_i.$$

For  $k = 1, 2, \dots, n-1$ , by (2.3) we get

$$\begin{aligned} B(k) &= \frac{1}{n^2} \left[ k \sum_{i=1}^k x_i + \left( \sum_{i=1}^n x_i^t \right) \left( \sum_{i=k+1}^n x_i^{1-t} \right) + \left( \sum_{i=k+1}^n x_i^t \right) \left( \sum_{i=1}^k x_i^{1-t} \right) \right] \\ &= \frac{1}{n^2} \left[ k \sum_{i=1}^k x_i + x_{k+1} + \left( \sum_{i=1}^n x_i^t \right) \left( \sum_{i=k+2}^n x_i^{1-t} \right) \right. \\ &\quad \left. + \left( \sum_{i=1}^k x_i^t + \sum_{i=k+2}^n x_i^t \right) x_{k+1} + \left( \sum_{i=k+2}^n x_i^t \right) \left( \sum_{i=1}^k x_i^{1-t} \right) + x_{k+1}^t \sum_{i=1}^k x_i^{1-t} \right] \end{aligned}$$



Title Page

Contents

◀ ▶

◀ ▶

Page 8 of 17

Go Back

Full Screen

Close

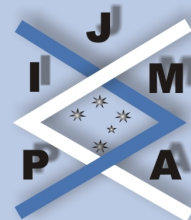
$$\begin{aligned}
 &= \frac{1}{n^2} \left[ k \sum_{i=1}^k x_i + x_{k+1} + x_{k+1}^t \sum_{i=1}^k x_i^{1-t} + x_{k+1}^{1-t} \sum_{i=1}^k x_i^t \right. \\
 &\quad \left. + \left( \sum_{i=1}^n x_i^t \right) \left( \sum_{i=k+2}^n x_i^{1-t} \right) + \left( \sum_{i=k+2}^n x_i^t \right) \left( \sum_{i=1}^{k+1} x_i^{1-t} \right) \right] \\
 &\leq \frac{1}{n^2} \left[ k \sum_{i=1}^k x_i + x_{k+1} + kx_{k+1} + \sum_{i=1}^k x_i \right. \\
 &\quad \left. + \left( \sum_{i=1}^n x_i^t \right) \left( \sum_{i=k+2}^n x_i^{1-t} \right) + \left( \sum_{i=k+2}^n x_i^t \right) \left( \sum_{i=1}^{k+1} x_i^{1-t} \right) \right] \\
 &= \frac{1}{n^2} \left[ (k+1) \sum_{i=1}^{k+1} x_i + \left( \sum_{i=1}^n x_i^t \right) \left( \sum_{i=k+2}^n x_i^{1-t} \right) + \left( \sum_{i=k+2}^n x_i^t \right) \left( \sum_{i=1}^{k+1} x_i^{1-t} \right) \right] \\
 &= B(k+1).
 \end{aligned}$$

By same arguments of proof for (1.2), we can also get inequalities in (1.3).

(2) For  $1 \leq j < k < l \leq n$ , from (1.1) we have

$$\begin{aligned}
 (2.4) \quad \left( \sum_{i=j}^{k-1} x_i^t \right) \left( \sum_{i=k+1}^l x_i^{1-t} \right) &= \sum_{i=j}^{k-1} \sum_{s=k+1}^l x_i^t x_s^{1-t} \\
 &\leq \sum_{i=j}^{k-1} \sum_{s=k+1}^l (tx_i + (1-t)x_s) \\
 &= (l-k) \sum_{i=j}^{k-1} tx_i + (k-j) \sum_{i=k+1}^l (1-t)x_i
 \end{aligned}$$





Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 9 of 17

Go Back

Full Screen

Close

and

$$(2.5) \quad \left( \sum_{i=k+1}^l x_i^t \right) \left( \sum_{i=j}^{k-1} x_i^{1-t} \right) \leq (l-k) \sum_{i=j}^{k-1} (1-t)x_i + (k-j) \sum_{i=k+1}^l tx_i.$$

Using (2.4) and (2.5), after a simple manipulation we have

$$(2.6) \quad \left( \sum_{i=j}^{k-1} x_i^t \right) \left( \sum_{i=k+1}^l x_i^{1-t} \right) + \left( \sum_{i=k+1}^l x_i^t \right) \left( \sum_{i=j}^{k-1} x_i^{1-t} \right) \\ \leq (l-k) \sum_{i=j}^{k-1} x_i + (k-j) \sum_{i=k+1}^l x_i.$$

From (2.6) we obtain

$$(k-j+1) \sum_{i=j}^k x_i + (l-k+1) \sum_{i=k}^l x_i + \left( \sum_{i=j}^l x_i^t \right) \left( \sum_{i=j}^l x_i^{1-t} \right) \\ = (k-j+1) \sum_{i=j}^k x_i + (l-k+1) \sum_{i=k+1}^l x_i + (l-k)x_k \\ + \left( \sum_{i=j}^k x_i^t \right) \left( \sum_{i=j}^k x_i^{1-t} \right) + \left( \sum_{i=k+1}^l x_i^t \right) \left( \sum_{i=k+1}^l x_i^{1-t} \right) \\ + x_k^t \sum_{i=k+1}^l x_i^{1-t} + x_k^{1-t} \sum_{i=k+1}^l x_i^t + x_k \\ + \left( \sum_{i=j}^{k-1} x_i^t \right) \left( \sum_{i=k+1}^l x_i^{1-t} \right) + \left( \sum_{i=k+1}^l x_i^t \right) \left( \sum_{i=j}^{k-1} x_i^{1-t} \right)$$



Title Page

Contents



Page 10 of 17

Go Back

Full Screen

Close

$$\begin{aligned} &\leq (k-j+1) \sum_{i=j}^k x_i + (l-k+1) \sum_{i=k+1}^l x_i + (l-k)x_k \\ &\quad + \left( \sum_{i=j}^k x_i^t \right) \left( \sum_{i=j}^k x_i^{1-t} \right) + \left( \sum_{i=k}^l x_i^t \right) \left( \sum_{i=k}^l x_i^{1-t} \right) \\ &\quad + (l-k) \sum_{i=j}^{k-1} x_i + (k-j) \sum_{i=k+1}^l x_i \\ &= (l-j+1) \sum_{i=j}^l x_i + \left( \sum_{i=j}^k x_i^t \right) \left( \sum_{i=j}^k x_i^{1-t} \right) + \left( \sum_{i=k}^l x_i^t \right) \left( \sum_{i=k}^l x_i^{1-t} \right), \end{aligned}$$

which implies (1.4).

This completes the proof of Theorem 1.1.  $\square$

*Proof of Corollary 1.2.* Replace  $t$ ,  $1-t$  and  $x_i$  in Theorem 1.1 by  $\frac{p}{p+q}$ ,  $\frac{q}{p+q}$  and  $x_i^{p+q}$ , respectively. We obtain Corollary 1.2.  $\square$



[Title Page](#)

[Contents](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 11 of 17

[Go Back](#)

[Full Screen](#)

[Close](#)

### 3. Applications

**Proposition 3.1.** Let  $x_{ir} > 0$  ( $i = 1, 2, \dots, n, n \geq 2; r = 1, 2, \dots, m, m \geq 2$ ) and  $t \in (0, 1)$ . For

$$F(k) \triangleq \frac{1}{n^2} \left[ k \sum_{i=1}^k \left( \sum_{r=1}^m x_{ir} \right) + \left( \sum_{i=1}^n \left( \sum_{r=1}^m x_{ir} \right)^t \right) \left( \sum_{i=k+1}^n \left( \sum_{r=1}^m x_{ir} \right)^{1-t} \right) + \left( \sum_{i=k+1}^n \left( \sum_{r=1}^m x_{ir} \right)^t \right) \left( \sum_{i=1}^k \left( \sum_{r=1}^m x_{ir} \right)^{1-t} \right) \right], \quad (k = 1, 2, \dots, n)$$

and

$$G(h) \triangleq \frac{1}{n^2} \left[ \sum_{i=1}^n \sum_{j=1}^n \left( \left( \sum_{r=1}^h x_{ir} \right)^t \left( \sum_{r=1}^h x_{jr} \right)^{1-t} + \sum_{r=h+1}^m x_{ir}^t x_{jr}^{1-t} \right) \right], \quad (h = 1, 2, \dots, m),$$

we have

$$\begin{aligned} (3.1) \quad & \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \left( \sum_{r=1}^m x_{ir}^t x_{jr}^{1-t} \right) \\ & = G(1) \leq G(2) \leq \dots \leq G(h) \leq G(h+1) \leq \dots \leq G(m) \\ & = \left( \frac{1}{n} \sum_{i=1}^n \left( \sum_{r=1}^m x_{ir} \right)^t \right) \left( \frac{1}{n} \sum_{i=1}^n \left( \sum_{r=1}^m x_{ir} \right)^{1-t} \right) \end{aligned}$$

$$\begin{aligned}
&= F(1) \leq F(2) \leq \dots \leq F(k) \leq F(k+1) \leq \dots \leq F(n) \\
&= \frac{1}{n} \sum_{i=1}^n \sum_{r=1}^m x_{ir}.
\end{aligned}$$

*Proof.* For  $x_{ir} > 0$ ,  $x_{jr} > 0$  ( $1 \leq i, j \leq n$ ,  $r = 1, 2, \dots, m$ ) and  $t \in (0, 1)$ . We write

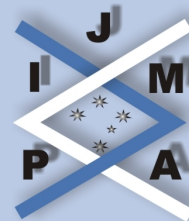
$$P(i, j; h) \triangleq \frac{\left( \sum_{r=1}^h x_{ir} \right)^t \left( \sum_{r=1}^h x_{jr} \right)^{1-t}}{\left( \sum_{r=1}^m x_{ir} \right)^t \left( \sum_{r=1}^m x_{jr} \right)^{1-t}} \quad (h = 1, 2, \dots, m).$$

The first named author of this paper showed in [8] that the following chain of Hölder's inequalities holds

$$\begin{aligned}
(3.2) \quad & \sum_{r=1}^m x_{ir}^t x_{jr}^{1-t} \\
&= P(i, j; 1) \\
&\leq P(i, j; 2) \leq \dots \leq P(i, j; h) \leq P(i, j; h+1) \leq \dots \leq P(i, j; m) \\
&= \left( \sum_{r=1}^m x_{ir} \right)^t \left( \sum_{r=1}^m x_{jr} \right)^{1-t}.
\end{aligned}$$

From the properties of inequality and (3.2), we have

$$\begin{aligned}
(3.3) \quad & \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \left( \sum_{r=1}^m x_{ir}^t x_{jr}^{1-t} \right) \\
&= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n P(i, j; 1)
\end{aligned}$$



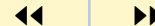
Some New Mean Value  
Inequalities

Liang-Cheng Wang and Cai-Liang Li

vol. 8, iss. 3, art. 87, 2007

Title Page

Contents



Page 12 of 17

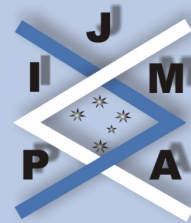
Go Back

Full Screen

Close

journal of **inequalities**  
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[Title Page](#)

[Contents](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 13 of 17

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

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$$\begin{aligned}
 &\leq \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n P(i, j; 2) \\
 &\leq \dots \leq \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n P(i, j; h) \\
 &\leq \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n P(i, j; h+1) \leq \dots \\
 &\leq \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n P(i, j; m) \\
 &= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \left( \sum_{r=1}^m x_{ir} \right)^t \left( \sum_{r=1}^m x_{jr} \right)^{1-t} \\
 &= \left( \frac{1}{n} \sum_{i=1}^n \left( \sum_{r=1}^m x_{ir} \right)^t \right) \left( \frac{1}{n} \sum_{i=1}^n \left( \sum_{r=1}^m x_{ir} \right)^{1-t} \right).
 \end{aligned}$$

It is easy to see that

$$(3.4) \quad G(h) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n P(i, j; h), \quad h = 1, 2, \dots, m.$$

(3.3) and (3.4) imply inequalities between the first equality and the second equality in (3.1).

Replacing  $x_i$  in (1.2) by  $\sum_{r=1}^m x_{ir}$ , we obtain inequalities between the third equality and the fourth equality in (3.1).

This completes the proof of Proposition 3.1. □



[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 14 of 17

[Go Back](#)

[Full Screen](#)

[Close](#)

**Proposition 3.2.** Let  $f_i : [a, b] \mapsto (0, +\infty)$  ( $a < b$ ) be continuous functions ( $i = 1, 2, \dots, n$ ,  $n \geq 2$ ) and  $t \in (0, 1)$ . For

$$\begin{aligned}
 H(k) = & \frac{1}{n^2} \left[ k \sum_{i=1}^k \left( \int_a^b f_i(x) dx \right) \right. \\
 & + \left( \sum_{i=1}^n \left( \int_a^b f_i(x) dx \right)^t \right) \left( \sum_{i=k+1}^n \left( \int_a^b f_i(x) dx \right)^{1-t} \right) \\
 & \left. + \left( \sum_{i=k+1}^n \left( \int_a^b f_i(x) dx \right)^t \right) \left( \sum_{i=1}^k \left( \int_a^b f_i(x) dx \right)^{1-t} \right) \right], \quad (k = 1, 2, \dots, n)
 \end{aligned}$$

and any  $y \in [a, b]$ , we have

$$\begin{aligned}
 (3.5) \quad & \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \left( \int_a^b (f_i(x))^t (f_j(x))^{1-t} dx \right) \\
 & \leq \frac{1}{n^2} \left[ \sum_{i=1}^n \sum_{j=1}^n \left( \left( \int_a^y f_i(x) dx \right)^t \left( \int_a^y f_j(x) dx \right)^{1-t} \right. \right. \\
 & \quad \left. \left. + \int_y^b (f_i(x))^t (f_j(x))^{1-t} dx \right) \right] \\
 & \leq \left( \frac{1}{n} \sum_{i=1}^n \left( \int_a^b f_i(x) dx \right)^t \right) \left( \frac{1}{n} \sum_{i=1}^n \left( \int_a^b f_i(x) dx \right)^{1-t} \right)
 \end{aligned}$$

$$\begin{aligned}
&= H(1) \leq H(2) \leq \dots \leq H(k) \leq H(k+1) \leq \dots \leq H(n) \\
&= \frac{1}{n} \sum_{i=1}^n \int_a^b f_i(x) dx.
\end{aligned}$$

*Proof.* For  $1 \leq i, j \leq n, t \in (0, 1), y \in [a, b]$  and continuous functions  $f_i : [a, b] \mapsto (0, +\infty)$  ( $i = 1, 2, \dots, n; n \geq 2$ ), in [8], Wang also obtained the following refinement for the integral form of Hölder's inequalities:

$$\begin{aligned}
(3.6) \quad & \int_a^b (f_i(x))^t (f_j(x))^{1-t} dx \\
& \leq \left( \int_a^y f_i(x) dx \right)^t \left( \int_a^y f_j(x) dx \right)^{1-t} + \int_y^b (f_i(x))^t (f_j(x))^{1-t} dx \\
& \leq \left( \int_a^b f_i(x) dx \right)^t \left( \int_a^b f_j(x) dx \right)^{1-t}.
\end{aligned}$$

Using the properties of inequality and (3.6), we have

$$\begin{aligned}
& \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \left( \int_a^b (f_i(x))^t (f_j(x))^{1-t} dx \right) \\
& \leq \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \left( \left( \int_a^y f_i(x) dx \right)^t \left( \int_a^y f_j(x) dx \right)^{1-t} + \int_y^b (f_i(x))^t (f_j(x))^{1-t} dx \right) \\
& \leq \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \left( \int_a^b f_i(x) dx \right)^t \left( \int_a^b f_j(x) dx \right)^{1-t} \\
& = \left( \frac{1}{n} \sum_{i=1}^n \left( \int_a^b f_i(x) dx \right)^t \right) \left( \frac{1}{n} \sum_{i=1}^n \left( \int_a^b f_i(x) dx \right)^{1-t} \right),
\end{aligned}$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 15 of 17

Go Back

Full Screen

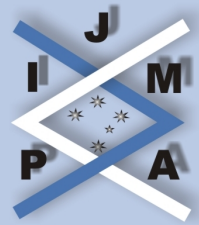
Close

which is two inequalities of left hand in (3.2).

Replacing  $x_i$  in (1.2) by  $\int_a^b f_i(x)dx$ , we obtain inequalities between the two equalities in (3.2).

This completes the proof of Proposition 3.2. □

*Remark 1.* (3.1) and (3.2) are extensions of Hölder's inequalities.



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**Some New Mean Value  
Inequalities**

Liang-Cheng Wang and Cai-Liang Li

vol. 8, iss. 3, art. 87, 2007

---

Title Page

Contents



Page 16 of 17

Go Back

Full Screen

Close

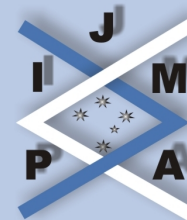
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Some New Mean Value  
Inequalities

Liang-Cheng Wang and Cai-Liang Li

vol. 8, iss. 3, art. 87, 2007

---

Title Page

Contents



Page 17 of 17

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756