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A GEOMETRIC INEQUALITY INVOLVING A MOBILE POINT IN THE PLACE OF THE TRIANGLE

XIAO-GUANG CHU AND YU-DONG WU

SUZHOU HENGTIAN TRADING CO. LTD.,
SUZHOU 215128, JIANGSU
PEOPLE'S REPUBLIC OF CHINA
srr345@163.com

DEPARTMENT OF MATHEMATICS
ZHEJIANG XINCHANG HIGH SCHOOL
SHAOXING 312500, ZHEJIANG
PEOPLE'S REPUBLIC OF CHINA
yudong.wu@yahoo.com.cn

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Dedicated to Professor Lu Yang on the occasion of his 73rd birthday.

ABSTRACT. By using **Bottema's inequality** and several identities in triangles, we prove a weighted inequality concerning the distances between a mobile point P and three vertexes A, B, C of $\triangle ABC$. As an application, a conjecture with regard to Fermat's sum $PA+PB+PC$ is proved.

Key words and phrases: Bottema's inequality, Euler's inequality, Fermat's sum, triangle, mobile point.

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1. INTRODUCTION AND MAIN RESULTS

For $\triangle ABC$, let a, b, c denote the side-lengths, A, B, C the angles, Δ the area, p the semi-perimeter, R the circumradius and r the inradius, respectively. In addition, supposing that P is a mobile point in the plane containing $\triangle ABC$, let PA, PB, PC denote the distances between P and A, B, C, respectively. We will customarily use the cyclic symbol, that is: $\sum f(a) = f(a) + f(b) + f(c)$, $\sum f(a,b) = f(a,b) + f(b,c) + f(c,a)$, $\prod f(a) = f(a)f(b)f(c)$, etc.

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The following inequality can be easily proved by making use of **Bottema's inequality**:

$$(PB+PC)\cos\frac{A}{2}+(PC+PA)\cos\frac{B}{2}+(PA+PB)\cos\frac{C}{2}\geq p\cdot\frac{p^2+2Rr+r^2}{4R^2}. \tag{1.1}$$

Here we choose to omit the details. From inequality (1.1) and the following known inequality (1.2) and identity (1.3) (see [3, 4, 6]):

$$PA\cos\frac{A}{2}+PB\cos\frac{B}{2}+PC\cos\frac{C}{2}\geq p, \tag{1.2}$$

and

$$q_1=\cos^2\frac{B-C}{2}+\cos^2\frac{C-A}{2}+\cos^2\frac{A-B}{2}=\frac{p^2+4R^2+2Rr+r^2}{4R^2}, \tag{1.3}$$

we easily get

$$PA+PB+PC\geq p\cdot\frac{\cos^2\frac{B-C}{2}+\cos^2\frac{C-A}{2}+\cos^2\frac{A-B}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}. \tag{1.4}$$

Considering the refinement of inequality (1.4), Chu [2] posed a conjecture as follows.

Conjecture 1.1. *For any $\triangle ABC$,*

$$PA+PB+PC\geq p\cdot\frac{\cos\frac{B-C}{2}+\cos\frac{C-A}{2}+\cos\frac{A-B}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}. \tag{1.5}$$

The main object of this paper is to prove Conjecture 1.1, which is easily seen to follow from the following stronger result.

Theorem 1.2. *In $\triangle ABC$, we have*

$$\begin{aligned}(PB+PC)\cos\frac{A}{2}&+(PC+PA)\cos\frac{B}{2}+(PA+PB)\cos\frac{C}{2}\\&\geq p\cdot\left[\cos\frac{B-C}{2}+\cos\frac{C-A}{2}+\cos\frac{A-B}{2}-1\right].\end{aligned} \tag{1.6}$$

2. Preliminary Results

In order to prove our main result, we shall require the following four lemmas.

Lemma 2.1. *In $\triangle ABC$, we have that*

$$q_2=\cos\frac{B-C}{2}\cdot\cos\frac{C-A}{2}\cdot\cos\frac{A-B}{2}=\frac{p^2+2Rr+r^2}{8R^2}, \tag{2.1}$$

$$\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}=\frac{p}{4R}, \tag{2.2}$$

$$\begin{aligned}q_3&=\sum\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{B-C}{2}\left(b^2+c^2-a^2\right)\\&=\frac{p^4+2Rrp^2-r\left(2R+r\right)\left(4R+r\right)^2}{4R^2},\end{aligned} \tag{2.3}$$

$$(2.4)\qquad \begin{aligned} q_4 &= \frac{1}{2}\sum\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\left(b^2+c^2-a^2\right)\\ &= \frac{(2R+3r)\,p^2-r\,(4R+r)^2}{2R}, \end{aligned}$$

and

$$(2.5)\qquad \begin{aligned} Q &= \sum\left(\cos\frac{B-C}{2}-\cos\frac{C-A}{2}\cos\frac{A-B}{2}\right)\\ &= q_1-3q_2-\frac{p\Delta^4\prod(b-c)^2}{a^2b^2c^2\prod(X+x)}, \end{aligned}$$

where

$$(2.6)\qquad \begin{aligned} &X = a\sqrt{bc\,(p-b)\,(p-c)}, \qquad Y = b\sqrt{ca\,(p-c)\,(p-a)},\\ &Z = c\sqrt{ab\,(p-a)\,(p-b)}, \end{aligned}$$

and

$$(2.7)\qquad \begin{aligned} &x = (b+c)\,(p-b)\,(p-c)\,, \qquad y = (c+a)\,(p-c)\,(p-a)\,,\\ &z = (a+b)\,(p-a)\,(p-b)\,. \end{aligned}$$

Proof. The proofs of identities (2.1) and (2.2) were given in [6]. Now, we present the proofs of identities (2.3) – (2.5). By utilizing the formulas

$$\cos\frac{A}{2}=\sqrt{\frac{p\,(p-a)}{bc}}, \qquad \sin\frac{A}{2}=\sqrt{\frac{(p-b)\,(p-c)}{bc}},$$
$$\cos\frac{B-C}{2}=\frac{b+c}{a}\sqrt{\frac{(p-b)\,(p-c)}{bc}},$$

and (see [5, pp.52])

$$\prod a = 4Rrp, \qquad \sum a = 2p \qquad \text{and} \qquad \sum bc = p^2+4Rr+r^2,$$

we get that

$$\begin{aligned} q_3 &= p\sum\frac{(b+c)\,(p-b)\,(p-c)}{a^2bc}\left(b^2+c^2-a^2\right)\\ &= \frac{p}{4a^2b^2c^2}\sum bc\,(b+c)\,(c+a-b)\,(a+b-c)\left(b^2+c^2-a^2\right)\\ &= \frac{p}{4a^2b^2c^2}[6(ab+bc+ca)^2(a+b+c)^3-8(ab+bc+ca)^3(a+b+c)\\ &\qquad -(ab+bc+ca)(a+b+c)^5-2abc(ab+bc+ca)(a+b+c)^2\\ &\qquad +8abc(ab+bc+ca)^2-abc(a+b+c)^4-4(a+b+c)a^2b^2c^2]\\ &= \frac{p^4+2Rrp^2-r\,(2R+r)\,(4R+r)^2}{4R^2} \end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\sum\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\left(b^2+c^2-a^2\right)\\
&\quad=\sum a^2\cos^2\frac{A}{2}\\
&\quad=\frac{p}{abc}\left(p\sum a^3-\sum a^4\right)\\
&\quad=\frac{p}{abc}\left[\frac{5}{2}(ab+bc+ca)(a+b+c)^2-2(ab+bc+ca)^2\right.\\
&\qquad\qquad\left.-\frac{1}{2}(a+b+c)^4-\frac{5}{2}(a+b+c)abc\right]\\
&\quad=\frac{(2R+3r)\,p^2-r\,(4R+r)^2}{2R}.
\end{aligned}$$

Thus, identities (2.3) and (2.4) hold true.

With (1.3), (2.1) and the formulas of half-angles, we obtain that

$$\begin{aligned}
q_1-3q_2&=\frac{-p^2+8R^2-2Rr-r^2}{8R^2}\\
&=\frac{1}{a^2b^2c^2}\sum x\,[bc\,(b+c)-(c+a)\,(a+b)\,(s-a)],
\end{aligned}$$

and

$$\begin{aligned}
Q&=\sum\left(\cos\frac{B-C}{2}-\cos\frac{C-A}{2}\cos\frac{A-B}{2}\right)\\
&=\frac{\sum X\,[bc\,(b+c)-(c+a)\,(a+b)\,(s-a)]}{a^2b^2c^2}.
\end{aligned}$$

It is easy to see that

$$X-x=\frac{\Delta^2\,(b-c)^2}{X+x},\quad Y-y=\frac{\Delta^2\,(c-a)^2}{Y+y},\quad\text{and}\quad Z-z=\frac{\Delta^2\,(a-b)^2}{Z+z}.$$

Then

$$\begin{aligned}
&a^2b^2c^2[Q-(q_1-3q_2)]\\
&\quad=\sum[bc(b+c)-(c+a)(a+b)(p-a)](X-x)\\
&\quad=\sum[bc(b+c)-(c+a)(a+b)(p-a)]\frac{\Delta^2(b-c)^2}{X+x}\\
&\quad=\sum p\Delta^2\frac{(a-b)(a-c)(b-c)^2}{(X+x)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
Q-(q_1-3q_2)&=\sum p\Delta^2\frac{(a-b)(a-c)(b-c)^2}{a^2b^2c^2(X+x)}\\
&=\frac{p\Delta^2(a-b)(a-c)(b-c)}{a^2b^2c^2}\sum\frac{b-c}{X+x},
\end{aligned}$$

where

$$
\begin{aligned}
&\frac{b-c}{X+x}+\frac{c-a}{Y+y}+\frac{a-b}{Z+z}\\
&=\frac{(b-c)(Y-X+y-x)}{(X+x)(Y+y)}+\frac{(a-b)(Y-Z+y-z)}{(Z+z)(Y+y)}\\
&=\frac{p(p-c)(b-c)(b-a)}{(X+x)(Y+y)}+\frac{pabc(p-c)(b-c)(b-a)}{(X+x)(Y+y)(X+Y)}\\
&\qquad+\frac{p(p-a)(a-b)(b-c)}{(Z+z)(Y+y)}+\frac{pabc(p-a)(a-b)(b-c)}{(Z+z)(Y+y)(Z+Y)}\\
&=\frac{p(b-c)(a-b)}{\prod(X+x)}\left[(p-a)(X+x)-(p-c)(Z+z)\right]\\
&\qquad+\frac{pabc(b-c)(a-b)}{(X+Y)(Y+Z)\prod(X+x)}\\
&\qquad\times\left[(p-a)(X+x)(X+Y)-(p-c)(Z+z)(Y+Z)\right]\\
&=\frac{-\Delta^2(b-c)(a-b)(a-c)}{\prod(X+x)}\\
&\qquad+\frac{p(b-c)(a-b)(a-c)}{(Z+X)\prod(X+x)}\left[abc\prod(p-a)-ca(p-b)Y\right]\\
&\qquad+\frac{pabc(b-c)(a-b)(a-c)}{(X+Y)(Z+Y)\prod(X+x)}\left[abc\prod(p-a)-Y\prod(p-a)\right.\\
&\qquad+\frac{abc(Y-abc)\prod(p-a)}{Z+X}\\
&\qquad\left.+\frac{abc(pb+ca)(p-b)\prod(p-a)-ca(p-b)Y\prod(p-a)}{Z+X}\right]\\
&=\frac{-\Delta^2(b-c)(a-b)(a-c)}{\prod(X+x)}+\frac{pabc(b-c)(a-b)(a-c)}{(X+Y)(Z+Y)\prod(X+x)}\\
&\qquad\cdot\left\{\left[\prod(p-a)-(p-b)\sqrt{ca(p-c)(p-a)}\right](X+Y)(Y+Z)\right.\\
&\qquad+abc\prod(p-a)\left[Y-abc+pb(p-b)+ca(p-b)-(p-b)\sqrt{ca(p-c)(p-a)}\right]\\
&\qquad\left.+(Z+X)(abc-Y)\prod(p-a)\right\}\\
&=\frac{-\Delta^2(b-c)(a-b)(a-c)}{\prod(X+x)},
\end{aligned}
$$

which implies the assertion (2.5). □

Lemma 2.2. *For any $\triangle ABC$,*

$$\sqrt{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)}\geq\frac{p}{2R}. \tag{2.8}$$

Proof. From **Euler's inequality** $R \geq 2r$, $abc = 4Rrp$, $a+b+c = 2p$ and the law of sines, we obtain that

$$2R^2p \geq 4Rrp \Longleftrightarrow R^2(a+b+c) \geq abc \tag{2.9}$$
$$\Longleftrightarrow \sin A + \sin B + \sin C \geq 4 \sin A \sin B \sin C.$$

Taking

$$A \to \frac{\pi - A}{2}, \qquad B \to \frac{\pi - B}{2}, \qquad \text{and} \qquad C \to \frac{\pi - C}{2},$$

we easily get

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \geq 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}. \tag{2.10}$$

Inequality (2.8) follows immediately in view of (2.10) and (2.2). □

Lemma 2.3. *In $\triangle ABC$, we have*

$$\sum \cos\frac{B}{2}\cos\frac{C}{2}\left(b^2+c^2-a^2\right) \geq \frac{p^4 + 2Rrp^2 - r\left(2R+r\right)\left(4R+r\right)^2}{4R^2}. \tag{2.11}$$

Proof. By employing (2.3) and the formulas of half-angles, inequality (2.11) is equivalent to

$$\sum \cos\frac{B}{2}\cos\frac{C}{2}\left(b^2+c^2-a^2\right) \geq \sum \cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{B-C}{2}\left(b^2+c^2-a^2\right), \tag{2.12}$$

or

$$\sum \frac{\left(b^2+c^2-a^2\right)}{a^2bc}\left[a\sqrt{bc\left(s-b\right)\left(s-c\right)} - \left(b+c\right)\left(s-b\right)\left(s-c\right)\right] \geq 0, \tag{2.13}$$

that is

$$\sum \frac{\Delta^2}{abc} \cdot \frac{\left(b^2+c^2-a^2\right)}{a\left(X+x\right)}\left(b-c\right)^2 \geq 0, \tag{2.14}$$

where X, Y, Z and x, y, z are given, just as in the proof of Lemma 2.1, by (2.6) and (2.7), respectively.

Without loss of generality, we can assume that $a \geq b \geq c$ to obtain

$$a\left(X+x\right) \geq b\left(Y+y\right) \geq c\left(Z+z\right),$$

and

$$\left(a-c\right)^2 \geq \left(b-c\right)^2,$$

and thus

$$\frac{\left(b-c\right)^2}{a\left(X+x\right)} \leq \frac{\left(c-a\right)^2}{b\left(Y+y\right)}.$$

Hence, in order to prove inequality (2.14), we only need to prove that

$$\frac{\Delta^2}{abc}\left[\frac{\left(b^2+c^2-a^2\right)}{a(X+x)}(b-c)^2 + \frac{\left(c^2+a^2-b^2\right)}{b(Y+y)}(c-a)^2\right] \geq 0. \tag{2.15}$$

We readily arrive at the following result for $a \geq b \geq c$,

$$\begin{aligned}\frac{\Delta^2}{abc}&\left[\frac{(b^2+c^2-a^2)}{a(X+x)}(b-c)^2+\frac{(c^2+a^2-b^2)}{b(Y+y)}(c-a)^2\right]\\&\geq \frac{\Delta^2}{abc}\left(b^2+c^2-a^2+c^2+a^2-b^2\right)\frac{(c-a)^2}{b\,(Y+y)}\\&=2c^2\cdot\frac{\Delta^2}{abc}\cdot\frac{(c-a)^2}{b\,(Y+y)}\geq 0.\end{aligned}$$

This shows that the inequality (2.15) or (2.11) holds true. The proof of Lemma 2.3 is thus complete. □

Lemma 2.4 (Bottema's inequality, see [1, pp. 118, Theorem 12.56])**.** *Let* Δ' *denote the area of* $\triangle A'B'C'$*, and* a', b', c' *the side-lengths of* $\triangle A'B'C'$*, respectively. Then*

$$\begin{aligned}(2.16)\quad &(a'PA+b'PB+c'PC)^2\\&\geq \frac{1}{2}[a'^2(b^2+c^2-a^2)+b'^2(c^2+a^2-b^2)+c'^2(a^2+b^2-c^2)]+8\Delta\Delta'.\end{aligned}$$

3. THE PROOF OF THEOREM 1.2

Proof. It is easy to show that

$$a'=\cos\frac{B}{2}+\cos\frac{C}{2},\qquad b'=\cos\frac{C}{2}+\cos\frac{A}{2},\qquad\text{and}$$
$$c'=\cos\frac{A}{2}+\cos\frac{B}{2}$$

are three side-lengths of a certain triangle. By using **Bottema's inequality** (2.16), in order to prove inequality (1.6), we only need to prove that

$$\begin{aligned}8\Delta\sqrt{\prod\cos\frac{A}{2}\sum\cos\frac{A}{2}}+\frac{1}{2}\sum\left(\cos\frac{B}{2}+\cos\frac{C}{2}\right)^2\left(b^2+c^2-a^2\right)\\ \geq p^2\left[\sum\cos\frac{B-C}{2}-1\right]^2\end{aligned}$$

or

$$\begin{aligned}(3.1)\qquad &8\Delta\sqrt{\prod\cos\frac{A}{2}\sum\cos\frac{A}{2}}+q_4\\&+\sum\cos\frac{B}{2}\cos\frac{C}{2}\left(b^2+c^2-a^2\right)+2p^2Q\geq p^2\left(q_1+1\right).\end{aligned}$$

With identities (1.3), (2.4), (2.5), together with Lemma 2.2 and Lemma 2.3, in order to prove inequality (3.1), we only need to prove that

$$8\Delta \cdot \frac{p}{2R} + \frac{(2R+3r)\,p^2 - r\,(4R+r)^2}{2R} + \frac{p^4 + 2Rrp^2 - r\,(2R+r)\,(4R+r)^2}{4R^2} + 2p^2 \left[\frac{-p^2+8R^2-2Rr-r^2}{8R^2} - \frac{p\Delta^4 \prod (b-c)^2}{a^2b^2c^2 \prod (X+x)}\right] \geq p^2 \left(\frac{p^2+4R^2+2Rr+r^2}{4R^2}+1\right)$$

or

$$\frac{-p^4 + (4R^2+20Rr-2r^2)\,p^2 - r\,(4R+r)^3}{4R^2} \geq \frac{2p^3\Delta^4 \prod (b-c)^2}{a^2b^2c^2 \prod (X+x)}. \tag{3.2}$$

From the known identities (see [5])

$$\Delta = rp \text{ and}$$
$$(b-c)^2(c-a)^2(a-b)^2 = 4r^2[-p^4 + (4R^2+20Rr-2r^2)p^2 - r(4R+r)^3],$$

inequality (3.2) is equivalent to

$$\prod (X+x) \geq 2r^4p^5. \tag{3.3}$$

For $X \geq x$, and with the following two known identities (see [5, pp.53])

$$\prod (b+c) = 2p(p^2+2Rr+r^2), \qquad \prod (p-a) = r^2p,$$

we obtain

$$\prod (X+x) \geq 8\prod x = 8\prod (b+c)\prod (p-a)^2 = 16r^4p^3(p^2+2Rr+r^2) > 16r^4p^5 > 2r^4p^5.$$

Therefore, inequality (3.3) holds. This completes the proof of Theorem 1.2. □

4. Remarks

Remark 1. From inequalities (1.2) and (1.6), it is easy to see that inequality (1.5) holds.

Remark 2. In view of

$$\sum \cos\frac{B-C}{2} \geq \sum \cos^2\frac{B-C}{2} = \frac{p^2+4R^2+2Rr+r^2}{4R^2} \iff \sum \cos\frac{B-C}{2} - 1 \geq \frac{p^2+2Rr+r^2}{4R^2},$$

it follows that inequality (1.6) is a refinement of inequality (1.1).

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