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ON STARLIKENESS AND CONVEXITY OF ANALYTIC FUNCTIONS SATISFYING A DIFFERENTIAL INEQUALITY

SUKHWINDER SINGH, SUSHMA GUPTA, AND SUKHJIT SINGH

DEPARTMENT OF APPLIED SCIENCES
BABA BANDA SINGH BAHADUR ENGINEERING COLLEGE
FATEHGARH SAHIB -140407 (PUNJAB)
INDIA.
ss_billing@yahoo.co.in

DEPARTMENT OF MATHEMATICS
SANT LONGOWAL INSTITUTE OF ENGINEERING & TECHNOLOGY
LONGOWAL-148106 (PUNJAB)
INDIA.
sushmagupta1@yahoo.com

sukhjit_d@yahoo.com

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ABSTRACT. In the present paper, the authors investigate a differential inequality defined by multiplier transformation in the open unit disk $E = \{z : |z| < 1\}$. As consequences, sufficient conditions for starlikeness and convexity of analytic functions are obtained.

Key words and phrases: Multivalent function, Starlike function, Convex function, Multiplier transformation.

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1. INTRODUCTION

Let $\mathcal{A}_p$ denote the class of functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, $p \in \mathbb{N} = \{1, 2, \dots\}$, which are analytic in the open unit disc $E = \{z : |z| < 1\}$. We write $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{A}_p$ is said to be p-valent starlike of order α $(0 \le \alpha < p)$ in E if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in E.$$

We denote by $S_p^*(\alpha)$, the class of all such functions. A function $f \in \mathcal{A}_p$ is said to be p-valent convex of order α $(0 \le \alpha < p)$ in E if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in E.$$

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Let $K_p(\alpha)$ denote the class of all those functions $f \in \mathcal{A}_p$ which are multivalently convex of order α in E. Note that $S_1^*(\alpha)$ and $K_1(\alpha)$ are, respectively, the usual classes of univalent starlike functions of order α and univalent convex functions of order α, $0 \le \alpha < 1$, and will be denoted here by $S^*(\alpha)$ and $K(\alpha)$, respectively. We shall use S^* and K to denote $S^*(0)$ and $K(0)$, respectively which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.

For $f \in \mathcal{A}_p$, we define the multiplier transformation $I_p(n, \lambda)$ as

$$I_p(n,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k, \quad (\lambda \ge 0, n \in \mathbb{Z}). \tag{1.1}$$

The operator $I_p(n, \lambda)$ has recently been studied by Aghalary et.al. [1]. Earlier, the operator $I_1(n, \lambda)$ was investigated by Cho and Srivastava [3] and Cho and Kim [2], whereas the operator $I_1(n, 1)$ was studied by Uralegaddi and Somanatha [11]. $I_1(n, 0)$ is the well-known Sălăgean [10] derivative operator D^n, defined as: $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}$.

A function $f \in \mathcal{A}_p$ is said to be in the class $S_n(p, \lambda, \alpha)$ for all z in E if it satisfies

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \frac{\alpha}{p}, \tag{1.2}$$

for some $\alpha\,(0 \le \alpha < p, p \in \mathbb{N})$. We note that $S_0(1, 0, \alpha)$ and $S_1(1, 0, \alpha)$ are the usual classes $S^*(\alpha)$ and $K(\alpha)$ of starlike functions of order α and convex functions of order α, respectively.

In 1989, Owa, Shen and Obradovič [8] obtained a sufficient condition for a function $f \in \mathcal{A}$ to belong to the class $S_n(1, 0, \alpha) = S_n(\alpha)$.

Recently, Li and Owa [4] studied the operator $I_1(n, 0)$.

In the present paper, we investigate the differential inequality

$$\Re\left(\frac{(1-\alpha)I_p(n+1,\lambda)f(z) + \alpha I_p(n+2,\lambda)f(z)}{(1-\beta)I_p(n,\lambda)f(z) + \beta I_p(n+1,\lambda)f(z)}\right) > M(\alpha,\beta,\gamma,\lambda,p)$$

where α and β are real numbers and $M(\alpha, \beta, \gamma, \lambda, p)$ is a certain real number given in Section 2, for starlikeness and convexity of $f \in \mathcal{A}_p$. We obtain sufficient conditions for $f \in \mathcal{A}_p$ to be a member of $S_n(p, \lambda, \gamma)$, for some $\gamma\,(0 \le \gamma < p, p \in \mathbb{N})$. Many known results for starlikeness appear as corollaries to our main result and some new results regarding convexity of analytic functions are obtained.

2. MAIN RESULT

We shall make use of the following lemma of Miller and Mocanu to prove our result.

Lemma 2.1 ([6, 7])**.** *Let Ω be a set in the complex plane $\mathbb{C}$ and let $\psi : \mathbb{C}^2 \times E \to \mathbb{C}$. For $u = u_1 + iu_2$, $v = v_1 + iv_2$, assume that ψ satisfies the condition $\psi(iu_2, v_1; z) \notin \Omega$, for all $u_2, v_1 \in \mathbb{R}$, with $v_1 \le -(1 + u_2^2)/2$ and for all $z \in E$. If the function p, $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, is analytic in E and if $\psi(p(z), zp'(z); z) \in \Omega$, then $\Re\, p(z) > 0$ in E.*

We, now, state and prove our main theorem.

Theorem 2.2. *Let $\alpha \ge 0$, $\beta \le 1$, $\lambda \ge 0$ and $0 \le \gamma < p$ be real numbers such that $\beta(1 - \frac{\gamma}{p}) < \frac{1}{2}$ and $\beta \le \alpha$. If $f \in \mathcal{A}_p$ satisfies the condition*

$$\Re\left(\frac{(1-\alpha)I_p(n+1,\lambda)f(z) + \alpha I_p(n+2,\lambda)f(z)}{(1-\beta)I_p(n,\lambda)f(z) + \beta I_p(n+1,\lambda)f(z)}\right) > M(\alpha,\beta,\gamma,\lambda,p), \tag{2.1}$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \frac{\gamma}{p}$$

i.e., $f(z) \in S_n(p,\lambda,\gamma)$ *where,*

$$M(\alpha,\beta,\gamma,\lambda,p) = \frac{\frac{(1-\alpha)\gamma}{p} + \frac{\alpha\gamma^2}{p^2} - \frac{\alpha\left(1-\frac{\gamma}{p}\right)}{2(p+\lambda)}}{1-\beta\left(1-\frac{\gamma}{p}\right)}.$$

Proof. Since $0 \leq \gamma < p$, let us write $\mu = \frac{\gamma}{p}$. Thus, we have $0 \leq \mu < 1$.

Now we define,

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = \mu + (1-\mu)r(z), \quad z \in E. \tag{2.2}$$

Therefore $r(z)$ is analytic in E and $r(0) = 1$.

Differentiating (2.2) logarithmically, we obtain

$$\frac{zI_p'(n+1,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} - \frac{zI_p'(n,\lambda)f(z)}{I_p(n,\lambda)f(z)} = \frac{(1-\mu)zr'(z)}{\mu+(1-\mu)r(z)}, \quad z \in E. \tag{2.3}$$

Using the fact that

$$zI_p'(n,\lambda)f(z) = (p+\lambda)I_p(n+1,\lambda)f(z) - \lambda I_p(n,\lambda)f(z).$$

Thus (2.3) reduces to

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} = \mu + (1-\mu)r(z) + \frac{(1-\mu)zr'(z)}{(\lambda+p)[\mu+(1-\mu)r(z)]}.$$

Now, a simple calculation yields

$$\begin{aligned}
&\frac{(1-\alpha)I_p(n+1,\lambda)f(z) + \alpha I_p(n+2,\lambda)f(z)}{(1-\beta)I_p(n,\lambda)f(z) + \beta I_p(n+1,\lambda)f(z)} \\
&= \frac{(1-\alpha) + \alpha\left(\mu + (1-\mu)r(z) + \frac{(1-\mu)zr'(z)}{(\lambda+p)[\mu+(1-\mu)r(z)]}\right)}{(1-\beta) + \beta[\mu+(1-\mu)r(z)]}[\mu+(1-\mu)r(z)] \\
&= \frac{(1-\alpha)[\mu+(1-\mu)r(z)] + \alpha\left([\mu+(1-\mu)r(z)]^2 + \frac{(1-\mu)zr'(z)}{(\lambda+p)}\right)}{(1-\beta)+\beta[\mu+(1-\mu)r(z)]} \\
&= \psi(r(z), zr'(z); z) \qquad (2.4)
\end{aligned}$$

where,

$$\psi(u,v;z) = \frac{(1-\alpha)[\mu+(1-\mu)u] + \alpha\left((\mu+(1-\mu)u)^2 + \frac{(1-\mu)v}{(\lambda+p)}\right)}{(1-\beta)+\beta[\mu+(1-\mu)u]}.$$

Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$, where u_1, u_2, v_1, v_2 are reals with $v_1 \leq -\frac{1+u_2^2}{2}$. Then, we have

$$\begin{aligned}
&\Re\, \psi(iu_2, v_1; z) \\
&= \frac{[(1-\alpha)\mu + \alpha\mu^2][1-\beta(1-\mu)]}{[1-\beta(1-\mu)]^2 + \beta^2(1-\mu)^2 u_2^2} \\
&\quad + \frac{(1-\mu)^2[(1-\alpha)\beta - \alpha(1-\beta(1-\mu)) + 2\alpha\beta\mu]u_2^2 + \frac{\alpha(1-\mu)[1-\beta(1-\mu)]v_1}{p+\lambda}}{[1-\beta(1-\mu)]^2 + \beta^2(1-\mu)^2 u_2^2} \\
&\leq \frac{\left[(1-\alpha)\mu + \alpha\mu^2 - \frac{\alpha(1-\mu)}{2(\lambda+p)}\right][1-\beta(1-\mu)]}{[1-\beta(1-\mu)]^2 + \beta^2(1-\mu)^2 u_2^2} \\
&\quad + \frac{\left[(1-\mu)^2[(1-\alpha)\beta - \alpha(1-\beta(1-\mu)) + 2\alpha\beta\mu] - \frac{\alpha(1-\mu)[1-\beta(1-\mu)]}{2(p+\lambda)}\right]u_2^2}{[1-\beta(1-\mu)]^2 + \beta^2(1-\mu)^2 u_2^2} \\
&= \frac{A + Bu_2^2}{[1-\beta(1-\mu)]^2 + \beta^2(1-\mu)^2 u_2^2} \\
&= \phi(u_2), \quad \text{say}
\end{aligned}$$

$$\leq \max \phi(u_2) \tag{2.5}$$

where,

$$A = \left[(1-\alpha)\mu + \alpha\mu^2 - \frac{\alpha(1-\mu)}{2(\lambda+p)}\right][1-\beta(1-\mu)]$$

and

$$B = (1-\mu)^2[(1-\alpha)\beta - \alpha(1-\beta(1-\mu)) + 2\alpha\beta\mu] - \frac{\alpha(1-\mu)[1-\beta(1-\mu)]}{2(p+\lambda)}.$$

It can be easily verified that $\phi^{'}(u_2) = 0$ implies that $u_2 = 0$. Under the given conditions, we observe that $\phi^{''}(0) < 0$. Therefore,

$$\max \phi(u_2) = \phi(0) = M(\alpha, \beta, \gamma, \lambda, p). \tag{2.6}$$

Let

$$\Omega = \{w : \Re\, w > M(\alpha, \beta, \gamma, \lambda, p)\}.$$

Then from (2.1) and (2.4), we have $\psi(r(z), zr'(z); z) \in \Omega$ for all $z \in E$, but $\psi(iu_2, v_1; z) \notin \Omega$, in view of (2.5) and (2.6). Therefore, by Lemma 2.1 and (2.2), we conclude that

$$\Re \left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \right) > \frac{\gamma}{p}.$$

□

3. COROLLARIES

By taking $p = 1$ and $\lambda = 0$ in Theorem 2.2. We have the following corollary.

Corollary 3.1. *Let $\alpha \geq 0$, $\beta \leq 1$ and $0 \leq \gamma < 1$ be real numbers such that $\beta(1-\gamma) < \frac{1}{2}$ and $\beta \leq \alpha$. If $f \in \mathcal{A}$ satisfies the condition*

$$\Re \left(\frac{(1-\alpha)D^{n+1}f(z) + \alpha D^{n+2}f(z)}{(1-\beta)D^n f(z) + \beta D^{n+1}f(z)} \right) > M(\alpha, \beta, \gamma, 0, 1),$$

then

$$\Re\frac{D^{n+1}f(z)}{D^n f(z)} > \gamma,$$

i.e. $f(z) \in S_n(\gamma)$, *where,*

$$M(\alpha,\beta,\gamma,0,1) = \frac{(1-\alpha)\gamma + \alpha\gamma^2 - \frac{\alpha(1-\gamma)}{2}}{1-\beta(1-\gamma)}.$$

By taking $p = 1, n = 0$ and $\lambda = 0$ in Theorem 2.2. We have the following corollary.

Corollary 3.2. *Let* $\alpha \geq 0$, $\beta \leq 1$ *and* $0 \leq \gamma < 1$ *be real numbers such that* $\beta(1-\gamma) < \frac{1}{2}$ *and* $\beta \leq \alpha$. *If* $f \in \mathcal{A}$ *satisfies the condition*

$$\Re\left(\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\beta)f(z) + \beta z f'(z)}\right) > M(\alpha,\beta,\gamma,0,1),$$

then

$$\Re\frac{zf'(z)}{f(z)} > \gamma,$$

i.e. $f(z) \in S^*(\gamma)$, *where,*

$$M(\alpha,\beta,\gamma,0,1) = \frac{(1-\alpha)\gamma + \alpha\gamma^2 - \frac{\alpha(1-\gamma)}{2}}{1-\beta(1-\gamma)}.$$

By taking $p = 1, n = 0, \lambda = 0$ and $\beta = 1$ in Theorem 2.2. We have the following corollary.

Corollary 3.3. *Let* $\alpha \geq 1$ *and* $\frac{1}{2} < \gamma < 1$ *be real numbers. If* $f \in \mathcal{A}$ *satisfies the condition*

$$\Re\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) > M(\alpha,1,\gamma,0,1),$$

then

$$\Re\frac{zf'(z)}{f(z)} > \gamma,$$

i.e. $f(z) \in S^*(\gamma)$, *where*

$$M(\alpha,1,\gamma,0,1) = 1 - \alpha(1-\gamma)\left(1 + \frac{1}{2\gamma}\right)$$

By taking $p = 1, n = 0, \lambda = 0$ and $\beta = 0$ in Theorem 2.2, we have the following result of Ravichandran et. al. [9].

Corollary 3.4. *Let* $\alpha \geq 0$ *and* $0 \leq \gamma < 1$ *be real numbers. If* $f \in \mathcal{A}$ *satisfies the condition*

$$\Re\frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) > M(\alpha,0,\gamma,0,1),$$

then

$$\Re\frac{zf'(z)}{f(z)} > \gamma,$$

i.e. $f(z) \in S^*(\gamma)$, *where,*

$$M(\alpha,0,\gamma,0,1) = (1-\alpha)\gamma + \alpha\gamma^2 - \frac{\alpha(1-\gamma)}{2}.$$

Remark 1. In the case when $\gamma = \frac{\alpha}{2}$, Corollary 3.4 reduces to the result of Li and Owa [5].

By taking $p = 1, n = 0$ and $\lambda = 1$ in Theorem 2.2, we have the following corollary.

Corollary 3.5. *Let $\alpha \geq 0$, $\beta \leq 1$ and $0 \leq \gamma < 1$ be real numbers such that $\beta(1-\gamma) < \frac{1}{2}$ and $\beta \leq \alpha$. If $f \in \mathcal{A}$ satisfies the condition*

$$\Re\frac{1}{2}\left(\frac{(2-\alpha)f(z)+(2+\alpha)zf'(z)+\alpha z^2 f''(z)}{(2-\beta)f(z)+\beta zf'(z)}\right) > M(\alpha,\beta,\gamma,1,1),$$

then

$$\Re\frac{1}{2}\left(1+\frac{zf'(z)}{f(z)}\right) > \gamma,$$

where,

$$M(\alpha,\beta,\gamma,1,1) = \frac{(1-\alpha)\gamma+\alpha\gamma^2-\frac{\alpha(1-\gamma)}{4}}{1-\beta(1-\gamma)}.$$

By taking $p = 1, n = 1$ and $\lambda = 0$ in Theorem 2.2, we have the following corollary.

Corollary 3.6. *Let $\alpha \geq 0$, $\beta \leq 1$ and $0 \leq \gamma < 1$ be real numbers such that $\beta(1-\gamma) < \frac{1}{2}$ and $\beta \leq \alpha$. If $f \in \mathcal{A}$ satisfies the condition*

$$\Re\left(\frac{zf'(z)+(2\alpha+1)z^2 f''(z)+\alpha z^3 f'''(z)}{zf'(z)+\beta z^2 f''(z)}\right) > M(\alpha,\beta,\gamma,0,1),$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \gamma,$$

i.e. $f(z) \in K(\gamma)$, where,

$$M(\alpha,\beta,\gamma,0,1) = \frac{(1-\alpha)\gamma+\alpha\gamma^2-\frac{\alpha(1-\gamma)}{2}}{1-\beta(1-\gamma)}.$$

By taking $p = 1, n = 1, \lambda = 0$ and $\beta = 0$ in Theorem 2.2, we have the following corollary.

Corollary 3.7. *Let $\alpha \geq 0$ and $0 \leq \gamma < 1$ be real numbers. If $f \in \mathcal{A}$ satisfies the condition*

$$\Re\left(1+(2\alpha+1)\frac{zf''(z)}{f'(z)}+\alpha\frac{z^2 f'''(z)}{f'(z)}\right) > M(\alpha,0,\gamma,0,1),$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \gamma,$$

i.e., $f(z) \in K(\gamma)$, where,

$$M(\alpha,0,\gamma,0,1) = (1-\alpha)\gamma+\alpha\gamma^2-\frac{\alpha(1-\gamma)}{2}.$$

Remark 2. In the main result, the real number $M(\alpha,\beta,\gamma,\lambda,p)$ may not be the best possible as authors have not obtained the extremal function for it. The problem is still open for the best possible real number $M(\alpha,\beta,\gamma,\lambda,p)$.

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