Journal of Inequalities in Pure and Applied Mathematics

EXTENSIONS OF SEVERAL INTEGRAL INEQUALITIES

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ISSN (electronic): 1443-5756
073-06

volume 7, issue 3, article 107, 2006.

Received 15 March, 2006; accepted 04 April, 2006.

Communicated by: J. Pecarić

Abstract

In this article, an open problem posed in [12] is studied once again, and, following closely theorems and methods from [5], some extensions of several integral inequalities are obtained.

2000 Mathematics Subject Classification: Primary 26D15.
Key words: Integral inequality, Cauchy's Mean Value Theorem.

The authors were supported in part by the Science Foundation of the Project for Fostering Innovation Talents at Universities of Henan Province, China.
The authors would like to heartily express their thanks to the anonymous referee for his/her valuable comments and corrections to this paper.

Contents

1. Introduction

In [12], the following interesting integral inequality is proved: Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = 0$. If $0 \leq f'(x) \leq 1$ for $x \in (a,b)$, then

$$\int_a^b [f(x)]^3 \,\mathrm{d}x \leq \left[\int_a^b f(x)\,\mathrm{d}x\right]^2. \tag{1.1}$$

If $f'(x) \geq 1$, then inequality (1.1) reverses. The equality in (1.1) holds only if $f(x) \equiv 0$ or $f(x) = x - a$.

As a generalization of inequality (1.1), the following more general result is also obtained in [12]: Let $n \in \mathbb{N}$ and suppose $f(x)$ has a continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) \geq 0$ for $0 \leq i \leq n-1$ and $f^{(n)}(x) \geq n!$. Then

$$\int_a^b [f(x)]^{n+2} \,\mathrm{d}x \geq \left[\int_a^b f(x)\,\mathrm{d}x\right]^{n+1}. \tag{1.2}$$

At the end of [12] an open problem is proposed: Under what conditions does the inequality

$$\int_a^b [f(x)]^t \,\mathrm{d}x \geq \left[\int_a^b f(x)\,\mathrm{d}x\right]^{t-1} \tag{1.3}$$

hold for $t > 1$?

This open problem has attracted some mathematicians' research interests and many generalizations, extensions and applications of inequality (1.2) or

(1.3) were investigated in recent years. For more detailed information, please refer to, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15] and the references therein.

In this paper, following closely theorems and methods from [5], we will establish some more extensions and generalizations of inequality (1.2) or (1.3) once again. Our main results are the following five theorems.

Theorem 1.1. *Let $f(x)$ be continuous and not identically zero on $[a,b]$, differentiable in (a,b), with $f(a)=0$, and let α,β be positive real numbers such that $\alpha>\beta>1$. If*

$$\left[f^{(\alpha-\beta)/(\beta-1)}(x)\right]' \gtreqless \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \tag{1.4}$$

for all $x\in(a,b)$, then

$$\int_a^b [f(t)]^\alpha \,\mathrm{d}t \gtreqless \left[\int_a^b f(t)\,\mathrm{d}t\right]^\beta . \tag{1.5}$$

Theorem 1.2. *Let $\alpha\in\mathbb{R}$ and $f(x)$ be continuous on $[a,b]$ and positive in (a,b).*

1. *For $\beta>1$, if*

$$\int_a^x f(t)\,\mathrm{d}t \lesseqgtr \beta^{1/(1-\beta)}[f(x)]^{(\alpha-1)/(\beta-1)} \tag{1.6}$$

for all $x\in(a,b)$, then inequality (1.5) is validated;

2. *For $0<\beta<1$, if inequality (1.6) is reversed, then inequality (1.5) holds;*

3. *For* $\beta = 1$, *if* $[f(x)]^{1-\alpha} \lesseqgtr 1$ *for all* $x \in (a,b)$, *then inequality* (1.5) *is valid.*

Theorem 1.3. *Suppose* $n \in \mathbb{N}$, $1 \le \beta \le n+1$, *and* $f(x)$ *has a derivative of the* n*-th order on the interval* $[a,b]$ *such that* $f^{(i)}(a) = 0$ *for* $0 \le i \le n-1$ *and* $f^{(n)}(x) \ge 0$.

1. *If* $f(x) \ge \left[\frac{(x-a)^{\beta-1}}{\beta^{\beta-2}}\right]^{1/(\alpha-\beta)}$ *and* $f^{(n)}(x)$ *is increasing, then the inequality with direction* $\ge$ *in* (1.5) *holds.*

2. *If* $0 \le f(x) \le \left[\frac{(x-a)^{\beta-1}}{\beta^{\beta-2}}\right]^{1/(\alpha-\beta)}$ *and* $f^{(n)}(x)$ *is decreasing, then the inequality with direction* $\le$ *in* (1.5) *is valid.*

Theorem 1.4. *Suppose* $n \in \mathbb{N}$, $1 < \beta \le n+1$, *and* $f(x)$ *has a derivative of the* n*-th order on the interval* $[a,b]$ *such that* $f^{(i)}(a) = 0$ *for* $0 \le i \le n-1$ *and* $f^{(n)}(x) \ge 0$.

1. *If* $f(x) \ge \left[\frac{\beta(x-a)^{(\beta-1)}}{(\beta-1)^{(\beta-1)}}\right]^{1/(\alpha-\beta)}$, *then the inequality with direction* $\ge$ *in* (1.5) *holds.*

2. *If* $0 \le f(x) \le \left[\frac{\beta(x-a)^{(\beta-1)}}{(\beta-1)^{(\beta-1)}}\right]^{1/(\alpha-\beta)}$, *then the inequality with direction* $\le$ *in* (1.5) *is valid.*

Theorem 1.5. *Let* α, β *be positive numbers,* $\alpha > \beta \ge 2$ *and* $f(x)$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *such that* $f(a) \ge 0$. *If*

$$[f^{(\alpha-\beta)}(x)]' \ge \frac{\beta(\beta-1)(\alpha-\beta)(x-a)^{\beta-2}}{\alpha-1}$$

for $x \in (a, b)$, *then the inequality with direction* $\geq$ *in* (1.5) *holds.*

Remark 1. *Theorem 1.5 generalizes a result obtained in [9, Theorem 2] by Pečarić and Pejković.*

Feng Qi, Ai-Jun Li,
Wei-Zhen Zhao, Da-Wei Niu and
Jian Cao

2. Proofs of Theorems

Proof of Theorem 1.1. If

$$\left[f^{(\alpha-\beta)/(\beta-1)}(x)\right]' \geq \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1}$$

for $x \in (a,b)$ and $\alpha > \beta > 1$, then $f(x) > 0$ for $x \in (a,b]$. Thus both sides of (1.5) do not equal zero. This allows us to consider the quotient of both sides of (1.5). Utilizing Cauchy's Mean Value Theorem consecutively yields

$$\frac{\left[\int_a^b f(t)\,\mathrm{d}t\right]^{\beta}}{\int_a^b [f(t)]^{\alpha}\,\mathrm{d}t} = \frac{\beta\left[\int_a^{\xi} f(t)\,\mathrm{d}t\right]^{\beta-1} f(\xi)}{[f(\xi)]^{\alpha}} \qquad \xi \in (a,b)$$

$$= \left\{\frac{\beta^{1/(\beta-1)}\int_a^{\xi} f(t)\,\mathrm{d}t}{[f(\xi)]^{(\alpha-1)/(\beta-1)}}\right\}^{\beta-1} \tag{2.1}$$

$$= \left\{\frac{\beta^{1/(\beta-1)} f(\theta)}{\frac{\alpha-1}{\beta-1}[f(\theta)]^{(\alpha-\beta)/(\beta-1)} f'(\theta)}\right\}^{\beta-1} \qquad \theta \in (a,\xi)$$

$$= \left\{\frac{(\alpha-\beta)\beta^{1/(\beta-1)}/(\alpha-1)}{\left[f^{(\alpha-\beta)/(\beta-1)}(\theta)\right]'}\right\}^{\beta-1} \tag{2.2}$$

$$\leq 1.$$

So the inequality with direction $\geq$ in (1.5) follows.

If

$$0 \leq \left[f^{(\alpha-\beta)/(\beta-1)}(x)\right]' \leq \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1}$$

for $x \in (a,b)$ and $\alpha > \beta > 1$, then $f^{(\alpha-\beta)/(\beta-1)}(x)$ is nondecreasing and $f(x) \geq 0$ for $x \in [a,b]$. Without loss of generality, we may assume $f(x) > 0$ for $x \in (a,b]$ (otherwise, we can find a point $a_1 \in (a,b)$ such that $f(a_1) = 0$ and $f(x) > 0$ for $x \in (a_1, b]$ and hence we only need to consider the inequality with direction $\leq$ in (1.5) on $[a_1, b]$). This means that both sides of inequality (1.5) are not zero. Therefore, the inequality with direction $\leq$ in (1.5) follows from (2.2). □

Proof of Theorem 1.2. The first and second conclusions are obtained easily by (2.1) of Theorem 1.1.

For $\beta = 1$, inequality (1.5) is reduced to

$$\int_a^b [f(t)]^\alpha \, \mathrm{d}t \gtreqless \int_a^b f(t) \, \mathrm{d}t. \tag{2.3}$$

Now consider the quotient of both sides of (2.3). By Cauchy's Mean Value Theorem, it is obtained that

$$\frac{\int_a^b [f(t)]^\alpha \, \mathrm{d}t}{\int_a^b f(t) \, \mathrm{d}t} = \frac{[f(\xi)]^\alpha}{f(\xi)} = [f(\xi)]^{\alpha-1}. \tag{2.4}$$

The third conclusion is proved. □

Proof of Theorem 1.3. Utilization of the condition that $f(x) \geq \left[\frac{(x-a)^{\beta-1}}{\beta^{\beta-2}}\right]^{1/(\alpha-\beta)}$

and Cauchy's Mean Value Theorem gives

$$\frac{\int_a^b [f(x)]^\alpha \,\mathrm{d}x}{\left[\int_a^b f(x)\,\mathrm{d}x\right]^\beta} = \frac{[f(b_1)]^{\alpha-1}}{\beta\left[\int_a^{b_1} f(x)\,\mathrm{d}x\right]^{\beta-1}} \qquad a < b_1 < b \tag{2.5}$$

$$\geq \frac{(b_1-a)^{\beta-1}[f(b_1)]^{\beta-1}/\beta^{\beta-2}}{\beta\left[\int_a^{b_1} f(x)\,\mathrm{d}x\right]^{\beta-1}} \tag{2.6}$$

$$= \left[\frac{(b_1-a)f(b_1)}{\beta\int_a^{b_1} f(x)\,\mathrm{d}x}\right]^{\beta-1}. \tag{2.7}$$

Now for the term in (2.7), by using Cauchy's Mean Value Theorem several times, we have

$$\frac{(b_1-a)f(b_1)}{\int_a^{b_1} f(x)\,\mathrm{d}x} = 1 + \frac{(b_2-a)f'(b_2)}{f(b_2)} \qquad a < b_2 < b_1$$

$$= 2 + \frac{(b_3-a)f''(b_3)}{f'(b_3)} \qquad a < b_3 < b_2$$

$$\dots$$

$$= n + \frac{(b_{n+1}-a)f^{(n)}(b_{n+1})}{f^{(n-1)}(b_{n+1})} \qquad a < b_{n+1} < b_n. \tag{2.8}$$

But $f^{(n-1)}(t) = f^{(n-1)}(t) - f^{(n-1)}(a) = (t-a)f^{(n)}(t_1)$ for some $t_1 \in (a,t)$. If $f^{(n)}(x)$ is increasing, then $f^{(n)}(t_1) \leq f^{(n)}(t)$. Therefore,

$$0 < f^{(n-1)}(t) \leq f^{(n)}(t)(t-a). \tag{2.9}$$

Applying (2.9) to (2.8) yields

$$\frac{(b_1-a)f(b_1)}{\int_a^{b_1} f(x)\,\mathrm{d}x} \geq n+1. \tag{2.10}$$

Hence,

$$\frac{\int_a^b [f(x)]^\alpha \,\mathrm{d}x}{\left[\int_a^b f(x)\,\mathrm{d}x\right]^\beta} \geq \left(\frac{n+1}{\beta}\right)^{\beta-1} \tag{2.11}$$

for $1 \leq \beta \leq n+1$. Then the inequality with direction $\geq$ in (1.5) holds.

Suppose that

$$0 \leq f(x) \leq \left[\frac{(x-a)^{\beta-1}}{\beta^{\beta-2}}\right]^{1/(\alpha-\beta)}$$

and $f^{(n)}(x)$ is decreasing. The statement of the theorem implies that the inequalities (2.6) and (2.9) reverse, this means that the inequalities (2.10) and (2.11) reverse also, so the inequality with direction $\leq$ in (1.5) holds. □

Proof of Theorem 1.4. If

$$f(x) \geq \left[\frac{\beta(x-a)^{(\beta-1)}}{(\beta-1)^{(\beta-1)}}\right]^{1/(\alpha-\beta)},$$

(2.5) becomes

$$\frac{\int_a^b [f(x)]^\alpha \,\mathrm{d}x}{\left[\int_a^b f(x)\,\mathrm{d}x\right]^\beta} \geq \left[\frac{(b_1-a)f(b_1)}{(\beta-1)\int_a^{b_1} f(x)\,\mathrm{d}x}\right]^{\beta-1}.$$

Note that if all the terms in (2.8) are positive, then $\frac{(b_1-a)f(b_1)}{\int_a^{b_1} f(x)\,\mathrm{d}x} \geq n$. Therefore, for $1 < \beta \leq n+1$, the inequality with direction $\geq$ in (1.5) holds.

If

$$0 \leq f(x) \leq \left[\frac{\beta(x-a)^{(\beta-1)}}{(\beta-1)^{(\beta-1)}}\right]^{1/(\alpha-\beta)},$$

the inequality with direction $\leq$ in (1.5) follows from a similar argument as above. □

Proof of Theorem 1.5. Suppose that

$$[f^{(\alpha-\beta)}(x)]' \geq \frac{\beta(\beta-1)(\alpha-\beta)(x-a)^{\beta-2}}{\alpha-1}.$$

Now consider the quotient of the two sides of (1.5). Applying Cauchy's Mean Value Theorem three times leads to

$$\begin{aligned}
\frac{\int_a^b [f(x)]^\alpha \,\mathrm{d}x}{\left[\int_a^b f(x)\,\mathrm{d}x\right]^\beta} &= \frac{[f(b_1)]^{\alpha-1}}{\beta\left[\int_a^{b_1} f(x)\,\mathrm{d}x\right]^{\beta-1}} \\
&\geq \frac{(\alpha-1)[f(b_2)]^{\alpha-3} f'(b_2)}{\beta(\beta-1)\left[\int_a^{b_2} f(x)\,\mathrm{d}x\right]^{\beta-2}} \qquad a < b_2 < b_1 \\
&\geq \left[\frac{f(b_2)(b_2-a)}{\int_a^{b_2} f(x)\,\mathrm{d}x}\right]^{\beta-2} \qquad a < b_3 < b_2 \\
&= \left[1 + \frac{f'(b_3)(b_3-a)}{f(b_3)}\right]^{\beta-2} \geq 1.
\end{aligned}$$

This completes the proof. □

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