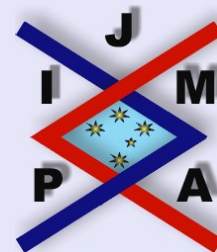


FENG QI, AI-JUN LI, WEI-ZHEN ZHAO, DA-WEI NIU AND JIAN CAO

Research Institute of Mathematical Inequality Theory  
Henan Polytechnic University  
Jiaozuo City, Henan Province, 454010, China.  
EMail: [qifeng@hpu.edu.cn](mailto:qifeng@hpu.edu.cn)  
URL: <http://rgmia.vu.edu.au/qi.html>

School of Mathematics and Informatics  
Henan Polytechnic University  
Jiaozuo City, Henan Province  
454010, China  
EMail: [liaijun72@163.com](mailto:liaijun72@163.com)  
EMail: [zhao\\_weizhen@sina.com](mailto:zhao_weizhen@sina.com)  
EMail: [nnddww@tom.com](mailto:nnddww@tom.com)  
EMail: [21caojian@163.com](mailto:21caojian@163.com)



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Abstract

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## Abstract

In this article, an open problem posed in [12] is studied once again, and, following closely theorems and methods from [5], some extensions of several integral inequalities are obtained.

*2000 Mathematics Subject Classification:* Primary 26D15.

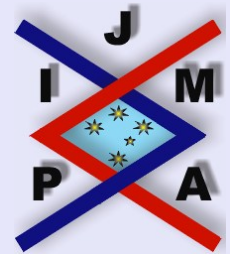
*Key words:* Integral inequality, Cauchy's Mean Value Theorem.

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# 1. Introduction

In [12], the following interesting integral inequality is proved: Let  $f(x)$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  such that  $f(a) = 0$ . If  $0 \leq f'(x) \leq 1$  for  $x \in (a, b)$ , then

$$(1.1) \quad \int_a^b [f(x)]^3 dx \leq \left[ \int_a^b f(x) dx \right]^2.$$

If  $f'(x) \geq 1$ , then inequality (1.1) reverses. The equality in (1.1) holds only if  $f(x) \equiv 0$  or  $f(x) = x - a$ .

As a generalization of inequality (1.1), the following more general result is also obtained in [12]: Let  $n \in \mathbb{N}$  and suppose  $f(x)$  has a continuous derivative of the  $n$ -th order on the interval  $[a, b]$  such that  $f^{(i)}(a) \geq 0$  for  $0 \leq i \leq n - 1$  and  $f^{(n)}(x) \geq n!$ . Then

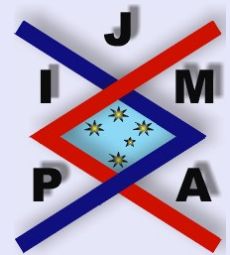
$$(1.2) \quad \int_a^b [f(x)]^{n+2} dx \geq \left[ \int_a^b f(x) dx \right]^{n+1}.$$

At the end of [12] an open problem is proposed: Under what conditions does the inequality

$$(1.3) \quad \int_a^b [f(x)]^t dx \geq \left[ \int_a^b f(x) dx \right]^{t-1}$$

hold for  $t > 1$ ?

This open problem has attracted some mathematicians' research interests and many generalizations, extensions and applications of inequality (1.2) or



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(1.3) were investigated in recent years. For more detailed information, please refer to, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15] and the references therein.

In this paper, following closely theorems and methods from [5], we will establish some more extensions and generalizations of inequality (1.2) or (1.3) once again. Our main results are the following five theorems.

**Theorem 1.1.** *Let  $f(x)$  be continuous and not identically zero on  $[a, b]$ , differentiable in  $(a, b)$ , with  $f(a) = 0$ , and let  $\alpha, \beta$  be positive real numbers such that  $\alpha > \beta > 1$ . If*

$$(1.4) \quad [f^{(\alpha-\beta)/(\beta-1)}(x)]' \begin{matrix} \geq \\ \leq \end{matrix} \frac{(\alpha - \beta)\beta^{1/(\beta-1)}}{\alpha - 1}$$

for all  $x \in (a, b)$ , then

$$(1.5) \quad \int_a^b [f(t)]^\alpha dt \begin{matrix} \geq \\ \leq \end{matrix} \left[ \int_a^b f(t) dt \right]^\beta.$$

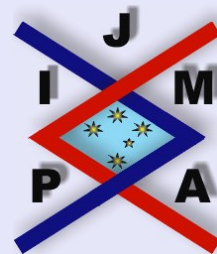
**Theorem 1.2.** *Let  $\alpha \in \mathbb{R}$  and  $f(x)$  be continuous on  $[a, b]$  and positive in  $(a, b)$ .*

1. For  $\beta > 1$ , if

$$(1.6) \quad \int_a^x f(t) dt \begin{matrix} \leq \\ \geq \end{matrix} \beta^{1/(1-\beta)} [f(x)]^{(\alpha-1)/(\beta-1)}$$

for all  $x \in (a, b)$ , then inequality (1.5) is validated;

2. For  $0 < \beta < 1$ , if inequality (1.6) is reversed, then inequality (1.5) holds;



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3. For  $\beta = 1$ , if  $[f(x)]^{1-\alpha} \lesseqgtr 1$  for all  $x \in (a, b)$ , then inequality (1.5) is valid.

**Theorem 1.3.** Suppose  $n \in \mathbb{N}$ ,  $1 \leq \beta \leq n + 1$ , and  $f(x)$  has a derivative of the  $n$ -th order on the interval  $[a, b]$  such that  $f^{(i)}(a) = 0$  for  $0 \leq i \leq n - 1$  and  $f^{(n)}(x) \geq 0$ .

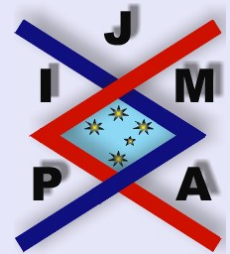
1. If  $f(x) \geq \left[ \frac{(x-a)^{\beta-1}}{\beta^{\beta-2}} \right]^{1/(\alpha-\beta)}$  and  $f^{(n)}(x)$  is increasing, then the inequality with direction  $\geq$  in (1.5) holds.
2. If  $0 \leq f(x) \leq \left[ \frac{(x-a)^{\beta-1}}{\beta^{\beta-2}} \right]^{1/(\alpha-\beta)}$  and  $f^{(n)}(x)$  is decreasing, then the inequality with direction  $\leq$  in (1.5) is valid.

**Theorem 1.4.** Suppose  $n \in \mathbb{N}$ ,  $1 < \beta \leq n + 1$ , and  $f(x)$  has a derivative of the  $n$ -th order on the interval  $[a, b]$  such that  $f^{(i)}(a) = 0$  for  $0 \leq i \leq n - 1$  and  $f^{(n)}(x) \geq 0$ .

1. If  $f(x) \geq \left[ \frac{\beta(x-a)^{(\beta-1)}}{(\beta-1)^{(\beta-1)}} \right]^{1/(\alpha-\beta)}$ , then the inequality with direction  $\geq$  in (1.5) holds.
2. If  $0 \leq f(x) \leq \left[ \frac{\beta(x-a)^{(\beta-1)}}{(\beta-1)^{(\beta-1)}} \right]^{1/(\alpha-\beta)}$ , then the inequality with direction  $\leq$  in (1.5) is valid.

**Theorem 1.5.** Let  $\alpha, \beta$  be positive numbers,  $\alpha > \beta \geq 2$  and  $f(x)$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  such that  $f(a) \geq 0$ . If

$$[f^{(\alpha-\beta)}(x)]' \geq \frac{\beta(\beta-1)(\alpha-\beta)(x-a)^{\beta-2}}{\alpha-1}$$



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for  $x \in (a, b)$ , then the inequality with direction  $\geq$  in (1.5) holds.

**Remark 1.** Theorem 1.5 generalizes a result obtained in [9, Theorem 2] by Pečarić and Pejković.



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## 2. Proofs of Theorems

*Proof of Theorem 1.1.* If

$$[f^{(\alpha-\beta)/(\beta-1)}(x)]' \geq \frac{(\alpha - \beta)\beta^{1/(\beta-1)}}{\alpha - 1}$$

for  $x \in (a, b)$  and  $\alpha > \beta > 1$ , then  $f(x) > 0$  for  $x \in (a, b]$ . Thus both sides of (1.5) do not equal zero. This allows us to consider the quotient of both sides of (1.5). Utilizing Cauchy's Mean Value Theorem consecutively yields

$$\frac{\left[\int_a^b f(t) dt\right]^\beta}{\int_a^b [f(t)]^\alpha dt} = \frac{\beta \left[\int_a^\xi f(t) dt\right]^{\beta-1} f(\xi)}{[f(\xi)]^\alpha} \quad \xi \in (a, b)$$

$$(2.1) \quad = \left\{ \frac{\beta^{1/(\beta-1)} \int_a^\xi f(t) dt}{[f(\xi)]^{(\alpha-1)/(\beta-1)}} \right\}^{\beta-1}$$

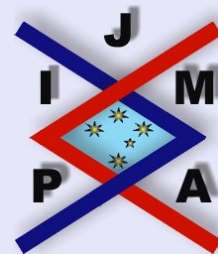
$$= \left\{ \frac{\beta^{1/(\beta-1)} f(\theta)}{\frac{\alpha-1}{\beta-1} [f(\theta)]^{(\alpha-\beta)/(\beta-1)} f'(\theta)} \right\}^{\beta-1} \quad \theta \in (a, \xi)$$

$$(2.2) \quad = \left\{ \frac{(\alpha - \beta)\beta^{1/(\beta-1)}/(\alpha - 1)}{[f^{(\alpha-\beta)/(\beta-1)}(\theta)]'} \right\}^{\beta-1} \leq 1.$$

So the inequality with direction  $\geq$  in (1.5) follows.

If

$$0 \leq [f^{(\alpha-\beta)/(\beta-1)}(x)]' \leq \frac{(\alpha - \beta)\beta^{1/(\beta-1)}}{\alpha - 1}$$



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for  $x \in (a, b)$  and  $\alpha > \beta > 1$ , then  $f^{(\alpha-\beta)/(\beta-1)}(x)$  is nondecreasing and  $f(x) \geq 0$  for  $x \in [a, b]$ . Without loss of generality, we may assume  $f(x) > 0$  for  $x \in (a, b]$  (otherwise, we can find a point  $a_1 \in (a, b)$  such that  $f(a_1) = 0$  and  $f(x) > 0$  for  $x \in (a_1, b]$  and hence we only need to consider the inequality with direction  $\leq$  in (1.5) on  $[a_1, b]$ ). This means that both sides of inequality (1.5) are not zero. Therefore, the inequality with direction  $\leq$  in (1.5) follows from (2.2).  $\square$

*Proof of Theorem 1.2.* The first and second conclusions are obtained easily by (2.1) of Theorem 1.1.

For  $\beta = 1$ , inequality (1.5) is reduced to

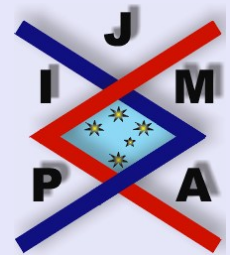
$$(2.3) \quad \int_a^b [f(t)]^\alpha dt \geq \int_a^b f(t) dt.$$

Now consider the quotient of both sides of (2.3). By Cauchy's Mean Value Theorem, it is obtained that

$$(2.4) \quad \frac{\int_a^b [f(t)]^\alpha dt}{\int_a^b f(t) dt} = \frac{[f(\xi)]^\alpha}{f(\xi)} = [f(\xi)]^{\alpha-1}.$$

The third conclusion is proved.  $\square$

*Proof of Theorem 1.3.* Utilization of the condition that  $f(x) \geq \left[ \frac{(x-a)^{\beta-1}}{\beta^{\beta-2}} \right]^{1/(\alpha-\beta)}$



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and Cauchy's Mean Value Theorem gives

$$(2.5) \quad \frac{\int_a^b [f(x)]^\alpha dx}{\left[\int_a^b f(x) dx\right]^\beta} = \frac{[f(b_1)]^{\alpha-1}}{\beta \left[\int_a^{b_1} f(x) dx\right]^{\beta-1}} \quad a < b_1 < b$$

$$(2.6) \quad \geq \frac{(b_1 - a)^{\beta-1} [f(b_1)]^{\beta-1} / \beta^{\beta-2}}{\beta \left[\int_a^{b_1} f(x) dx\right]^{\beta-1}}$$

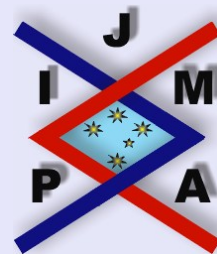
$$(2.7) \quad = \left[ \frac{(b_1 - a) f(b_1)}{\beta \int_a^{b_1} f(x) dx} \right]^{\beta-1}.$$

Now for the term in (2.7), by using Cauchy's Mean Value Theorem several times, we have

$$(2.8) \quad \begin{aligned} \frac{(b_1 - a) f(b_1)}{\int_a^{b_1} f(x) dx} &= 1 + \frac{(b_2 - a) f'(b_2)}{f(b_2)} & a < b_2 < b_1 \\ &= 2 + \frac{(b_3 - a) f''(b_3)}{f'(b_3)} & a < b_3 < b_2 \\ &\dots \\ &= n + \frac{(b_{n+1} - a) f^{(n)}(b_{n+1})}{f^{(n-1)}(b_{n+1})} & a < b_{n+1} < b_n. \end{aligned}$$

But  $f^{(n-1)}(t) = f^{(n-1)}(t) - f^{(n-1)}(a) = (t - a) f^{(n)}(t_1)$  for some  $t_1 \in (a, t)$ . If  $f^{(n)}(x)$  is increasing, then  $f^{(n)}(t_1) \leq f^{(n)}(t)$ . Therefore,

$$(2.9) \quad 0 < f^{(n-1)}(t) \leq f^{(n)}(t)(t - a).$$



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Applying (2.9) to (2.8) yields

$$(2.10) \quad \frac{(b_1 - a)f(b_1)}{\int_a^{b_1} f(x) dx} \geq n + 1.$$

Hence,

$$(2.11) \quad \frac{\int_a^b [f(x)]^\alpha dx}{\left[\int_a^b f(x) dx\right]^\beta} \geq \left(\frac{n+1}{\beta}\right)^{\beta-1}$$

for  $1 \leq \beta \leq n + 1$ . Then the inequality with direction  $\geq$  in (1.5) holds.

Suppose that

$$0 \leq f(x) \leq \left[\frac{(x-a)^{\beta-1}}{\beta^{\beta-2}}\right]^{1/(\alpha-\beta)}$$

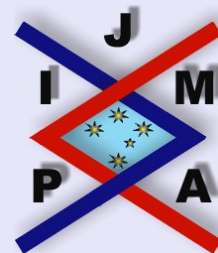
and  $f^{(n)}(x)$  is decreasing. The statement of the theorem implies that the inequalities (2.6) and (2.9) reverse, this means that the inequalities (2.10) and (2.11) reverse also, so the inequality with direction  $\leq$  in (1.5) holds.  $\square$

*Proof of Theorem 1.4.* If

$$f(x) \geq \left[\frac{\beta(x-a)^{(\beta-1)}}{(\beta-1)^{(\beta-1)}}\right]^{1/(\alpha-\beta)},$$

(2.5) becomes

$$\frac{\int_a^b [f(x)]^\alpha dx}{\left[\int_a^b f(x) dx\right]^\beta} \geq \left[\frac{(b_1 - a)f(b_1)}{(\beta - 1) \int_a^{b_1} f(x) dx}\right]^{\beta-1}.$$



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Note that if all the terms in (2.8) are positive, then  $\frac{(b_1-a)f(b_1)}{\int_a^{b_1} f(x) dx} \geq n$ . Therefore, for  $1 < \beta \leq n + 1$ , the inequality with direction  $\geq$  in (1.5) holds.

If

$$0 \leq f(x) \leq \left[ \frac{\beta(x-a)^{(\beta-1)}}{(\beta-1)^{(\beta-1)}} \right]^{1/(\alpha-\beta)},$$

the inequality with direction  $\leq$  in (1.5) follows from a similar argument as above.  $\square$

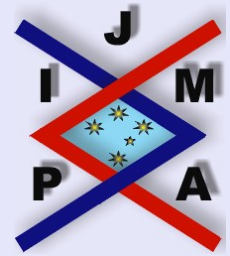
*Proof of Theorem 1.5.* Suppose that

$$[f^{(\alpha-\beta)}(x)]' \geq \frac{\beta(\beta-1)(\alpha-\beta)(x-a)^{\beta-2}}{\alpha-1}.$$

Now consider the quotient of the two sides of (1.5). Applying Cauchy's Mean Value Theorem three times leads to

$$\begin{aligned} \frac{\int_a^b [f(x)]^\alpha dx}{\left[ \int_a^b f(x) dx \right]^\beta} &= \frac{[f(b_1)]^{\alpha-1}}{\beta \left[ \int_a^{b_1} f(x) dx \right]^{\beta-1}} \\ &\geq \frac{(\alpha-1)[f(b_2)]^{\alpha-3} f'(b_2)}{\beta(\beta-1) \left[ \int_a^{b_2} f(x) dx \right]^{\beta-2}} && a < b_2 < b_1 \\ &\geq \left[ \frac{f(b_2)(b_2-a)}{\int_a^{b_2} f(x) dx} \right]^{\beta-2} && a < b_3 < b_2 \\ &= \left[ 1 + \frac{f'(b_3)(b_3-a)}{f(b_3)} \right]^{\beta-2} \geq 1. \end{aligned}$$

This completes the proof.  $\square$



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## References

- [1] M. AKKOUCHI, On an integral inequality of Feng Qi, *Divulg. Mat.*, **13**(1) (2005), 11–19.
- [2] M. AKKOUCHI, Some integral inequalities, *Divulg. Mat.*, **11**(2) (2003), 121–125.
- [3] L. BOUGOFFA, An integral inequality similar to Qi's inequality, *J. Inequal. Pure Appl. Math.*, **6**(1) (2005), Art. 27. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=496>]
- [4] L. BOUGOFFA, Notes on Qi type integral inequalities, *J. Inequal. Pure Appl. Math.*, **4**(4) (2003), Art. 77. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=318>]
- [5] Y. CHEN AND J. KIMBALL, Note on an open problem of Feng Qi, *J. Inequal. Pure Appl. Math.*, **7**(1) (2006), Art. 4. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=621>]
- [6] V. CSISZÁR AND T.F. MÓRI, The convexity method of proving moment-type inequalities, *Statist. Probab. Lett.*, **66** (2004), 303–313.
- [7] S. MAZOUZI AND F. QI, On an open problem by Feng Qi regarding an integral inequality, *RGMIA Res. Rep. Coll.*, **6**(1) (2003), Art. 6. [ONLINE: <http://rgmia.vu.edu.au/v6n1.html>]
- [8] S. MAZOUZI AND F. QI, On an open problem regarding an integral inequality, *J. Inequal. Pure Appl. Math.*, **4**(2) (2003), Art. 31. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=269>]



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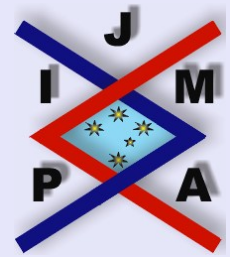
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- [9] J. PEČARIĆ AND T. PEJKOVIĆ, Note on Feng Qi's integral inequality, *J. Inequal. Pure Appl. Math.*, **5**(3) (2004), Art. 51. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=418>]
- [10] J. PEČARIĆ AND T. PEJKOVIĆ, On an integral inequality, *J. Inequal. Pure Appl. Math.*, **5**(2) (2004), Art. 47. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=401>]
- [11] T. K. POGÁNY, On an open problem of F. Qi, *J. Inequal. Pure Appl. Math.*, **3**(4) (2002), Art. 54. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=206>]
- [12] F. QI, Several integral inequalities, *J. Inequal. Pure Appl. Math.*, **1**(2) (2000), Art. 19. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=113>]
- [13] J.-Sh. SUN, A note on an open problem for integral inequality, *RGMI Res. Rep. Coll.*, **7**(3) (2004), Art. 21. [ONLINE: <http://rgmia.vu.edu.au/v7n3.html>]
- [14] N. TOWGHI, Notes on integral inequalities, *RGMI Res. Rep. Coll.*, **4**(2) (2001), Art. 12, 277–278. [ONLINE: <http://rgmia.vu.edu.au/v4n2.html>]
- [15] A. WITKOWSKI, On F. Qi integral inequality, *J. Inequal. Pure Appl. Math.*, **6**(2) (2005), Art. 36. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=505>]



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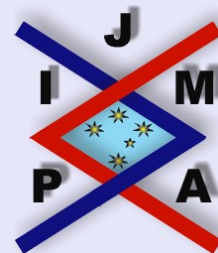
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- [16] K.-W. YU AND F. QI, A short note on an integral inequality, *RGMI Res. Rep. Coll.*, **4**(1) (2001), Art. 4, 23–25. [ONLINE: <http://rgmia.vu.edu.au/v4n1.html>]



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