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CESÁRO MEANS OF N-MULTIPLE TRIGONOMETRIC FOURIER SERIES

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ABSTRACT. Zhizhiashvili proved sufficient condition for the Cesáro summability by negative order of N-multiple trigonometric Fourier series in the space $L^p, 1 \leq p \leq \infty$. In this paper we show that this condition cannot be improved .

Key words and phrases: Trigonometric system, Cesáro means, Summability.

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Let R^N be N-dimensional Euclidean space. The elements of R^N are denoted by $x = (x_1, \dots, x_N)$, $y = (y_1, \dots, y_N)$, ... For any $x, y \in R^N$ the vector $(x_1 + y_1, \dots, x_N + y_N)$ of the space R^N is denoted by $x + y$. Let $\|x\| = \left(\sum_{i=1}^N x_i^2\right)^{1/2}$.

Denote by $C\left([0,2]^N\right)$ the space of continuous on $[0, 2\pi]^N$, 2π-periodic relative to each variable functions with the following norm

$$\|f\|_C = \sup_{x \in [0,2\pi]^N} |f(x)|$$

and $L^p\left([0,2\pi]^N\right)$, $(1 \leq p \leq \infty)$ are the collection of all measurable, 2π-periodic relative to each variable functions f defined on $[0, 2\pi]^N$, with the norms

$$\|f\|_p = \left(\int_{[0,2\pi]^N} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

For the case $p = \infty$, by $L^p\left([0,2\pi]^N\right)$ we mean $C\left([0,2\pi]^N\right)$.

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Let $M := \{1,2,\ldots,N\}$, $B := \{s_1,\ldots,s_r\}$, $s_k < s_{k+1}$, $k = 1,\ldots,r-1$, $B \subset M$, $B' := M \backslash B$. Let

$$\Delta^{\{s_i\}}(f,x,h_{s_i}) := f(x_1,\ldots,x_{s_i-1},x_{s_i}+h_{s_i},x_{s_i+1},\ldots,x_N) - f(x_1,\ldots,x_{s_i-1},x_{s_i},x_{s_i+1},\ldots,x_N).$$

The expression we get by successive application of operators $\Delta^{\{s_1\}}(f,x,h_{s_1}),\ldots,$ $\Delta^{\{s_r\}}(f,x,h_{s_r})$ will be denoted by $\Delta^B(f,x,h_{s_1},\ldots,h_{s_r})$, i. e.

$$\Delta^B(f,x,h_{s_1},\ldots,h_{s_r}) := \Delta^{\{s_r\}}\left(\Delta^{B\backslash\{s_r\}},x,h_{s_r}\right).$$

Let $f \in L^p\left([0,2\pi]^N\right)$. The expression

$$\omega_B(\delta_{s_1},\ldots,\delta_{s_r};f) := \sup_{\left|h_{s_i}\right|\le\delta_{s_i},i=1,\ldots,r} \left\|\Delta^B(f,\cdot,h_{s_1},\ldots,h_{s_r})\right\|_p$$

is called a mixed or a particular modulus of continuity in the L^p norm, when card$(B) \in [2,N]$ or card$(B) = 1$.

The total modulus of continuity of the function $f \in L^p\left([0,2\pi]^N\right)$ in the L^p norm is defined by

$$\omega(\delta,f)_p = \sup_{\|h\|\le\delta} \|f(\cdot+h) - f(\cdot)\|_p \qquad (1 \le p \le \infty).$$

Suppose that f is a Lebesgue integrable function on $[0,2\pi]^N$, 2π periodic relative to each variable.Then its N-dimensional Fourier series with respect to the trigonometric system is defined by

$$\sum_{i_1=0}^{\infty}\cdots\sum_{i_N=0}^{\infty} 2^{-\lambda(i)} \sum_{B\subset M} a_{i_1,\ldots,i_N}^{(B)} \prod_{j\in B'} \cos i_j x_j \prod_{k\in B} \sin i_k x_k,$$

where

$$a_{i_1,\ldots,i_N}^{(B)} = \frac{1}{\pi^N} \int_{[0,2\pi]^N} f(x) \prod_{j\in B'} \cos i_j x_j \prod_{k\in B} \sin i_k x_k dx$$

is the Fourier coefficient of f and $\lambda(i)$ is the number of those coordinates of the vector $i := (i_1,\ldots,i_N)$ which are equal to zero.

Let $S_{p_1,\ldots,p_N}(f,x)$ denote the $(p_1,\ldots,p_N)$-th rectangular partial sums of the N-dimensional Fourier series with respect to the trigonometric system, i. e.

$$S_{p_1,\ldots,p_N}(f,x) := \sum_{i_1=0}^{p_1}\cdots\sum_{i_N=0}^{p_N} A_{i_1,\ldots,i_N}(f,x),$$

where

$$A_{i_1,\ldots,i_N}(f,x) := 2^{-\lambda(i)} \sum_{B\subset M} a_{i_1,\ldots,i_N}^{(B)} \prod_{j\in B'} \cos i_j x_j \prod_{k\in B} \sin i_k x_k.$$

The Cesáro $(C;\alpha_1,\ldots,\alpha_N)$-means of N-multiple trigonometric Fourier series defined by

$$\sigma_{m_1,\ldots,m_N}^{\alpha_1,\ldots,\alpha_N}(f,x) = \left(\prod_{i=1}^{N} A_{m_i}^{\alpha_i}\right)^{-1} \sum_{p_1=0}^{m_1}\cdots\sum_{p_N=0}^{m_N} \prod_{j=1}^{N} A_{m_j-p_j}^{\alpha_j-1} S_{p_1,\ldots,p_N}(f,x),$$

where

$$A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}, \qquad \alpha \ne -1,-2,\ldots, \quad n = 0,1,\ldots.$$

It is well-known that [4]

$$c_1(\alpha)\,n^{\alpha} \le A_n^{\alpha} \le c_2(\alpha)\,n^{\alpha}. \tag{1}$$

For the uniform summability of Cesáro means of negative order of one-dimensional trigonometric Fourier series the following result of Zygmund [3] is well-known: if

$$\omega(\delta, f)_C = o(\delta^{\alpha})$$

and $\alpha \in (0,1)$, then the trigonometric Fourier series of the function f is uniformly $(C,-\alpha)$ summable to f.

In [2] Zhizhiashvili proved sufficient conditions for the convergence of Cesáro means of negative order of N-multiple trigonometric Fourier series in the space $L^p\left([0,2\pi]^N\right)$, $(1 \le p \le \infty)$. The following is proved.

Theorem A (Zhizhiashvili). *Let* $f \in L^p\left([0,2\pi]^N\right)$ *for some* $p \in [1,+\infty]$ *and* $\alpha_1 + \cdots + \alpha_N < 1$*, where* $\alpha_i \in (0,1)$*,* $i = 1,2,\ldots,N$*. If*

$$\omega(\delta, f)_p = o\left(\delta^{\alpha_1+\cdots+\alpha_N}\right),$$

then

$$\left\|\sigma_{m_1,\ldots,m_N}^{-\alpha_1,\ldots,-\alpha_N}(f) - f\right\|_p \to 0 \qquad \textit{as} \quad m_i \to \infty,\ i = 1,\ldots,N.$$

In case $p = \infty$ the sharpness of Theorem A has been proved by Zhizhiashvili [2]. The following theorem shows that Theorem A cannot be improved in cases $1 \le p < \infty$. Moreover, we prove the following

Theorem 1 (for $N = 1$ see [1]). *Let* $\alpha_1 + \cdots + \alpha_N < 1$ *and* $\alpha_i \in (0,1)$*,* $i = 1,2,\ldots,N$*, then there exists the function* $f_0 \in C\left([0,2\pi]^N\right)$ *for which*

$$\omega(\delta, f_0)_C = O\left(\delta^{\alpha_1+\cdots+\alpha_N}\right) \tag{2}$$

and

$$\overline{\lim_{m\to\infty}} \left\|\sigma_{m,\ldots,m}^{-\alpha_1,\ldots,-\alpha_N}(f_0) - f_0\right\|_1 > 0.$$

Proof. We can define the sequence $\{n_k : k \le 1\}$ satisfying the properties

$$\sum_{j=k+1}^{\infty} \frac{1}{n_j^{\alpha_1+\cdots+\alpha_N}} = O\left(\frac{1}{n_k^{\alpha_1+\cdots+\alpha_N}}\right), \tag{3}$$

$$\sum_{j=1}^{k-1} n_j^{1-(\alpha_1+\cdots+\alpha_N)} = O\left(n_k^{1-(\alpha_1+\cdots+\alpha_N)}\right), \tag{4}$$

$$\frac{n_{k-1}}{n_k} < \frac{1}{k}. \tag{5}$$

Consider the function f_0 defined by

$$f_0(x_1,\ldots,x_N) := \sum_{j=1}^{\infty} f_j(x_1,\ldots,x_N),$$

where

$$f_j(x_1,\ldots,x_N) := \frac{1}{n_j^{\alpha_1+\cdots+\alpha_N}} \prod_{i=1}^{N} \sin n_j x_i.$$

From (3) it is easy to show that $f_0 \in C\left([0,2\pi]^N\right)$. First we shall prove that

$$\omega_i(\delta, f)_C = O\left(\delta^{\alpha_1+\cdots+\alpha_N}\right), \qquad i = 1,\ldots,N. \tag{6}$$

Let $\frac{1}{n_k} \le \delta < \frac{1}{n_{k-1}}$. Then from (3) and (4) we can write that

$$\begin{aligned}
&\left|f_0(x_1,\ldots,x_{i-1},x_i+\delta,x_{i+1},\ldots,x_N) - f_0(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_N)\right| \\
&\le \sum_{j=1}^{\infty} \frac{1}{n_j^{\alpha_1+\cdots+\alpha_N}} \left|\sin n_j(x_i+\delta) - \sin n_j x_i\right| \\
&\le \sum_{j=1}^{k-1} \frac{1}{n_j^{\alpha_1+\cdots+\alpha_N}} \left|\sin n_j(x_i+\delta) - \sin n_j x_i\right| + 2\sum_{j=k}^{\infty} \frac{1}{n_j^{\alpha_1+\cdots+\alpha_N}} \\
&\le \sum_{j=1}^{k-1} \frac{n_j\delta}{n_j^{\alpha_1+\cdots+\alpha_N}} + O\left(\frac{1}{n_k^{\alpha_1+\cdots+\alpha_N}}\right) \\
&= O\left(\delta n_{k-1}^{1-(\alpha_1+\cdots+\alpha_N)}\right) + O\left(\frac{1}{n_k^{\alpha_1+\cdots+\alpha_N}}\right) \\
&= O\left(\delta^{\alpha_1+\cdots\alpha_N}\right),
\end{aligned}$$

which proves (6).

Since

$$\omega(\delta, f)_C \le \sum_{i=1}^{N} \omega_i(\delta, f)_C,$$

we obtain the proof of estimation (2).

Next we shall prove that $\sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_0)$ diverge in the metric of $L^1\left([0,2\pi]^N\right)$. It is clear that

$$\begin{aligned}
&\left\|\sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_0) - f_0\right\|_1 \\
&\ge \left\|\sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_k)\right\|_1 \\
&\quad - \sum_{j=1}^{k-1} \left\|\sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_j) - f_j\right\|_C - \sum_{j=k+1}^{\infty} \left\|\sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_j)\right\|_C - \sum_{j=k}^{\infty} \|f_j\|_C \\
&= I - II - III - IV.
\end{aligned} \tag{7}$$

It is evident that

$$\sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_j) = 0, \qquad j = k+1, k+2, \ldots. \tag{8}$$

Using (3) for IV we have

$$IV \le \sum_{j=k}^{\infty} \frac{1}{n_j^{\alpha_1+\cdots+\alpha_N}} = O\left(\frac{1}{n_k^{\alpha_1+\cdots+\alpha_N}}\right). \tag{9}$$

Since [2]

$$\left\|\sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_j) - f_j\right\|_C = O\left(\sum_{B\subset M} \omega_B\left(\frac{1}{n_k}, f_j\right)_C n_k^{\sum_{s\in B}\alpha_s}\right)$$

and

$$\omega_i\left(\frac{1}{n_k}, f_j\right) = O\left(\frac{1}{n_j^{\alpha_1+\cdots+\alpha_N}} \frac{n_j}{n_k}\right),$$

from (4) and (5) we get

$$II = O\left(\frac{1}{n_k^{1-(\alpha_1+\cdots+\alpha_N)}}\sum_{j=1}^{k-1} n_j^{1-(\alpha_1+\cdots+\alpha_N)}\right) \tag{10}$$
$$= O\left(\frac{1}{n_k^{1-(\alpha_1+\cdots+\alpha_N)}}\sum_{j=1}^{k-2} n_j^{1-(\alpha_1+\cdots+\alpha_N)} + \frac{n_{k-1}^{1-(\alpha_1+\cdots+\alpha_N)}}{n_k^{1-(\alpha_1+\cdots+\alpha_N)}}\right)$$
$$= O\left(\frac{n_{k-1}^{1-(\alpha_1+\cdots+\alpha_N)}}{n_k^{1-(\alpha_1+\cdots+\alpha_N)}}\right)$$
$$= O\left(\left(\frac{1}{k}\right)^{1-(\alpha_1+\cdots+\alpha_N)}\right) = o(1) \quad \text{as} \quad k\to\infty.$$

Since

$$a_{i_1,\dots,i_N}^{(B)}(f_k) = 0, \quad \text{for} \quad B\subset M, B\neq M$$

and

$$a_{i_1,\dots,i_N}^{(M)}(f_k) = \begin{cases} n_k^{-\alpha_1-\cdots-\alpha_N}, & \text{for} \quad i_1=\cdots=i_N=n_k; \\ 0, & \text{otherwise,} \end{cases}$$

from (1) we have

$$\left\|\sigma_{n_k,\dots,n_k}^{-\alpha_1,\dots,-\alpha_N}(f_k)\right\|_1 \tag{11}$$
$$= \int_0^{2\pi}\cdots\int_0^{2\pi}\left|\sigma_{n_k,\dots,n_k}^{-\alpha_1,\dots,-\alpha_N}(f_k;x_1,\dots,x_N)\right| dx_1\cdots dx_N$$
$$\geq \left|\int_0^{2\pi}\cdots\int_0^{2\pi}\sigma_{n_k,\dots,n_k}^{-\alpha_1,\dots,-\alpha_N}(f_k;x_1,\dots,x_N)\prod_{i=1}^N \sin n_k x_i dx_1\cdots dx_N\right|$$
$$= \left|\frac{1}{A_{n_k}^{-\alpha_1}}\cdots\frac{1}{A_{n_k}^{-\alpha_N}}\sum_{i_1=0}^{n_k}\cdots\sum_{i_N=0}^{n_k}\prod_{j=1}^N A_{n_k-i_j}^{-\alpha_1-1}\right.$$
$$\left.\times\int_0^{2\pi}\cdots\int_0^{2\pi} S_{i_1,\dots,i_N}(f_k;x_1,\dots,x_N)\prod_{i=1}^N \sin n_k x_i dx_1\cdots dx_N\right|$$
$$= \pi^N \frac{1}{A_{n_k}^{-\alpha_1}}\cdots\frac{1}{A_{n_k}^{-\alpha_N}} a_{n_k,\dots,n_k}^{(M)}(f_k)$$
$$= \pi^N \frac{1}{A_{n_k}^{-\alpha_1}}\cdots\frac{1}{A_{n_k}^{-\alpha_N}} n_k^{-\alpha_1-\cdots-\alpha_N} \geq c(\alpha_1,\dots,\alpha_N) > 0.$$

Combining (7) – (11) we complete the proof of Theorem 1. □

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