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POINCARÉ TYPE INEQUALITIES FOR VARIABLE EXPONENTS

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ABSTRACT. We consider Poincaré type inequalities of integral form for variable exponents. We give conditions under which these inequalities do not hold as well as conditions under which they hold.

Key words and phrases: Poincaré inequality, variable exponent.

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1. INTRODUCTION AND PRELIMINARIES

One of the classical Poincaré inequalities states

$$\int_G |\varphi(x)|^p\, dx \le C(N,p,|G|) \int_G |\nabla\varphi(x)|^p\, dx, \qquad \forall \varphi \in C_0^1(G),$$

where G is a bounded open set in $\mathbb{R}^N$ ($N \ge 1$) and $p \ge 1$.

In Fu [2], this inequality with p replaced by a bounded variable exponent $p(x)$ is given as a lemma. Namely, let $p(x)$ be a bounded measurable function on G such that $p(x) \ge 1$ for all $x \in G$. We shall say that the Poincaré inequality (PI, for short) holds on G for $p(\cdot)$ if there exists a constant $C > 0$ such that

(PI) $$\int_G |\varphi(x)|^{p(x)}\, dx \le C \int_G |\nabla\varphi(x)|^{p(x)}\, dx$$

for all $\varphi \in C_0^1(G)$. Fu's lemma asserts that (PI) always holds. However, as was already remarked in [1, pp. 444-445, Example] in the one dimensional case, this is false. We shall give some types of $p(\cdot)$ for which (PI) does not hold.

We remark here that the following norm-form of the Poincaré inequality holds for variable exponents (cf. [3, Theorem 3.10]):

$$\|\varphi\|_{L^{p(\cdot)}(G)} \le C \||\nabla\varphi|\|_{L^{p(\cdot)}(G)}$$

for all $\varphi \in C_0^1(G)$ provided that $p(x)$ is continuous on $\overline{G}$, where $\|\cdot\|_{L^{p(\cdot)}(G)}$ denotes the (Luxemburg) norm in the variable exponent Lebesgue space $L^{p(\cdot)}(G)$ (see [3] for definition). Thus, our results show that we must distiguish between norm-form and integral-form when we consider the Poincaré inequalities for variable exponents.

We also consider a slightly weaker form: we shall say that the weak Poincaré inequality (wPI, for short) holds on G for $p(\cdot)$ if there exists a constant $C > 0$ such that

$$\text{(wPI)} \qquad \int_G |\varphi(x)|^{p(x)}\,dx \le C\left(1 + \int_G |\nabla\varphi(x)|^{p(x)}\,dx\right)$$

for all $\varphi \in C_0^1(G)$. We shall see that this weak Poincaré inequality does not always hold either.

The main purpose of this paper is to give some sufficient conditions on $p(\cdot)$ under which (PI) or (wPI) holds, and our results show that (PI) holds for a fairly large class of non-constant $p(x)$ and (wPI) holds for $p(x)$ in a larger class.

2. INVALIDITY OF POINCARÉ TYPE INEQUALITIES

For a measurable function $p(x)$ on G and $E \subset G$, let

$$p_E^+ = \operatorname*{ess\,sup}_{x\in E} p(x) \quad \text{and} \quad p_E^- = \operatorname*{ess\,inf}_{x\in E} p(x).$$

Lemma 2.1. *Let $p(x)$ and $q(x)$ be measurable functions on G such that $0 < p_G^- \le p_G^+ < \infty$ and $0 < q_G^- \le q_G^+ < \infty$.*

(1) *If there exist a compact set K and open sets G_1, G_2 such that $K \subset G_1 \Subset G_2 \subset G$, $|K| > 0$ and $q_K^- > p^+_{G_2\setminus\overline{G_1}}$, then there exists a sequence $\{\varphi_n\}$ in $C_0^1(G)$ such that*

$$\int_G |\varphi_n(x)|^{q(x)}\,dx \to \infty \qquad \textit{and} \qquad \frac{\int_G |\nabla\varphi_n(x)|^{p(x)}\,dx}{\int_G |\varphi_n(x)|^{q(x)}\,dx} \to 0$$

as $n \to \infty$.

(2) *If there exist a compact set K and open sets G_1, G_2 such that $K \subset G_1 \Subset G_2 \subset G$, $|K| > 0$ and $q_K^+ < p^-_{G_2\setminus\overline{G_1}}$, then there exists a sequence $\{\psi_n\}$ in $C_0^1(G) \setminus \{0\}$ such that*

$$\int_G |\nabla\psi_n(x)|^{p(x)}\,dx \to 0 \qquad \textit{and} \qquad \frac{\int_G |\nabla\psi_n(x)|^{p(x)}\,dx}{\int_G |\psi_n(x)|^{q(x)}\,dx} \to 0$$

as $n \to \infty$.

Proof. Choose $\varphi_1 \in C_0^1(G)$ such that $\varphi_1 = 1$ on $\overline{G_1}$ and Spt $\varphi_1 \subset G_2$.
(1) Suppose $q_K^- > p^+_{G_2\setminus\overline{G_1}}$. For simplicity, write $q_1 = q_K^-$ and $p_2 = p^+_{G_2\setminus\overline{G_1}}$. Let $\varphi_n = n\varphi_1,\ n = 1, 2, \ldots$. Then

$$\int_G |\nabla\varphi_n|^{p(x)}\,dx = \int_{G_2\setminus\overline{G_1}} n^{p(x)}|\nabla\varphi_1|^{p(x)}\,dx \le n^{p_2}\int_G |\nabla\varphi_1|^{p(x)}\,dx$$

and

$$\int_G |\varphi_n|^{q(x)}\,dx \ge \int_K n^{q(x)}\,dx \ge n^{q_1}|K|.$$

These inequalities show that the sequence $\{\varphi_n\}$ has the required properties.
(2) Suppose $q_K^+ < p^-_{G_2\setminus\overline{G_1}}$. Write $q_2 = q_K^+$ and $p_1 = p^-_{G_2\setminus\overline{G_1}}$. Let $\psi_n = (1/n)\varphi_1,\ n = 1, 2, \ldots$. Then

$$\int_G |\nabla\psi_n|^{p(x)}\,dx = \int_{G_2\setminus\overline{G_1}} n^{-p(x)}|\nabla\varphi_1|^{p(x)}\,dx \le n^{-p_1}\int_G |\nabla\varphi_1|^{p(x)}\,dx$$

and

$$\int_G |\psi_n|^{q(x)}\,dx \ge \int_K n^{-q(x)}\,dx \ge n^{-q_2}|K|.$$

Thus the sequence $\{\psi_n\}$ has the required properties. □

By taking $p(x) = q(x)$ in this lemma, we readily obtain

Proposition 2.2.

(1) *If there exist a compact set* K *and open sets* G_1, G_2 *such that* $K \subset G_1 \Subset G_2 \subset G$, $|K| > 0$ *and* $p_K^- > p^+_{G_2 \setminus \overline{G_1}}$, *then (wPI) does not hold for* $p(\cdot)$ *on* G.

(2) *If there exist a compact set* K *and open sets* G_1, G_2 *such that* $K \subset G_1 \Subset G_2 \subset G$, $|K| > 0$ *and* $p_K^+ < p^-_{G_2 \setminus \overline{G_1}}$, *then (PI) does not hold for* $p(\cdot)$ *on* G.

3. VALIDITY OF POINCARÉ TYPE INEQUALITIES IN ONE-DIMENSIONAL CASE

We shall say that $f(t)$ on (t_0, t_1) is of type (L) if there is $\tau \in (t_0, t_1)$ such that $f(t)$ is non-increasing on (t_0, τ) and non-decreasing on (τ, t_1).

Proposition 3.1. *Let* $N = 1$ *and* $G = (a, b)$.

(1) *If* $p(t)$ *is monotone (i.e., non-decreasing or non-increasing) or of type (L) on* G, *then*

$$\int_a^b |f(t)|^{p(t)}\, dx \le \frac{|G|}{2} + \max(|G|, |G|^{p^+}) \int_a^b |f'(t)|^{p(t)}\, dt$$

for $f \in C_0^1(G)$, *where* $|G| = b - a$ *and* $p^+ = p_G^+$.

(2) *If* $p(t)$ *is monotone on* G, *then*

$$\int_a^b |f(t)|^{p(t)}\, dx \le C \int_a^b |f'(t)|^{p(t)}\, dt$$

for $f \in C_0^1(G)$, *where the constant* C *depends only on* p^+ *and* $|G|$.

Proof. (I) First, we consider the case $G = (0, 1)$. Let $f \in C_0^1(G)$.

(I-1) Suppose $p(t)$ is non-increasing on $(0, \tau)$, $0 < \tau \le 1$. Then, for $0 < t < \tau$,

$$\begin{aligned} |f(t)|^{p(t)} &\le \left(\int_0^t |f'(s)|\, ds \right)^{p(t)} \le \int_0^t |f'(t)|^{p(t)}\, ds \\ &\le \int_0^t \left(1 + |f'(s)|^{p(s)}\right)\, ds \le t + \int_0^1 |f'(s)|^{p(s)}\, ds. \end{aligned}$$

Hence

$$\int_0^\tau |f(t)|^{p(t)}\, dt \le \frac{\tau^2}{2} + \tau \int_0^1 |f'(s)|^{p(s)}\, ds.$$

Similarly, if $p(t)$ is non-decreasing on $(\tau, 1)$, $0 \le \tau < 1$, then

$$\int_\tau^1 |f(t)|^{p(t)}\, dt \le \frac{(1-\tau)^2}{2} + (1-\tau) \int_0^1 |f'(s)|^{p(s)}\, ds.$$

Hence, if $p(t)$ is monotone or of type (L) on G, then

$$\int_0^1 |f(t)|^{p(t)}\, dt \le \frac{1}{2} + \int_0^1 |f'(t)|^{p(t)}\, dt. \tag{3.1}$$

(I-2) The case $\|f'\|_1 := \int_0^1 |f'(t)|\, dt \ge 1$.

In this case,

$$\begin{aligned} 1 &\le \int_0^1 |f'(t)|\, dt = \frac{1}{2} \int_0^1 |2f'(t)|\, dt \\ &\le \frac{1}{2} + \frac{1}{2} \int_0^1 |2f'(t)|^{p(t)}\, dt \le \frac{1}{2} + 2^{p^+ - 1} \int_0^1 |f'(t)|^{p(t)} dt, \end{aligned}$$

so that

$$\frac{1}{2} \le 2^{p^+-1} \int_0^1 |f'(t)|^{p(t)} dt.$$

Hence, by (3.1), we have

$$\int_0^1 |f(t)|^{p(t)}\, dt \le (1 + 2^{p^+-1}) \int_0^1 |f'(t)|^{p(t)} dt \tag{3.2}$$

in case $\|f'\|_1 \ge 1$.

(I-3) The case $p(t)$ is monotone and $\|f'\|_1 < 1$.

We may assume that $p(t)$ is non-decreasing. Set

$$E_1 = \{t \in (0,1);\, |f'(t)| \le 1\}, \quad E_2 = \{t \in (0,1);\, |f'(t)| > 1\},$$

$$g_1(t) = \int_{(0,t)\cap E_1} |f'(s)|\, ds \quad \text{and} \quad g_2(t) = \int_{(0,t)\cap E_2} |f'(s)|\, ds.$$

Then for $0 < t < 1$

$$\begin{aligned} |f(t)|^{p(t)} &\le \left(\int_0^t |f'(s)|\, ds \right)^{p(t)} = \big(g_1(t) + g_2(t)\big)^{p(t)} \\ &\le 2^{p^+-1}\big(g_1(t)^{p(t)} + g_2(t)^{p(t)}\big). \end{aligned}$$

Since $p(s) \le p(t)$ for $0 < s < t$ and $|f(s)| \le 1$ for $s \in E_1$,

$$g_1(t)^{p(t)} \le \int_{(0,t)\cap E_1} |f'(s)|^{p(t)}\, ds \le \int_{(0,t)\cap E_1} |f'(s)|^{p(s)}\, ds \le \int_{E_1} |f'(s)|^{p(s)}\, ds.$$

On the other hand, since $g_2(t) \le \|f'\|_1 < 1$ and $|f'(s)| > 1$ for $s \in E_2$,

$$g_2(t)^{p(t)} \le g_2(t) = \int_{(0,t)\cap E_2} |f'(s)|\, ds \le \int_{E_2} |f'(s)|^{p(s)}\, ds.$$

Hence

$$|f(t)|^{p(t)} \le 2^{p^+-1} \int_0^1 |f'(s)|^{p(s)}\, ds$$

for all $0 < t < 1$, and hence

$$\int_0^1 |f(t)|^{p(t)}\, dt \le 2^{p^+-1} \int_0^1 |f'(s)|^{p(s)}\, ds$$

in case $\|f'\|_1 < 1$.

(I-4) Combining (I-2) and (I-3), we have (3.2) for all $f \in C_0^1(G)$ if $p(t)$ is monotone.

(II) The general case: Let $G = (a,b)$ and $f \in C_0^1(G)$. Let

$$g(t) = f(a + t(b-a)) \quad \text{and} \quad q(t) = p(a + t(b-a))$$

for $0 < t < 1$. Then

$$\int_a^b |f(s)|^{p(s)}\, ds = (b-a) \int_0^1 |g(t)|^{q(t)}\, dt$$

and

$$\begin{aligned} \int_0^1 |g'(t)|\, dt &= \frac{1}{b-a} \int_a^b |(b-a) f'(s)|^{p(s)}\, ds \\ &\le \max(1, (b-a)^{p^+-1}) \int_a^b |f'(s)|^{p(s)}\, ds. \end{aligned}$$

Hence, applying (3.1) and (3.2) to $g(t)$ and $q(t)$, we obtain the required inequalities of the proposition. (In fact, we can take $C = (1+2^{p^+-1})\max(|G|, |G|^{p^+})$.) $\square$

4. VALIDITY OF POINCARÉ TYPE INEQUALITIES IN HIGHER-DIMENSIONAL CASE

Theorem 4.1. *Let $N \geq 2$ and $G \subset G' \times (a, b)$ with a bounded open set $G' \subset \mathbb{R}^{N-1}$ and set $G_{x'} = \{t \in (a, b) : (x', t) \in G\}$ for $x' \in G'$.*

(1) *If $t \mapsto p(x', t)$ is monotone or of type (L) on each component of $G_{x'}$ for a.e. $x' \in G'$ (with respect to the $(N-1)$-dimensional Lebesgue measure), then (wPI) holds for $p(\cdot)$ on G.*

(2) *If $t \mapsto p(x', t)$ is monotone on each component of $G_{x'}$ for a.e. $x' \in G'$ (with respect to the $(N-1)$-dimensional Lebesgue measure), then (PI) holds for $p(\cdot)$ on G.*

Proof. Fix $x' \in G'$ for a moment and let I_j be the components of $G_{x'}$. If $\varphi \in C_0^1(G)$, then $t \mapsto \varphi(x', t)$ belongs to $C_0^1(I_j)$ for each j. Thus, by Proposition 3.1, if $t \mapsto p(x', t)$ is monotone or of type (L) on each I_j, then

$$\int_{I_j} |\varphi(x', t)|^{p(x',t)}\, dt \leq |I_j| + \max(1, |I_j|^{p^+}) \int_{I_j} |\nabla\varphi(x', t)|^{p(x',t)}\, dt,$$

so that

$$\int_{G_{x'}} |\varphi(x', t)|^{p(x',t)}\, dt \leq |G_{x'}| + \max(1, (b-a)^{p^+}) \int_{G_{x'}} |\nabla\varphi(x', t)|^{p(x',t)}\, dt;$$

and if $t \mapsto p(x', t)$ is monotone on each I_j then

$$\int_{I_j} |\varphi(x', t)|^{p(x',t)}\, dt \leq C(p^+, I_j) \int_{I_j} |\nabla\varphi(x', t)|^{p(x',t)}\, dt,$$

so that

$$\int_{G_{x'}} |\varphi(x', t)|^{p(x',t)}\, dt \leq C(p^+, b-a) \int_{G_{x'}} |\nabla\varphi(x', t)|^{p(x',t)}\, dt.$$

Hence, integrating over G' with respect to x', we obtain the assertion of the theorem. $\square$

The following proposition is easily seen by a change of variables:

Proposition 4.2. *(PI) and (wPI) are diffeomorphically invariant. More precisely, let G_1 and G_2 be bounded open sets and $\Phi(x) = (\phi_1(x), \dots, \phi_N(x))$ be a (C^1-)diffeomorphism of G_1 onto G_2. Suppose $|\nabla\phi_j|$, $j = 1, \dots, N$ and $|\nabla\psi_j|$, $j = 1, \dots, N$ are all bounded, where $\Phi^{-1}(y) = (\psi_1(y), \dots, \psi_N(y))$, and suppose $0 < \alpha \leq J_\Phi(x) \leq \beta$ for all $x \in G_1$. Let $p_1(x) = p_2(\Phi(x))$ for $x \in G_1$. Then, (PI) (resp. (wPI)) holds for $p_1(\cdot)$ on G_1 if and only if it holds for $p_2(\cdot)$ on G_2.*

Combining Theorem 4.1 with this Proposition, we can find a fairly large class of $p(x)$ for which (PI) (as well as (wPI)) holds.

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