

## STARLIKENESS AND CONVEXITY CONDITIONS FOR CLASSES OF FUNCTIONS DEFINED BY SUBORDINATION

R. AGHALARY, JAY M. JAHANGIRI AND S.R. KULKARNI

University of Urmia, Urmia, Iran.

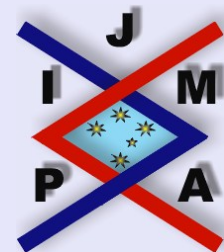
*EMail:* [raghalary@yahoo.com](mailto:raghalary@yahoo.com)

Kent State University, Ohio, USA.

*EMail:* [jay@geauga.kent.edu](mailto:jay@geauga.kent.edu)

Fergusson College, Pune, India.

*EMail:* [srkulkarni40@hotmail.com](mailto:srkulkarni40@hotmail.com)



---

volume 5, issue 2, article 31,  
2004.

*Received 21 May, 2003;  
accepted 18 April, 2004.*

*Communicated by: H. Silverman*

---

Abstract

Contents

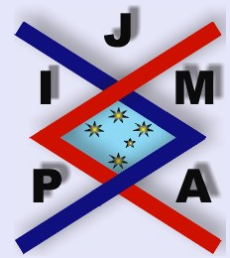


Home Page

Go Back

Close

Quit



## Abstract

We consider the family  $\mathcal{P}(1, b)$ ,  $b > 0$ , consisting of functions  $p$  analytic in the open unit disc  $U$  with the normalization  $p(0) = 1$  which have the disc formulation  $|p - 1| < b$  in  $U$ . Applying the subordination properties to certain choices of  $p$  using the functions  $f_n(z) = z + \sum_{k=1+n}^{\infty} a_k z^k$ ,  $n = 1, 2, \dots$ , we obtain inclusion relations, sufficient starlikeness and convexity conditions, and coefficient bounds for functions in these classes. In some cases our results improve the corresponding results appeared in print.

*2000 Mathematics Subject Classification:* Primary 30C45.

*Key words:* Subordination, Hadamard product, Starlike, Convex.

## Contents

1	Introduction .....	3
2	The Family $\mathcal{F}(1, b)$ .....	6
3	The Family $\mathcal{M}_{\chi}^{\nu}(1, b)$ .....	12
4	Coefficient Bounds .....	19
	References	

---

### Starlikeness and Convexity Conditions for Classes of Functions Defined by Subordination

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

---

Title Page

Contents



Go Back

Close

Quit

Page 2 of 25

# 1. Introduction

Let  $\mathcal{A}$  denote the class of functions that are analytic in the open unit disc  $U = \{z \in \mathcal{C} : |z| < 1\}$  and let  $\mathcal{A}_n$  be the subclass of  $\mathcal{A}$  consisting of functions  $f_n$  of the form

$$(1.1) \quad f_n(z) = z + \sum_{k=1+n}^{\infty} a_k z^k, \quad n = 1, 2, 3, \dots$$

The function  $p \in \mathcal{A}$  and normalized by  $p(0) = 1$  is said to be in  $\mathcal{P}(1, b)$  if

$$(1.2) \quad |p(z) - 1| < b, \quad b > 0, \quad z \in U.$$

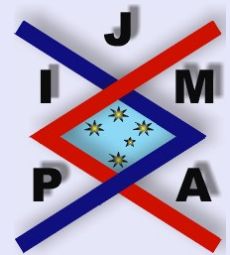
The class  $\mathcal{P}(1, b)$  which is defined using the disc formulation (1.2) was studied by Janowski [6] and has an alternative characterization in terms of subordination (see [5] or [14]), that is, for  $z \in U$ , we have

$$(1.3) \quad p \in \mathcal{P}(1, b) \iff p(z) \prec 1 + bz.$$

For the functions  $\phi$  and  $\psi$  in  $\mathcal{A}$ , we say that the  $\phi$  is subordinate to  $\psi$  in  $U$ , denoted by  $\phi \prec \psi$ , if there exists a function  $w(z)$  in  $\mathcal{A}$  with  $w(0) = 0$  and  $|w(z)| < 1$ , such that  $\phi(z) = \psi(w(z))$  in  $U$ . For further references see Duren [3].

The family  $\mathcal{P}(1, b)$  contains many interesting classes of functions which have close inter-relations with different well-known classes of analytic univalent functions. For example, for  $f_n \in \mathcal{A}_n$  if

$$\left( \frac{z f'_n}{f_n} \right) \in \mathcal{P}(1, 1 - \alpha), \quad 0 \leq \alpha \leq 1,$$



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 3 of 25

then  $f_n$  is starlike of order  $\alpha$  in  $U$  and if

$$\left(1 + \frac{zf_n''}{f_n'}\right) \in \mathcal{P}(1, 1 - \alpha), \quad 0 \leq \alpha \leq 1,$$

then  $f_n$  is convex of order  $\alpha$  in  $U$ .

For  $0 \leq \alpha \leq 1$  we let  $\mathcal{S}^*(\alpha)$  be the class of functions  $f_n \in \mathcal{A}_n$  which are starlike of order  $\alpha$  in  $U$ , that is,

$$\mathcal{S}^*(\alpha) \equiv \left\{ f_n \in \mathcal{A}_n : \Re \left( \frac{zf_n'}{f_n} \right) \geq \alpha, \quad |z| < 1 \right\},$$

and let  $\mathcal{K}(\alpha)$  be the class of functions  $f_n \in \mathcal{A}_n$  which are convex of order  $\alpha$  in  $U$ , that is,

$$\mathcal{K}(\alpha) \equiv \left\{ f_n \in \mathcal{A}_n : \Re \left( 1 + \frac{zf_n''}{f_n'} \right) \geq \alpha, \quad |z| < 1 \right\}.$$

Alexander [1] showed that  $f_n$  is convex in  $U$  if and only if  $zf_n'$  is starlike in  $U$ .

In this paper we investigate inclusion relations, starlikeness, convexity, and coefficient conditions on  $f_n$  and its related classes for two choices of  $p(f_n)$  in  $\mathcal{P}(1, b)$ . In some cases, we improve the related known results appeared in the literature.

Define  $\mathcal{F}(1, b)$  be the subclass of  $\mathcal{P}(1, b)$  consisting of functions  $p(f_1)$  so that

$$(1.4) \quad p(f_1(z)) = \frac{zf_1'(z)}{f_1(z)} \left( 1 + \frac{zf_1''(z)}{f_1'(z)} \right)$$



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 4 of 25

where  $f_1 \in \mathcal{A}_1$  is given by (1.1).

For fixed  $v > -1$ ,  $n \geq 1$ , and for  $\lambda \geq 0$ , define  $\mathcal{M}_\lambda^v(1, b)$  be the subclass of  $\mathcal{P}(1, b)$  consisting of functions  $p(f_n)$  so that

$$(1.5) \quad p(f_n(z)) = (1 - \lambda) \frac{D^v f_n(z)}{z} + \lambda (D^v f_n(z))'$$

where  $f_n \in \mathcal{A}_n$  and  $D^v f$  is the  $v$ -th order Ruscheweyh derivative [10].

The  $v$ -th order Ruscheweyh derivative  $D^v$  of a function  $f_n$  in  $\mathcal{A}_n$  is defined by

$$(1.6) \quad D^v f_n(z) = \frac{z}{(1-z)^{1+v}} * f_n(z) = z + \sum_{k=1+n}^{\infty} B_k(v) a_k z^k,$$

where

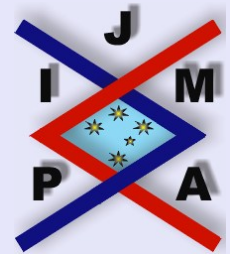
$$B_k(v) = \frac{(1+v)(2+v) \cdots (v+k-1)}{(k-1)!}$$

and the operator “ $*$ ” stands for the convolution or Hadamard product of two power series

$$f(z) = \sum_{i=1}^{\infty} a_i z^i \quad \text{and} \quad g(z) = \sum_{i=1}^{\infty} b_i z^i$$

defined by

$$(f * g)(z) = f(z) * g(z) = \sum_{i=1}^{\infty} a_i b_i z^i.$$



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 5 of 25

## 2. The Family $\mathcal{F}(1, b)$

The class  $\mathcal{F}(1, b)$  for certain values of  $b$  yields a sufficient starlikeness condition for the functions  $f_1 \in \mathcal{A}_1$ .

**Theorem 2.1.** *If  $0 < b \leq \frac{9}{4}$  and  $p(f_1) \in \mathcal{F}(1, b)$  then*

$$\frac{zf_1'}{f_1} \in \mathcal{P} \left( 1, \frac{3 - \sqrt{9 - 4b}}{2} \right).$$

We need the following lemma, which is due to Jack [4].

**Lemma 2.2.** *Let  $w(z)$  be analytic in  $U$  with  $w(0) = 0$ . If  $|w|$  attains its maximum value on the circle  $|z| = r$  at a points  $z_0$ , we can write  $z_0 w'(z_0) = kw(z_0)$  for some real  $k$ ,  $k \geq 1$ .*

*Proof of Theorem 2.1.* For  $b_1 = \frac{3 - \sqrt{9 - 4b}}{2}$  write  $\frac{zf_1'(z)}{f_1(z)} = 1 + b_1 w(z)$ . Obviously,  $w$  is analytic in  $U$  and  $w(0) = 0$ . The proof is complete if we can show that  $|w| < 1$  in  $U$ . On the contrary, suppose that there exists  $z_0 \in U$  such that  $|w(z_0)| = 1$ . Then, by Lemma 2.2, we must have  $z_0 w'(z_0) = kw(z_0)$  for some real  $k$ ,  $k \geq 1$  which yields

$$\begin{aligned} \left| \frac{z_0 f_1'(z_0)}{f_1(z_0)} \left( 1 + \frac{z_0 f_1''(z_0)}{f_1'(z_0)} \right) - 1 \right| &= |(1 + b_1 w(z_0))^2 + b_1 z_0 w'(z_0) - 1| \\ &= |b_k + 2b_1 + b_1^2 w(z_0)| \\ &\geq 3b_1 - b_1^2 = b. \end{aligned}$$

This contradicts the hypothesis, and so the proof is complete.  $\square$



Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 6 of 25

**Corollary 2.3.** For  $0 < b \leq 2$  let  $p(f_1) \in \mathcal{F}(1, b)$ . Then  $f_1 \in \mathcal{S}^* \left( \frac{-1+\sqrt{9-4b}}{2} \right)$ .

**Corollary 2.4.** If  $p(f_1) \in \mathcal{F}(1, b)$  and  $0 < b \leq 2$ , then

$$\left| \arg \frac{z f_1'(z)}{f_1(z)} \right| < \arcsin \left( \frac{3 - \sqrt{9 - 4b}}{2} \right).$$

It is not known if the above corollaries can be extended to the case when  $b > 2$ .

**Corollary 2.5.** If  $\Re \left( \frac{f_1(z)}{z f_1'(z) + z^2 f_1''(z)} \right) > \frac{1}{2}$  then  $f_1 \in \mathcal{S}^* \left( \frac{-1+\sqrt{5}}{2} \right)$ .

**Remark 2.1.** For  $0 < b < 2$ , Theorem 2.1 is an improvement to Theorem 1 obtained by Obradović, Joshi, and Jovanović [8].

**Corollary 2.6.** If  $p(f_1) \in \mathcal{F}(1, b)$  then  $f_1$  is convex in  $U$  for  $0 < b \leq 0.935449$ .

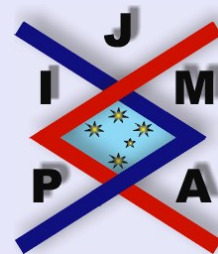
*Proof.* For  $p(f_1) \in \mathcal{F}(1, b)$  we can write  $|\arg p(f_1)| < \arcsin b$ . Therefore,

$$\left| \arg \left( 1 + \frac{z f_1''(z)}{f_1'(z)} \right) \right| < \arcsin b + \arcsin \left( \frac{3 - \sqrt{9 - 4b}}{2} \right).$$

Now the proof is complete upon noting that the right hand side of the above inequality is less than  $\frac{\pi}{2}$  for  $b = 0.935449$ .  $\square$

**Remark 2.2.** It is not known if the above Corollary 2.6 is sharp but it is an improvement to Corollary 2 obtained by Obradovic, Joshi, and Jovanovic [8].

**Corollary 2.7.** If  $p(f_1) \in \mathcal{F}(1, b)$  then  $f_1$  is convex in the disc  $|z| < \frac{0.935449}{b}$  for  $0.935449 \leq b \leq 1$ .



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 7 of 25

*Proof.* We write  $p(f_1) = 1 + bw(z)$  where  $w$  is a Schwarz function. Let  $|z| \leq \rho$ . Then  $|w(z)| \leq \rho$  and so  $|p(f_1) - 1| < b\rho$  for  $|z| \leq \rho$ . Upon choosing  $b\rho = 0.935449$  it follows from the above Corollary 2.6 that  $|\arg(1 + zf_1''/f_1')| < \pi/2$  for  $|z| \leq \rho = 0.935449/b$ . Therefore the proof is complete.  $\square$

In the following example we show that there exist functions  $f$  which are not necessarily starlike or univalent in  $U$  for  $p(f_1) \in \mathcal{F}(1, b)$  if  $b$  is sufficiently large.

**Example 2.1.** For the spirallike function  $g(z) = z/(1 - z)^{1+i}$  we have

$$\Re\left(e^{-\frac{\pi}{4}i} \frac{zg'(z)}{g(z)}\right) = \frac{1}{\sqrt{2}} \left(\frac{1 - |z|^2}{|1 - z|^2}\right) > 0, \quad z \in U.$$

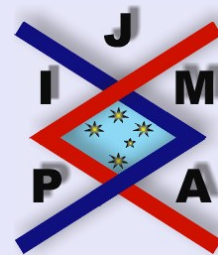
Since  $\frac{zg'(z)}{g(z)} = \frac{1+iz}{1-z}$ , we obtain

$$\Re\left(\frac{zg'(z)}{g(z)}\right) = \frac{1 - r(\cos\theta + \sin\theta)}{1 - 2r\cos\theta + r^2}$$

for  $z = re^{i\theta}$ . Thus  $g(z)$  is not starlike for  $|z| < t$ ,  $\frac{1}{\sqrt{2}} < t < 1$ . This means that  $f(z) = \frac{g(rz)}{r}$  is not starlike in  $U$ . Now set

$$h(z) = \int_0^z \frac{g(\zeta)}{\zeta} d\zeta = i((1 - z)^{-i} - 1)$$

and let  $z_0 = \frac{e^{2\pi-1}}{e^{2\pi+1}} \approx 0.996$ . Therefore,  $h(z_0) = h(-z_0)$  and so  $h$  is not univalent in  $U$ . Consequently,  $f(z) = \frac{h(z_0z)}{z_0}$  is not univalent in  $U$  for sufficiently large



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 8 of 25



values of  $b$ . On the other hand,  $p(g) \in \mathcal{F}(1, b)$  for sufficiently large  $b$ , since,

$$|p(g(z)) - 1| = \left| \frac{1 + 3iz}{(1 - z)^2} + \frac{z}{1 - z} - 1 \right| < b$$

for sufficiently large  $b$ .

The following theorem is the converse of Theorem 2.1 for a special case.

**Theorem 2.8.** *If  $\frac{zf'_1}{f_1} \in \mathcal{P}\left(1, \frac{3-\sqrt{5}}{2}\right)$  then  $p(f_1) \in \mathcal{F}(1, 1)$  for  $|z| < r_0 = 0.7851$ .*

To prove our theorem, we need the following lemma due to Dieudonné [2].

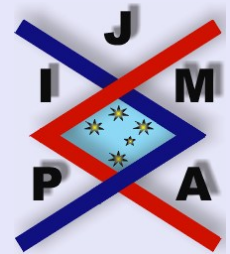
**Corollary 2.9.** *Let  $z_0$  and  $w_0$  be given points in  $U$ , with  $z_0 \neq 0$ . Then for all functions  $f$  analytic and satisfying  $|f(z)| < 1$  in  $U$ , with  $f(0) = 0$  and  $f(z_0) = w_0$ , the region of values of  $f'(z_0)$  is the closed disc*

$$\left| w - \frac{w_0}{z_0} \right| \leq \frac{|z_0|^2 - |w_0|^2}{|z_0|(1 - |z_0|^2)}.$$

*Proof of Theorem 2.8.* Write

$$q(z) = \frac{zf'_1(z)}{f_1(z)} = 1 + \left( \frac{3 - \sqrt{5}}{2} \right) w(z),$$

where  $w$  is a Schwarz function. We need to find the largest disc  $|z| < \rho$  for



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 9 of 25

which

$$\begin{aligned} & \left| \left[ 1 + \left( \frac{3 - \sqrt{5}}{2} \right) w(z) \right]^2 + \left( \frac{3 - \sqrt{5}}{2} \right) zw'(z) - 1 \right| \\ &= \left| \left( \frac{3 - \sqrt{5}}{2} \right)^2 w^2(z) + (3 - \sqrt{5})w(z) + \left( \frac{3 - \sqrt{5}}{2} \right) zw'(z) \right| < 1. \end{aligned}$$

For fixed  $r = |z|$  and  $R = |w(z)|$  we have  $R \leq r$ . Therefore, by Lemma 2.9, we obtain

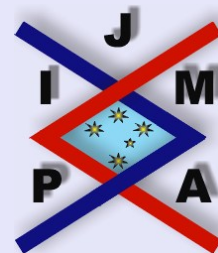
$$|w'(z)| \leq \frac{R}{r} + \frac{r^2 - R^2}{r(1 - r^2)}$$

and so

$$\begin{aligned} |p(f_1) - 1| &= \left| \frac{zf_1'(z)}{f_1(z)} \left( 1 + \frac{zf_1''(z)}{f_1'(z)} \right) - 1 \right| \\ &= \left| \left( \frac{3 - \sqrt{5}}{2} \right)^2 w^2(z) + (3 - \sqrt{5})w(z) + \left( \frac{3 - \sqrt{5}}{2} \right) zw'(z) \right| \\ &\leq t^2 R^2 + 3tR + t \frac{r^2 - R^2}{1 - r^2} \\ &= \frac{t}{1 - r^2} \psi(R), \end{aligned}$$

where

$$\psi(R) = R^2(t - tr^2 - 1) + 3R(1 - r^2) + r^2 \quad \text{and} \quad t = \frac{3 - \sqrt{5}}{2}.$$



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

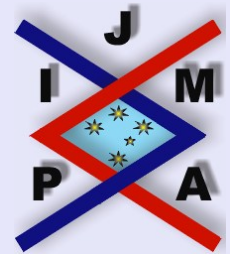
Close

Quit

Page 10 of 25

We note that  $\psi(R)$  attains its maximum at  $R_0 = \frac{3(1-r^2)}{2(1+tr^2-t)}$ . So the theorem follows for  $r_0 \approx 0.7851$  which is the root of the equation  $\frac{t}{1-r^2}\psi(R_0) = 1$ .

Letting  $z_0$  and  $w_0$  in Lemma 2.9 be so that  $|z_0| = r_0$  and  $|w_0| = \frac{3(1-r_0^2)}{2(1+tr_0^2-t)}$  we conclude that the bound given by Theorem 2.8 is sharp.  $\square$




---

**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

---

Title Page

Contents



Go Back

Close

Quit

Page 11 of 25

### 3. The Family $\mathcal{M}_\lambda^v(1, b)$

We begin with stating and proving some properties of the family  $\mathcal{M}_\lambda^v(1, b)$ .

**Theorem 3.1.** *If  $p(f_n) \in \mathcal{M}_\lambda^v(1, b)$  then*

$$\frac{D^v f_n(z)}{z} \in \mathcal{P}\left(1, \frac{b}{1 + \lambda n}\right).$$

We need the following lemma, which is due to Miller and Mocanu [7].

**Lemma 3.2.** *Let  $q(z) = 1 + q_n z^n + \dots$  ( $n \geq 1$ ) be analytic in  $U$  and let  $h(z)$  be convex univalent in  $U$  with  $h(0) = 1$ . If  $q(z) + \frac{1}{c} z q'(z) \prec h(z)$  for  $c > 0$ , then*

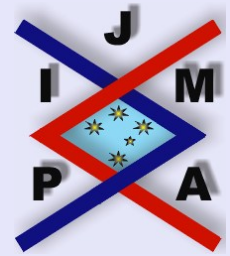
$$q(z) \prec \frac{c}{n} z^{-c/n} \int_0^z h(t) t^{\frac{c}{n}-1} dt.$$

*Proof of Theorem 3.1.* For  $p(f_n) \in \mathcal{M}_\lambda^v(1, b)$  set  $q(z) = \frac{D^v f_n(z)}{z}$ . Then we can write  $q(z) + \lambda z q'(z) \prec 1 + bz$ . Now, applying Lemma 3.2, we obtain

$$q(z) \prec +1 + \frac{b}{1 + \lambda n} z.$$

Substituting back for  $q(z)$  and choosing  $w(z)$  to be analytic in  $U$  with  $|w(z)| \leq |z|^n$ , by the definition of subordination we have

$$\frac{D^v f_n(z)}{z} = 1 + \frac{b}{(1 + \lambda n)} w(z).$$



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 12 of 25

Now the theorem follows using the necessary and sufficient condition (1.3). The estimates in Theorem 3.1 are sharp for  $p(f_n)$  where  $f_n$  is given by

$$\frac{D^v f_n(z)}{z} = 1 + \frac{b}{(1 + \lambda n)} z^n.$$

□

**Corollary 3.3.** *If  $p(f_n) \in \mathcal{M}_\lambda^v(1, b)$  then*

$$\left| \frac{D^v f_n(z)}{z} \right| \leq 1 + \frac{b}{1 + \lambda n} |z|^n.$$

**Corollary 3.4.** *If  $|f'_n(z) + \lambda z f''_n(z) - 1| < b$  then*

$$f'_n(z) \prec 1 + \frac{b}{1 + \lambda n} z.$$

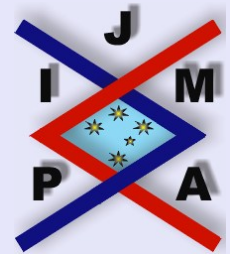
**Corollary 3.5.** *If  $\left| (1 - \lambda) \frac{f_n(z)}{z} + \lambda f'_n(z) - 1 \right| < b$  then*

$$\frac{f_n(z)}{z} \prec 1 + \frac{b}{1 + \lambda n} z.$$

In the next two theorems we investigate the inclusion relations for classes of  $\mathcal{M}_\lambda^v$ .

**Theorem 3.6.** *For  $0 \leq \lambda_1 < \lambda$  and  $v \geq 0$ , let  $b_1 = \frac{1 + \lambda_1 n}{1 + \lambda n} b$ . Then*

$$\mathcal{M}_\lambda^v(1, b) \subset \mathcal{M}_{\lambda_1}^v(1, b_1).$$



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 13 of 25

*Proof.* The case for  $\lambda_1 = 0$  is trivial. For  $\lambda_1 \neq 0$  suppose that  $p(f_n) \in \mathcal{M}_\lambda^v(1, b)$ . Therefore, we can write

$$\begin{aligned} & (1 - \lambda_1) \frac{D^v f_n(z)}{z} + \lambda_1 (D^v f_n(z))' \\ &= \frac{\lambda_1}{\lambda} \left[ (1 - \lambda) \frac{D^v f_n(z)}{z} + \lambda (D^v f_n(z))' \right] + \left( 1 - \frac{\lambda_1}{\lambda} \right) \left( \frac{D^v f_n(z)}{z} \right). \end{aligned}$$

Now, by definition,  $p(f_n) \in \mathcal{M}_{\lambda_1}^v(1, b_1)$  and so the proof is complete. □

**Theorem 3.7.** Let  $v \geq 0$  and  $b_1 = \frac{b(1+v)}{n+1+v}$ . Then

$$\mathcal{M}_{\lambda}^{v+1}(1, b) \subset \mathcal{M}_{\lambda}^v(1, b_1).$$

*Proof.* For  $f_n \in \mathcal{A}_n$  suppose that  $p_1(f_n) \in \mathcal{M}_{\lambda}^{v+1}(1, b)$  where

$$p_1(f_n(z)) = (1 - \lambda) \frac{D^{1+v} f_n(z)}{z} + \lambda (D^{v+1} f_n(z))'.$$

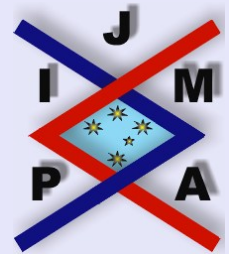
Set

$$p_2(f_n(z)) = (1 - \lambda) \frac{D^v f_n(z)}{z} + \lambda (D^v f_n(z))'.$$

An elementary differentiation yields

$$\begin{aligned} p_1(f_n(z)) &= (1 - \lambda) \frac{D^{1+v} f_n(z)}{z} + \lambda (D^{v+1} f_n(z))' \\ &= p_2(f_n(z)) + \frac{1}{1+v} z p_2'(f_n(z)). \end{aligned}$$

From this and Lemma 3.2, we conclude that  $p_1(f_n) \in \mathcal{M}_{\lambda}^v(1, b_1)$ . □



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 14 of 25

### Corollary 3.8.

$$f'_n(z) + \lambda z f''_n(z) \in \mathcal{P}(1, b) \implies (1 - \lambda) \frac{f_n(z)}{z} + \lambda f'_n(z) \in \mathcal{P}\left(1, \frac{b}{1 + n}\right).$$

**Theorem 3.9.** For  $v \geq 0$  and  $\lambda > 0$  let  $b < 1 + \lambda n$ . If  $p(f_n) \in \mathcal{M}_\lambda^v(1, b)$  then

$$\left| \frac{z(D^v f_n(z))'}{D^v f_n(z)} - 1 \right| < \frac{b(2 + \lambda n)}{\lambda[(1 + \lambda n) - b]}.$$

*Proof.* First note that, we can write

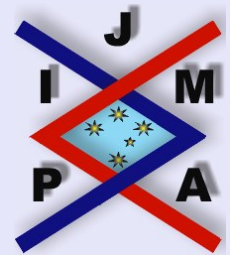
$$\left| (1 - \lambda) \frac{D^v f_n(z)}{z} + \lambda(D^v f_n(z))' - 1 \right| < b; \quad \left| \frac{D^v f_n(z)}{z} - 1 \right| < \frac{b}{1 + \lambda n}.$$

For  $b_1 = \frac{b(2 + \lambda n)}{\lambda[(1 + \lambda n) - b]}$  we define  $w(z)$  by

$$1 + b_1 w(z) = \frac{[z(D^v f_n(z))']}{[D^v f_n(z)]}.$$

One can easily verify that  $w(z)$  is analytic in  $U$  and  $w(0) = 0$ . To conclude the proof, it suffices to show that  $|w(z)| < 1$  in  $U$ . If this is not the case, then by Lemma 2.2, there exists a point  $z_0 \in U$  such that  $|w(z_0)| = 1$  and  $z_0 w'(z_0) = kw(z_0)$ . Therefore

$$\begin{aligned} |p(f_n(z_0)) - 1| &= \left| (1 - \lambda) \frac{D^v f(z_0)}{z_0} + \lambda(D^v f(z_0))' - 1 \right| \\ &= \left| \frac{D^v f_n(z_0)}{z_0} \left[ (1 - \lambda) + \lambda \frac{z_0(D^v f_n(z_0))'}{D^v f_n(z_0)} \right] - 1 \right| \end{aligned}$$



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 15 of 25

$$= \left| \lambda \left( \frac{z_0 (D^v f_n(z_0))'}{D^v f_n(z_0)} - 1 \right) \frac{D^v f_n(z_0)}{z_0} + \left( \frac{D^v f_n(z_0)}{z_0} - 1 \right) \right|$$

$$\geq \lambda b_1 \left( 1 - \frac{b}{1+n\lambda} \right) - \frac{b}{1+n\lambda} = b.$$

This is a contradiction to the hypothesis and so  $|w(z)| < 1$  in  $U$ . □

**Corollary 3.10.** *i) If  $f'_n(z) \in \mathcal{P}(1, \frac{1+n}{3+n})$  then  $\frac{zf'_n(z)}{f_n(z)} \in \mathcal{P}(1, 1)$ .*

*ii) If  $f'_n(z) + zf''_n(z) \in \mathcal{P}(1, \frac{1+n}{3+n})$  then  $\frac{zf''_n(z)}{f'_n(z)} \in \mathcal{P}(1, 1)$ .*

**Theorem 3.11.** *Let  $p(f_n) \in \mathcal{M}_\lambda^v(1, b)$  for some  $\lambda > 0$ . If*

$$b = \begin{cases} \frac{\lambda(1+\lambda n)}{2+\lambda(n-1)}; & 0 < \lambda \leq \frac{(n-3) + \sqrt{n^2 + 2n + 9}}{2n} \\ (1+\lambda n) \sqrt{\frac{2\lambda-1}{\lambda^2 n^2 + 2\lambda(1+n)}}; & \frac{(n-3) + \sqrt{n^2 + 2n + 9}}{2n} \leq \lambda \leq 1 \end{cases}$$

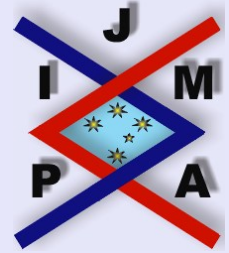
then

$$\Re \left( \frac{D^{v+1} f_n(z)}{D^v f_n(z)} \right) > \frac{v}{1+v}.$$

We need the following lemma, which is due to Ponnusamy and Singh [9].

**Lemma 3.12.** *Let  $0 < \lambda_1 < \lambda < 1$  and let  $Q$  be analytic in  $U$  satisfying  $Q(z) \prec 1 + \lambda_1 z$ , and  $Q(0) = 1$ . If  $q(z)$  is analytic in  $U$ ,  $q(0) = 1$  and satisfies*

$$Q(z)[c + (1-c)q(z)] \prec 1 + \lambda z,$$



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 16 of 25



where

$$c = \begin{cases} \frac{1 - \lambda}{1 + \lambda_1}, & 0 < \lambda + \lambda_1 \leq 1 \\ \frac{1 - (\lambda^2 + \lambda_1^2)}{2(1 - \lambda_1^2)}, & \lambda^2 + \lambda_1^2 \leq 1 \leq \lambda + \lambda_1 \end{cases}$$

then  $\operatorname{Re}\{q(z)\} > 0$ ,  $z \in U$ .

*Proof of Theorem 3.11.* From Theorem 3.1 and the fact  $0 < b < 1 < 1 + \lambda n$  we conclude that

$$\frac{D^v f_n(z)}{z} \prec 1 + b_1 z, \quad 0 < b_1 = \frac{b}{1 + n\lambda} < b < 1.$$

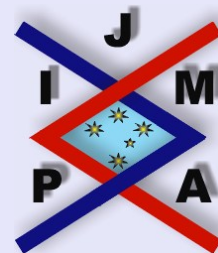
On the other hand, we may write

$$\frac{D^v f_n(z)}{z} \left[ (1 - \lambda) + \lambda \left( \frac{z(D^v f_n(z))'}{D^v f_n(z)} \right) \right] \prec 1 + bz.$$

Letting  $Q(z) = \frac{D^v f_n(z)}{z}$ ,  $q(z) = \frac{z(D^v f_n(z))'}{D^v f_n(z)}$ , and  $c = 1 - \lambda$ , we see that all conditions in Lemma 3.12 are satisfied. This implies that  $\operatorname{Re} q(z) > 0$  and so the proof is complete.  $\square$

**Corollary 3.13.** Let  $p(f_n) \in \mathcal{M}_\lambda^v(1, b)$  for some  $\lambda > 0$ . Then  $D^v f_n$  is starlike in the disc

$$|z| \leq \begin{cases} \frac{\lambda(1 + n\lambda)}{(2 + \lambda(n - 1))b} & \text{if } 0 < \lambda < \lambda_1 \text{ and } b_1 \leq b \leq 1 \\ \frac{(1 + \lambda n)}{b} \sqrt{\frac{2\lambda - 1}{\lambda^2 n^2 + 2\lambda(1 + n)}} & \text{if } \lambda_1 \leq \lambda \leq 1 \text{ and } b_2 \leq b \leq 1, \end{cases}$$



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 17 of 25

where

$$\lambda_1 = \frac{(n-3) + \sqrt{n^2 + 2n + 9}}{2n}, \quad b_1 = \frac{\lambda(1+n\lambda)}{[2 + \lambda(n-1)]}, \quad \text{and}$$

$$b_2 = (1 + \lambda_n) \sqrt{\frac{2\lambda - 1}{\lambda^2 n^2 + 2\lambda(1+n)}}.$$

i) If  $f'_n \in \mathcal{P}\left(1, \frac{(1+n)}{\sqrt{1+(1+n)^2}}\right)$  then  $f_n$  is starlike in  $U$ .

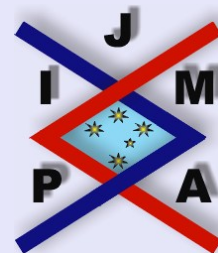
ii) If  $f'_n + z f''_n \in \mathcal{P}\left(1, \frac{(1+n)}{\sqrt{1+(1+n)^2}}\right)$  then  $f_n$  is convex in  $U$ .

If we let  $\lambda = 1$  and  $v = 0, 1$  in Corollary 3.13, then we obtain

**Corollary 3.14.** Let  $\frac{(1+n)}{\sqrt{1+(1+n)^2}} \leq b \leq 1$  and  $f_n \in \mathcal{A}_n$ .

i) If  $f'_n \in \mathcal{P}(1, b)$  then  $f$  is starlike for  $|z| < \frac{(1+n)}{b\sqrt{1+(1+n)^2}}$ .

ii) If  $f'_n + z f''_n \in \mathcal{P}(1, b)$  then  $f$  is convex for  $|z| < \frac{1+n}{b\sqrt{1+(1+n)^2}}$ .



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 18 of 25

## 4. Coefficient Bounds

Sufficient coefficient conditions for  $\mathcal{F}(1, b)$  and  $\mathcal{M}_\lambda^v(1, b)$  are given next.

**Theorem 4.1.** Let  $p(f_1)$  be given by (1.4) for  $f_1$  as in (1.1). If

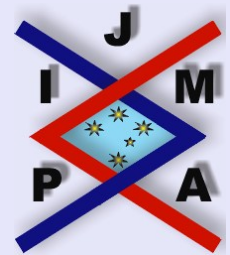
$$(4.1) \quad \sum_{k=2}^{\infty} (k^2 + b - 1) |a_k| < b,$$

then  $p(f_1) \in \mathcal{F}(1, b)$ .

*Proof.* We need to show that if (4.1) then  $|p(f_1(z)) - 1| < b$ . For  $p(f_1)$  we can write

$$\begin{aligned} |p(f_1(z)) - 1| &= \left| \frac{z f_1'}{f_1} \left( 1 + \frac{z f_1''}{f_1'} \right) - 1 \right| \\ &= \left| \frac{\sum_{k=2}^{\infty} (k^2 - 1) a_k z^k}{z + \sum_{k=2}^{\infty} a_k z^k} \right| \\ &\leq \frac{\sum_{k=2}^{\infty} (k^2 - 1) |a_k| |z|^{k-1}}{1 - \sum_{k=2}^{\infty} |a_k| |z|^{k-1}} \\ &< \frac{\sum_{k=2}^{\infty} (k^2 - 1) |a_k|}{1 - \sum_{k=2}^{\infty} |a_k|}. \end{aligned}$$

The above right hand inequality is less than  $b$  by (4.1) and so  $p(f_1) \in \mathcal{F}(1, b)$ .  $\square$



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 19 of 25

**Theorem 4.2.** Let  $p(f_n)$  be given by (1.5) for  $f_n$  as in (1.1). If

$$(4.2) \quad \sum_{k=1+n}^{\infty} (\lambda k - \lambda + 1) B_k(v) |a_k| < b,$$

then  $p(f_n) \in \mathcal{M}_{\lambda}^v(1, b)$ .

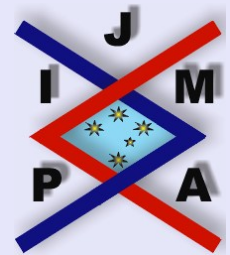
*Proof.* Apply the Ruscheweyh derivative (1.6) to the function  $f_n(z)$  and substitute in (1.5) to obtain

$$\begin{aligned} |p(f_n(z)) - 1| &= \left| (1 - \lambda) \frac{D^v f_n(z)}{z} + \lambda (D^v f_n(z))' - 1 \right| \\ &= \left| \sum_{k=1+n}^{\infty} (\lambda k - \lambda + 1) B_k(v) a_k z^{k-1} \right| \\ &< \sum_{k=1+n}^{\infty} (\lambda k - \lambda + 1) B_k(v) |a_k|. \end{aligned}$$

Now this latter inequality is less than  $b$  by (4.2) and so  $p(f_n) \in \mathcal{M}_{\lambda}^v(1, b)$ .  $\square$

Next, by judiciously varying the arguments of the coefficients of the functions  $f_n$  given by (1.1), we shall show that the sufficient coefficient conditions (4.1) and (4.2) are also necessary for their respective classes with varying arguments.

A function  $f_n$  given by (1.1) is said to be in  $\mathcal{V}(\theta_k)$  if  $\arg(a_k) = \theta_k$  for all  $k$ . If, further, there exists a real number  $\beta$  such that  $\theta_k + (k - 1)\beta \equiv \pi \pmod{2\pi}$  then  $f_n$  is said to be in  $\mathcal{V}(\theta_k; \beta)$ . The union of  $\mathcal{V}(\theta_k; \beta)$  taken over all possible



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 20 of 25

$\{\theta_k\}$  and all possible real  $\beta$  is denoted by  $\mathcal{V}$ . For more details see Silverman [13].

Some examples of functions in  $\mathcal{V}$  are

- i)  $\mathcal{T} \equiv \mathcal{V}(\pi; 0) \subset \mathcal{V}$  where  $\mathcal{T}$  is the class of analytic univalent functions with negative coefficients studied by Schild [11] and Silverman [12].
- ii) Univalent functions of the form  $z + \sum_{k=2}^{\infty} |a_k| e^{i\theta_k} z^k$  are in  $\mathcal{V}(\theta_k; 2\pi/k) \subset \mathcal{V}$  for  $\theta_k = \pi - 2(k-1)\pi/k$ .

Note that the family  $\mathcal{V}$  is rotationally invariant since  $f_n \in \mathcal{V}(\theta_k; \beta)$  implies that

$$e^{-i\gamma} f_n(z e^{i\gamma}) \in \mathcal{V}(\theta_k + (k-1)\gamma; \beta - \gamma).$$

Finally, we let

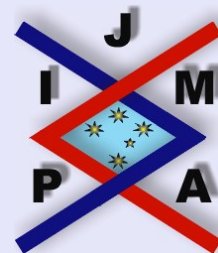
$$\mathcal{V}\mathcal{F}(1, b) \equiv \mathcal{V} \cap \mathcal{F}(1, b) \quad \text{and} \quad \mathcal{V}\mathcal{M}_{\lambda}^v(1, b) \equiv \mathcal{V} \cap \mathcal{M}_{\lambda}^v(1, b).$$

**Theorem 4.3.**

$$p(f_1) \in \mathcal{V}\mathcal{F}(1, b) \iff \sum_{k=2}^{\infty} (k^2 + b - 1) |a_k| < b.$$

*Proof.* In light of Theorem 4.1, we only need to prove the “only if” part of the theorem. Suppose  $p(f_1) \in \mathcal{V}\mathcal{F}(1, b)$ , then

$$|p(f_1) - 1| = \left| \frac{\sum_{k=2}^{\infty} (k^2 - 1) a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}} \right| < b$$



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page
Contents
⏪
⏩
◀
▶
Go Back
Close
Quit
Page 21 of 25

or

$$(4.3) \quad \left| \sum_{k=2}^{\infty} (k^2 - 1)a_k z^{k-1} \right| < b \left| 1 + \sum_{k=2}^{\infty} a_k z^{k-1} \right|.$$

The condition (4.3) must hold for all values of  $z$  in  $U$ . Therefore, for  $f_1 \in \mathcal{V}(\theta_k; \beta)$  we set  $z = re^{i\beta}$  in (4.3) and let  $r \rightarrow 1^-$ . Upon clearing the inequality (4.3) we obtain the condition

$$\sum_{k=2}^{\infty} (k^2 - 1)|a_k| < b \left( 1 - \sum_{k=2}^{\infty} |a_k| \right)$$

as required. □

**Corollary 4.4.** *If  $0 < b \leq 1$  and  $p(f_1) \in \mathcal{VF}(1, b)$  then  $f_1$  is convex in  $U$ .*

**Corollary 4.5.** *If  $1 < b \leq 3$  and  $p(f_1) \in \mathcal{VF}(1, b)$  then  $f_1$  is starlike in  $U$ .*

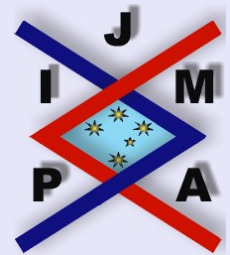
The above two corollaries can be justified using Theorem 4.3 and the following lemma due to Silverman [12].

**Lemma 4.6.** *For  $f_1$  of the form (1.1) and univalent in  $U$  we have*

*i) If  $\sum_{k=2}^{\infty} k^2 |a_k| \leq 1$ , then  $f_1$  is convex in  $U$ .*

*ii) If  $\sum_{k=2}^{\infty} k |a_k| \leq 1$ , then  $f_1$  is starlike in  $U$ .*

Next, we show that the above sufficient coefficient condition (4.2) is also necessary for functions in  $\mathcal{VM}_{\lambda}^{\nu}(1, b)$ .



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

Close

Quit

Page 22 of 25

**Theorem 4.7.**

$$p(f_n) \in \mathcal{VM}_\lambda^v(1, b) \iff \sum_{k=1+n}^{\infty} (\lambda k - \lambda + 1) B_k(v) |a_k| < b.$$

*Proof.* Suppose that  $p(f_n) \in \mathcal{VM}_\lambda^v(1, b)$ . Then, by (1.5), we have

$$|p(f_n(z)) - 1| = \left| (1 - \lambda) \frac{D^v f_n(z)}{z} + \lambda (D^v f_n(z))' - 1 \right| < b.$$

On the other hand, for  $f_n \in \mathcal{V}(\theta_k; \beta)$  we have

$$f_n(z) = z + \sum_{k=1+n}^{\infty} |a_k| e^{i\theta_k} z^k.$$

The condition required for  $p(f_n) \in \mathcal{VM}_\lambda^v(1, b)$  must hold for all values of  $z$  in  $U$ . Setting  $z = r e^{i\beta}$  yields

$$\sum_{k=1+n}^{\infty} (\lambda k - \lambda + 1) B_k(v) |a_k| r^{k-1} < b.$$

The required coefficient condition follows upon letting  $z \rightarrow 1^-$ . □

From the above Theorem 4.7 and Lemma 4.6.ii, we obtain

**Corollary 4.8.** *If  $\lambda \geq 2b - 1$  and  $p(f_n) \in \mathcal{VM}_\lambda^v(1, b)$  then  $f$  is starlike in  $U$ .*



**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

Title Page

Contents



Go Back

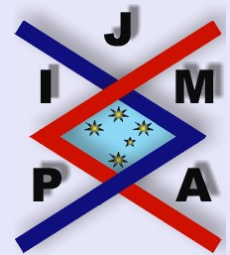
Close

Quit

Page 23 of 25

## References

- [1] J.W. ALEXANDER, Functions which map the interior of the unit circle upon simple regions, *Ann. of Math. Soc.*, **17** (1915-1916), 12–22.
- [2] J. DIEUDONNÉ, Recherches sur quelques problèmes relatifs aux polynômes et aux fonctions bornées d'une variable complexe, *Ann. Sci. Ecole Norm. Sup.*, **48** (1931), 247–358.
- [3] P.L. DUREN, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften 259, A series of Comprehensive Studies in Mathematics, Springer-Verlag New York Inc, 1983.
- [4] I.S. JACK, Functions starlike and convex of order  $\alpha$ , *J. London Math. Soc.*, **3** (1971), 469–474.
- [5] J.M. JAHANGIRI, H. SILVERMAN AND E.M. SILVIA, Inclusion relations between classes of functions defined by subordination, *J. Math. Anal. Appl.*, **151** (1990), 318–329.
- [6] W. JANOWSKI, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.*, **23** (1970), 159–177.
- [7] S.S. MILLER AND P.T. MOCANU, Differential subordination and univalent functions, *Michigan Math. J.*, **28** (1981), 157–171.
- [8] M. OBRADOVIĆ, S.B. JOSHI AND I. JOVANOVIĆ, On certain sufficient conditions for starlikeness and convexity, *Indian J. Pure Appl. Math.*, **29** (1998), 271–275.



---

**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

---

Title Page

Contents



Go Back

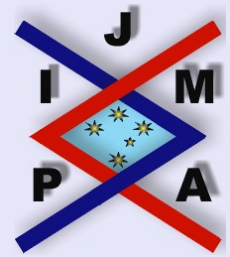
Close

Quit

Page 24 of 25



- [9] S. PONNUSAMY AND V. SINGH, Convolution properties of some classes of analytic functions, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.*, (POMI) **226** (1996), 138–154.
- [10] St. RUSCHEWEYH, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, **49** (1975), 109–115.
- [11] A. SCHILD, On a class of functions schlicht in the unit circle, *Proc. Amer. Math. Soc.*, **5** (1954), 115–120.
- [12] H. SILVERMAN, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51** (1975), 109–116.
- [13] H. SILVERMAN, Univalent functions with varying arguments, *Houston J. Math.*, **7** (1981), 283–287.
- [14] H. SILVERMAN AND E. M. SILVIA, Subclasses of starlike functions subordinate to convex functions, *Canad. J. Math.*, **37** (1985), 48–61.




---

**Starlikeness and Convexity  
Conditions for Classes of  
Functions Defined by  
Subordination**

R. Aghalary, Jay M. Jahangiri and  
S.R. Kulkarni

---

Title Page

Contents



Go Back

Close

Quit

Page 25 of 25