

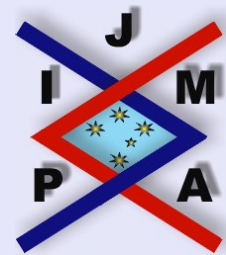
Journal of Inequalities in Pure and Applied Mathematics

ON THE SYMMETRY OF SQUARE-FREE SUPPORTED ARITHMETICAL FUNCTIONS IN SHORT INTERVALS

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ISSN (electronic): 1443-5756
059-04



volume 5, issue 2, article 33,
2004.

*Received 17 March, 2003;
accepted 02 April, 2004.*

Communicated by: A. Fiorenza

Abstract

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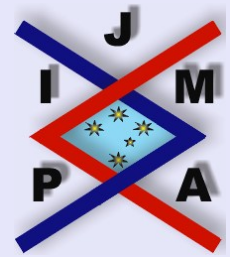


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Abstract

We study the links between additive and multiplicative arithmetical functions, say f , and their square-free supported counterparts, i.e. $\mu^2 f$ (here μ^2 is the square-free numbers characteristic function), regarding the (upper bound) estimate of their symmetry around x in almost all short intervals $[x - h, x + h]$.

2000 Mathematics Subject Classification: 11N37, 11N36

Key words: Symmetry, Square-free, Short intervals.

The author wishes to thank Professor Saverio Salerno and Professor Alberto Perelli for friendly and helpful comments. Also, he wants to express his sincere thanks to Professor Henryk Iwaniec, for his warm and familiar welcome during his stay in Rutgers University as a Visiting Scholar (the present work was conceived and written during this period).

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1. Introduction and Statement of the Results

In this paper we study the symmetry, in almost all short intervals, of square-free supported arithmetical functions.

In our previous paper [3] we applied elementary methods, i.e. the Large Sieve, in order to study the symmetry of distribution (around x) of the square-free numbers in "almost all" the "short" intervals $[x-h, x+h]$ (as usual, "almost all" means for all $x \in [N, 2N]$, except at most $o(N)$ of them; "short" means that $h = h(N)$ and $h \rightarrow \infty$, $h = o(N)$, as $N \rightarrow \infty$).

As in [1], [2], [4], and [5] on (respectively) the prime-divisors function, von Mangoldt function, the divisor function and a wide class of arithmetical functions, we study the symmetry of our arithmetical function f .

We define the "symmetry sum" of f as (here $\text{sgn}(t) \stackrel{\text{def}}{=} t/|t|$, $\text{sgn}(0) \stackrel{\text{def}}{=} 0$)

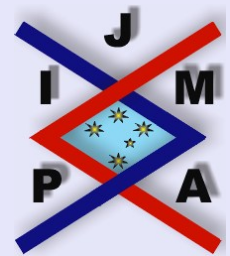
$$S_f^\pm(x) \stackrel{\text{def}}{=} \sum_{|n-x| \leq h} f(n) \text{sgn}(n-x),$$

and its mean-square as the "symmetry integral" of f :

$$I_f(N, h) \stackrel{\text{def}}{=} \sum_{x \sim N} \left| \sum_{|n-x| \leq h} f(n) \text{sgn}(n-x) \right|^2.$$

Here and hereafter $x \sim N$ stands for $N < x \leq 2N$.

We will connect (in Theorem 1.1 and Theorem 1.2) $I_f(N, h)$ and $I_{\mu^2 f}(N, h)$, for suitable f ; thus relating the symmetry of f to that of f on the square-free numbers (μ^2 being their characteristic function). Thus, we can estimate just one



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symmetry integral for two arithmetical functions, whenever they agree on the square-free numbers.

As an example, for $d(n)$ the divisor function, [4] estimates $I_d(N, h)$; then (using Theorem 1.4 to check the symmetry of $d(n)$ in arithmetic progressions) in Theorem 1.3 we bound $I_{\mu^2 d}(N, h) = I_{\mu^2 2\Omega}(N, h)$, and then obtain information on $I_{2\Omega}(N, h)$ by Theorem 1.1 (here the function $2^{\Omega(n)}$ is completely multiplicative, with $2^{\Omega(p)} = 2$).

We denote with \mathcal{F} the set of arithmetical functions $f : \mathbb{N} \rightarrow \mathbb{C}$ and with \mathcal{B} the set of $f \in \mathcal{F}$, with $|f|$ bounded (by an absolute constant); \mathcal{M} denotes the multiplicative $f \in \mathcal{F}$ and \mathcal{A} the additive ones.

Also, we can define ($\forall \alpha \in]1, 2[$) the set of "symmetric" arithmetical functions f as (where we **assume**: $\forall E > 0 \sup_{\mathbb{N}} |f| \ll N^E$):

$$\mathcal{S}_\alpha \stackrel{def}{=} \left\{ f \in \mathcal{F} : \sup_{q \leq N^{c\varepsilon}} I_f(N, h, k, q) \ll \frac{Nh^\alpha}{k^2 N^\varepsilon} \forall k \leq N^{c\varepsilon}, \text{ for some } c, \varepsilon > 0 \right\}$$

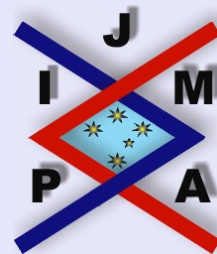
(the \ll -constant is absolute, as well as $c > 0$), where we have set

$$I_f(N, h, k, q) \stackrel{def}{=} \sum_{x \sim N} \left| \sum_{\substack{|n-x/k| \leq h/k \\ n \equiv 0(q)}} f(n) \operatorname{sgn} \left(n - \frac{x}{k} \right) \right|^2 ;$$

in the following, as here, we will abbreviate $n \equiv a(q)$ to mean $n \equiv a \pmod{q}$.

We start giving a first link between f and $\mu^2 f$ (in the sequel $L \stackrel{def}{=} \log N$):

Theorem 1.1. *Let $N, h \in \mathbb{N}$, where $h = h(N)$, $h/L^2 \rightarrow \infty$ and $h = o(N)$ as $N \rightarrow \infty$. Assume $J \ll \frac{\sqrt{h}}{L}$, $J \rightarrow \infty$ as $N \rightarrow \infty$. Let $\|f\|_\infty := \sup_{\mathbb{N}} |f|$.*



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If f is **completely multiplicative** then

$$(i) \quad I_f(N, h) \ll L^2 \max_{D \ll J} \sum_{d \sim D} d^2 I_{\mu^2 f} \left(\frac{N}{d^2}, \frac{h}{d^2} \right) + \frac{Nh^2}{J^2} \|f\|_\infty^2$$

and

$$(ii) \quad I_{\mu^2 f}(N, h) \ll L^2 \max_{D \ll J} \sum_{d \sim D} d^2 I_f \left(\frac{N}{d^2}, \frac{h}{d^2} \right) + \frac{Nh^2}{J^2} \|f\|_\infty^2.$$

If f is **completely additive** then

$$(i) \quad I_f(N, h) \ll L^2 \max_{D \ll J} \sum_{d \sim D} d^2 I_{\mu^2 f} \left(\frac{N}{d^2}, \frac{h}{d^2} \right) + \left(\frac{Nh^2}{J^2} + NJ\sqrt{h}L^2 \right) \|f\|_\infty^2$$

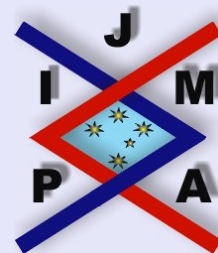
and

$$(ii) \quad I_{\mu^2 f}(N, h) \ll L^2 \max_{D \ll J} \sum_{d \sim D} d^2 I_f \left(\frac{N}{d^2}, \frac{h}{d^2} \right) + \left(\frac{Nh^2}{J^2} + NJL^2 \right) \|f\|_\infty^2.$$

We generalize Theorem 1.1 to **additive** and to **multiplicative** functions:

Theorem 1.2. Let $f \in \mathcal{A} \cup \mathcal{M}$. Let N, h be natural numbers, with $h = N^\theta$ (for $0 < \theta < 1$). Assume that f is supported over the cube-free numbers and that $\forall \varepsilon > 0, \|f\|_\infty \ll N^\varepsilon$, as $N \rightarrow \infty$. Choose $\forall \alpha \in]1, 2]$ $\varepsilon = \frac{\theta(\alpha-1)}{3} > 0$. Then

$$f \in \mathcal{S}_\alpha \Leftrightarrow \mu^2 f \in \mathcal{S}_\alpha.$$



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We give a **concrete example**: the function $f(n) = 2^{\Omega(n)}$ (where $\Omega(n)$ is the total number of prime divisors of n); in this case $f \in \mathcal{S}_\alpha$ and $\mu^2 f \in \mathcal{S}_\alpha \forall \alpha > \frac{3}{2}$, as we will prove directly, also to detail the (more delicate) estimates

Theorem 1.3. *Let $N, h \in \mathbb{N}$, $h = h(N) \geq L$ and $h = o\left(\frac{\sqrt{N}}{L}\right)$ as $N \rightarrow \infty$. Then*

$$\sum_{x \sim N} \left| \sum_{|n-x| \leq h} 2^{\Omega(n)} \operatorname{sgn}(n-x) \right|^2 \ll Nh^{3/2} N^\varepsilon$$

and

$$\sum_{x \sim N} \left| \sum_{|n-x| \leq h} \mu^2(n) 2^{\Omega(n)} \operatorname{sgn}(n-x) \right|^2 \ll Nh^{3/2} N^\varepsilon.$$

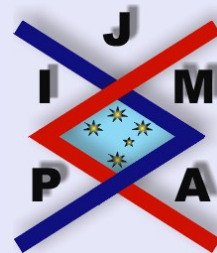
Remark 1.1. *We explicitly remark that these bounds are non-optimal.*

This result is obtained directly upon estimating the mean-square of the symmetry sum for the divisor function over the **arithmetic progressions**:

Theorem 1.4. *Let $N, h \in \mathbb{N}$, with $h = h(N) \rightarrow \infty$ and $h = o\left(\frac{\sqrt{N}}{L}\right)$ as $N \rightarrow \infty$. Then, uniformly $\forall q \in \mathbb{N}$,*

$$\sum_{x \sim N} \left| \sum_{\substack{|n-x| \leq h \\ n \equiv 0(q)}} d(n) \operatorname{sgn}(n-x) \right|^2 \ll NhL^3 + NL^2 \log^2 q,$$

where the \mathcal{O} -constant does not depend on q .



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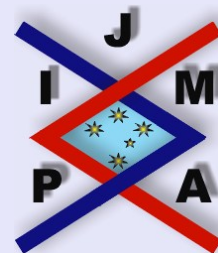
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The paper is organized as follows

- In Section 2 we give the necessary lemmas;
- In Section 3 we prove our theorems.



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2. Lemmas

Lemma 2.1. Let $f \in \mathcal{F}$ be an arithmetical function, $\|f\|_\infty \stackrel{\text{def}}{=} \sup_{\mathbb{N}} |f(n)|$.

Then, for N , $h = h(N) \in \mathbb{N}$ and $h \rightarrow \infty$, $h = o(N)$ as $N \rightarrow \infty$:

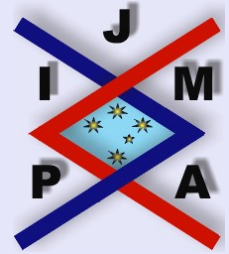
$$\sum_{x \sim N} \left| \sum_{\sqrt{2h} < d \leq \sqrt{x+h}} a(d) \sum_{\left| m - \frac{x}{d^2} \right| \leq \frac{h}{d^2}} b(m) f(md^2) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 \ll N h L^2 \|f\|_\infty^2,$$

uniformly $\forall a, b \in \mathcal{B}$.

(Actually, for our purposes, $\|f\|_\infty = \max_{N-h \leq n \leq 2N+h} |f(n)|$.)

Proof. Let Σ be the LHS. By a dyadic dissection and Cauchy inequality

$$\begin{aligned} \Sigma &\ll L^2 \max_{\sqrt{h} \ll D \ll \sqrt{N}} \sum_{x \sim N} \left| \sum_{d \sim D} a(d) \sum_{\left| m - \frac{x}{d^2} \right| \leq \frac{h}{d^2}} b(m) f(md^2) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 \\ &\ll L^2 \max_{\sqrt{h} \ll D \ll \sqrt{N}} D \sum_{x \sim N} \sum_{d \sim D} \left| \sum_{\left| m - \frac{x}{d^2} \right| \leq \frac{h}{d^2}} b(m) f(md^2) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 \\ &\ll \|f\|_\infty^2 L^2 \max_{\sqrt{h} \ll D \ll \sqrt{N}} D \sum_{d \sim D} \sum_{\substack{N-h \leq m_1, m_2 \leq 2N+h \\ m_1 d^2 - h \leq x \leq m_1 d^2 + h \\ m_2 d^2 - h \leq x \leq m_2 d^2 + h}} 1. \end{aligned}$$



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Clearly, the limitations on x imply $m_1 - \frac{2h}{d^2} \leq m_2 \leq m_1 + \frac{2h}{d^2}$ (here we "reflect" the "sporadicity") and this in turn, due to $D \gg \sqrt{h} \Rightarrow d^2 \gg h$, gives ($\forall m_1$ FIXED) $\mathcal{O}(1)$ possible values to m_2 . Hence Σ is bounded by

$$\|f\|_\infty^2 hL^2 \max_{\sqrt{h} \ll D \ll \sqrt{N}} D \sum_{d \sim D} \sum_{\substack{N-h \leq m_1 \leq \frac{2N+h}{d^2} \\ |m_2 - m_1| \ll 1}} 1 \ll NhL^2 \|f\|_\infty^2.$$

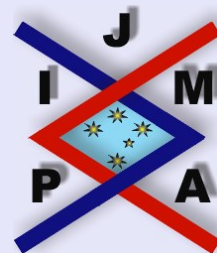
□

Lemma 2.2. Assume $f \in \mathcal{F}$ is completely additive and $\|f\|_\infty \stackrel{def}{=} \sup_N |f|$. Let $N, h \in \mathbb{N}$ with $h = h(N) \rightarrow \infty$, $h = o(N)$, as $N \rightarrow \infty$. Then $\forall J \leq \sqrt{2h}$

$$\begin{aligned} & \sum_{x \sim N} \left| \sum_{d \leq \sqrt{2h}} a(d) \sum_{\substack{|m - \frac{x}{d^2}| \leq \frac{h}{d^2}}} b(m) f(md^2) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 \\ & \ll L^2 \max_{D \ll J} D \left(\|f\|_\infty^2 \sum_{d \sim D} \sum_{x \sim N} \left| \sum_{\substack{|m - \frac{x}{d^2}| \leq \frac{h}{d^2}}} b(m) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 \right. \\ & \quad \left. + \sum_{d \sim D} \sum_{x \sim N} \left| \sum_{\substack{|m - \frac{x}{d^2}| \leq \frac{h}{d^2}}} b(m) f(m) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 \right) + \frac{Nh^2}{J^2} \|f\|_\infty^2, \end{aligned}$$

uniformly $\forall a, b \in \mathcal{B}$ (bounded arithmetical functions).

Proof. Let us call the left mean-square Σ . Then Σ is at most



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$$\sum_{x \sim N} L^2 \max_{D \ll J} \left| \sum_{d \sim D} a(d) \sum_{\left| m - \frac{x}{d^2} \right| \leq \frac{h}{d^2}} b(m) f(md^2) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 + \frac{Nh^2}{J^2} \|f\|_\infty^2.$$

Since f is completely additive

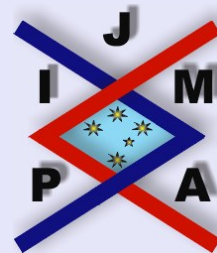
$$\begin{aligned} \Sigma &\ll L^2 \max_{D \ll J} D \left(\sum_{x \sim N} \sum_{d \sim D} \left| \sum_{\left| m - \frac{x}{d^2} \right| \leq \frac{h}{d^2}} b(m) f(m) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 \right. \\ &\quad \left. + \|f\|_\infty^2 \sum_{x \sim N} \sum_{d \sim D} \left| \sum_{\left| m - \frac{x}{d^2} \right| \leq \frac{h}{d^2}} b(m) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 \right) + \frac{Nh^2}{J^2} \|f\|_\infty^2, \end{aligned}$$

by the Cauchy inequality. The lemma is thus proved. \square

Lemma 2.3. *Let f be completely multiplicative. Then, if $N, h \in \mathbb{N}$, with $h = h(N) \rightarrow \infty$ and $h = o(N)$ (as $N \rightarrow \infty$), we have $\forall J \leq \sqrt{2h}$*

$$\begin{aligned} &\sum_{x \sim N} \left| \sum_{d \leq \sqrt{2h}} a(d) \sum_{\left| m - \frac{x}{d^2} \right| \leq \frac{h}{d^2}} b(m) f(md^2) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 \\ &\ll \|f\|_\infty^2 \left(L^2 \max_{D \ll J} D \sum_{d \sim D} \sum_{x \sim N} \left| \sum_{\left| m - \frac{x}{d^2} \right| \leq \frac{h}{d^2}} b(m) f(m) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 + \frac{Nh^2}{J^2} \right) \end{aligned}$$

uniformly $\forall a, b \in \mathcal{B}$.



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Proof. Let us call the left mean-square Σ . Then

$$\Sigma \ll \sum_{x \sim N} L^2 \max_{D \ll J} \left| \sum_{d \sim D} a(d) \sum_{\left| m - \frac{x}{d^2} \right| \leq \frac{h}{d^2}} b(m) f(md^2) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 + \frac{Nh^2}{J^2} \|f\|_\infty^2,$$

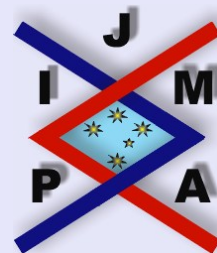
and being f completely multiplicative we get

$$\Sigma \ll \|f\|_\infty^2 L^2 \max_{D \ll J} D \sum_{d \sim D} \sum_{x \sim N} \left| \sum_{\left| m - \frac{x}{d^2} \right| \leq \frac{h}{d^2}} b(m) f(m) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 + \frac{Nh^2}{J^2} \|f\|_\infty^2,$$

by the Cauchy inequality. The lemma is thus proved. \square

Lemma 2.4. *Let N, h, J and D be as in Lemma 2.2, with $D = o(\sqrt{h})$. Then*

$$\sum_{d \sim D} \sum_{x \sim N} \left| \sum_{\left| m - \frac{x}{d^2} \right| \leq \frac{h}{d^2}} f(m) \operatorname{sgn} \left(m - \frac{x}{d^2} \right) \right|^2 \ll \sum_{d \sim D} d^2 \sum_{y \sim \frac{N}{d^2}} \left| \sum_{\left| m - y \right| \leq h/d^2} f(m) \operatorname{sgn}(m - y) \right|^2 + \left(\frac{h^2}{D} + ND \right) \|f\|_\infty^2.$$



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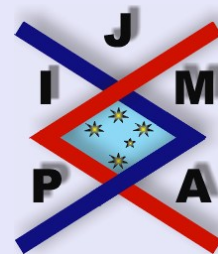
Proof. Write $x = yd^2 + r$ ($0 \leq r < d^2$) and let Σ be the left mean-square; since we have $\sum_{x \sim N} = \sum_{y \sim \frac{N}{d^2}} + \mathcal{O}(d^2)$, then

$$\Sigma \ll \sum_{d \sim D} \sum_{0 \leq r < d^2} \sum_{y \sim \frac{N}{d^2}} \left| \sum_{\left| m - y - \frac{r}{d^2} \right| \leq \frac{h}{d^2}} f(m) \operatorname{sgn} \left(m - y - \frac{r}{d^2} \right) \right|^2 + \frac{h^2}{D} \|f\|_\infty^2$$

(thus $\frac{h^2}{D}$ is due to x -range remainders); then correcting $\mathcal{O}(1)$ values of the m -sum gives as a remainder (due to h -range)

$$\mathcal{O} \left(\sum_{d \sim D} d^2 \frac{N}{d^2} \|f\|_\infty^2 \right) = \mathcal{O}(ND \|f\|_\infty^2).$$

Gathering the estimates we then obtain the lemma. □



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3. Proof of the Theorems

We start by proving Theorem 1.1.

Proof. In both cases (f completely additive or completely multiplicative) we use the hypothesis on f to "separate variables" after having expressed the symmetry of f by that of $\mu^2 f$ (for i), say) and the symmetry of $\mu^2 f$ by that of f (for ii), say). Thus, to prove i) it will suffice to remember that each natural number $n = md^2$, where m and d are natural and $\mu^2(m) = 1$, i.e. m is square-free:

$$\sum_{|n-x|\leq h} f(n)\operatorname{sgn}(n-x) = \sum_{d\leq\sqrt{x+h}} \sum_{\left|m-\frac{x}{d^2}\right|\leq\frac{h}{d^2}} \mu^2(m)f(md^2)\operatorname{sgn}\left(m-\frac{x}{d^2}\right).$$

Instead, to prove ii) we simply use the following formula (see [7]):

$$\mu^2(n) = \sum_{d^2|n} \mu(d) \quad \forall n \in \mathbb{N}$$

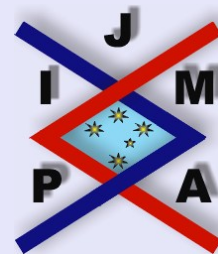
to get

$$\sum_{|n-x|\leq h} \mu^2(n)f(n)\operatorname{sgn}(n-x) = \sum_{d\leq\sqrt{x+h}} \mu(d) \sum_{\left|m-\frac{x}{d^2}\right|\leq\frac{h}{d^2}} f(md^2)\operatorname{sgn}\left(m-\frac{x}{d^2}\right).$$

As for the additional terms in the completely additive case, they come from the estimate of the square-free symmetry sum as in [3].

Putting together Lemmas 2.1, 2.2, 2.3 and 2.4, the theorem is proved. \square

We now come to the proof of Theorem 1.2.



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Proof. We first prove that $f \in \mathcal{S} \Rightarrow \mu^2 f \in \mathcal{S}$.

As before, we split at D (to be chosen); say (here $[a, b]$ is the l.c.m. of a, b)

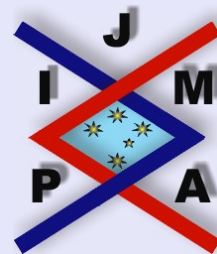
$$\begin{aligned} \Sigma &\stackrel{\text{def}}{=} \sum_{d \leq D} \mu(d) \sum_{\substack{|n - \frac{x}{k}| \leq \frac{h}{k} \\ n \equiv 0 \pmod{[q, d^2]}}} f(n) \operatorname{sgn}(n - x) \\ &= \sum_{d \leq D} \mu(d) \sum_{\substack{t|[q, d^2] \\ g=[q, d^2]/t}} \sum_{\substack{|m - \frac{x}{kt^2g}| \leq \frac{h}{kt^2g} \\ (m, g)=1}} f(mt^2g) \operatorname{sgn}\left(m - \frac{x}{kt^2g}\right) \end{aligned}$$

and observe that, since f is supported over the cube-free numbers, Σ is

$$\begin{aligned} &\sum_{d \leq D} \mu(d) \sum_{\substack{t|[q, d^2] \\ g=[q, d^2]/t}} f(t^2g) \sum_{j|g} \mu(j) \sum_{\substack{|m - \frac{x}{kt^2g}| \leq \frac{h}{kt^2g} \\ m \equiv 0 \pmod{j}}} f(m) \operatorname{sgn}\left(m - \frac{x}{kt^2g}\right) \\ &\ll \|f\|_\infty N^\delta \sum_{d \leq D} \frac{1}{d} \max_{j, t \leq qd^2} \left| \sum_{\substack{|m - \frac{x}{kt[q, d^2]}| \leq \frac{h}{kt[q, d^2]} \\ m \equiv 0 \pmod{j}}} f(m) \operatorname{sgn}\left(m - \frac{x}{kt[q, d^2]}\right) \right|, \end{aligned}$$

by (see [7]) the estimate $\forall \delta > 0 \ d(n) \ll n^\delta$; using the hypothesis $f \in \mathcal{S}_\alpha$ we get, by Cauchy inequality

$$\sum_{x \sim N} |\Sigma|^2 \ll \|f\|_\infty^2 N^{2\delta} \sum_{d \leq D} \frac{1}{d^2} \sum_{d \leq D} d^2 \frac{Nh^\alpha}{k^2 d^4 N^\varepsilon} \ll \frac{Nh^\alpha}{k^2 N^\varepsilon}$$



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Hence, it remains to prove that the mean-square of, say

$$\Sigma' \stackrel{def}{=} \sum_{D < d \leq \sqrt{x+h}} \mu(d) \sum_{\substack{|n-\frac{x}{k}| \leq \frac{h}{k} \\ n \equiv 0 \pmod{(q, d^2)}}} f(n) \operatorname{sgn}(n-x)$$

is

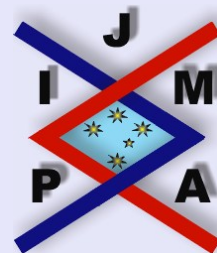
$$\sum_{x \sim N} |\Sigma'|^2 \ll \frac{Nh^\alpha}{k^2 N^\varepsilon}.$$

By the Cauchy inequality and a "sporadicity" argument as in the proof of Lemma 2.1,

$$\begin{aligned} \sum_{x \sim N} |\Sigma'|^2 &\ll \|f\|_\infty^2 \sum_{x \sim N} \left(\sum_{D < d \leq \sqrt{\frac{h}{k}}} \left(\frac{h}{kd^2} + 1 \right) \right)^2 \\ &\quad + \|f\|_\infty^2 L^2 \max_{\sqrt{\frac{h}{k}} \ll J \ll \sqrt{N}} J \sum_{d \sim J} \sum_{x \sim N} \left(\sum_{\left| m - \frac{x}{k[d^2, q]} \right| \leq \frac{h}{k[d^2, q]}} 1 \right)^2 \\ &\ll N^\delta N \left(\frac{h^2}{k^2 D^2} + \frac{h}{k} \right) + N^\delta \max_{\sqrt{\frac{h}{k}} \ll J \ll \sqrt{N}} J \sum_{d \sim J} \sum_{\substack{\frac{N-h}{k[d^2, q]} < m \leq \frac{2N+h}{k[d^2, q]}}} h. \end{aligned}$$

Hence

$$\sum_{x \sim N} |\Sigma'|^2 \ll \frac{Nh^\alpha}{k^2 N^\varepsilon} \left(\frac{N^{\delta+\varepsilon} h^{2-\alpha}}{D^2} + h^{1-\alpha} N^{\delta+\varepsilon} k \right).$$



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In order to obtain the above required estimate we need $\varepsilon \leq \frac{\theta(\alpha-1)}{3}$ (for the II term in brackets) and, comparing the mean-squares of Σ and of Σ' , we come to the choice $D = N^{\frac{4-\alpha}{2(\alpha-1)}\varepsilon}$ (I term). This proves the first implication.

As for the reverse implication $\mu^2 f \in \mathcal{S} \Rightarrow f \in \mathcal{S}$ we do not need the hypothesis on the support of f and we use the same method (but using $n = md^2$ instead of the identity for μ^2). This finally proves Theorem 1.2. \square

We now prove Theorem 1.4.

Proof. First of all, let us call $I_q(N, h)$ the mean-square to evaluate.

We will closely follow the proof of Theorem 1 in [4].

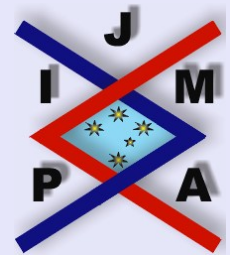
In fact, we start from the "flipping" property to write:

$$\sum_{\substack{|n-x| \leq h \\ n \equiv 0(q)}} d(n) \operatorname{sgn}(n-x) = \frac{1}{q} \sum_{r \leq q} \sum_{|n-x| \leq h} e_q(rn) \left(2 \sum_{\substack{d|n \\ d \leq \sqrt{n}}} 1 \right) \operatorname{sgn}(n-x) + \mathcal{O}\left(\frac{h}{\sqrt{N}} + 1\right),$$

having used the orthogonality of the additive characters (see [7]). By our hypothesis on h (see [4] for the details)

$$\sum_{\substack{|n-x| \leq h \\ n \equiv 0(q)}} d(n) \operatorname{sgn}(n-x) = \frac{2}{q} \sum_{r \leq q} \sum_{d \leq \sqrt{x}} \sum_{\substack{|n-x| \leq h \\ n \equiv 0(d)}} e_q(rn) \operatorname{sgn}(n-x) + \mathcal{O}(1)$$

(here the constant is independent of q , like all the others following).



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Next, write $n - x = s$ to get (again by orthogonality)

$$\begin{aligned} \sum_{\substack{|n-x| \leq h \\ n \equiv 0(d)}} e_q(rn) \operatorname{sgn}(n-x) &= e_q(rx) \sum_{\substack{|s| \leq h \\ s \equiv -x(d)}} e_q(rs) \operatorname{sgn}(s) \\ &= \frac{e_q(rx)}{d} \sum_{j \leq d} e_d(jx) \sum_{|s| \leq h} e_q(rs) e_d(js) \operatorname{sgn}(s) \\ &= e_q(rx) \sum_{j \leq d} c_{j,d}(q, r) e_d(jx), \end{aligned}$$

say, where

$$c_{j,d}(q, r) \stackrel{\text{def}}{=} \frac{2i}{d} \sum_{s \leq h} \sin \left(2\pi s \left(\frac{r}{q} + \frac{j}{d} \right) \right).$$

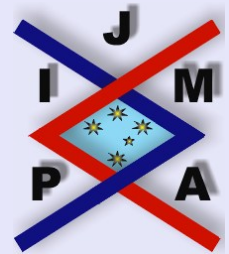
Here (w.r.t. the quoted [4, Theorem 1]) we have the dependence of the Fourier coefficients on q and r ; also, while $c_{d,d} = 0$ there, here (by the estimate in of [6, Chap. 25])

$$c_{d,d}(q, r) = \frac{2i}{d} \sum_{s \leq h} \sin \frac{2\pi sr}{q} \ll \frac{q}{rd}.$$

Hence, this term's contribute to the mean-square $I_q(N, h)$ is:

$$\sum_{x \sim N} \left| \frac{1}{q} \sum_{r \leq q} e_q(rx) \sum_{d \leq \sqrt{x}} c_{d,d}(q, r) e_d(jx) \right|^2 \ll \sum_{x \sim N} \left(\sum_{r \leq q} \frac{1}{r} L \right)^2 \ll NL^2 \log^2 q$$

(that is why we have this additional remainder, here!).



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Henceforth, we can rely upon the proof of [4, Theorem 1], the only difference being the r, s dependence:

$$(*) \sum_{x \sim N} \left| \frac{1}{q} \sum_{r \leq q} e_q(rx) \sum_{d \leq \sqrt{x}} \sum_{j < d} c_{j,d}(q, r) e_d(jx) \right|^2$$

$$\ll \frac{1}{q} \sum_{r \leq q} \sum_{x \sim N} \left| \sum_{d \leq \sqrt{x}} \sum_{j < d} c_{j,d}(q, r) e_d(jx) \right|^2$$

(we have used the Cauchy inequality).

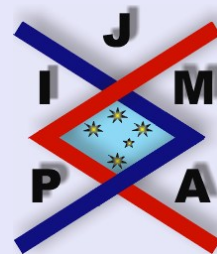
We apply, then, exactly the same estimates; while there we get (we are quoting inequalities to ease comparison)

$$\sum_{j < d} |c_{j,d}|^2 \leq \sum_{j \leq d} |c_{j,d}|^2 \leq \frac{2h}{d},$$

here we have (the constant $c > 0$ is ininfluent)

$$\sum_{j \leq d} |c_{j,d}(q, r)|^2 = c \frac{1}{d^2} \sum_{|s_1|, |s_2| \leq h} \operatorname{sgn}(s_1) \operatorname{sgn}(s_2) \sum_{j \leq d} e \left((s_1 - s_2) \left(\frac{r}{q} + \frac{j}{d} \right) \right)$$

$$= \frac{c}{d} \sum_{|s_1| \leq h} \operatorname{sgn}(s_1) \sum_{\substack{|s_2| \leq h \\ s_2 \equiv s_1(d)}} \operatorname{sgn}(s_2) e_q(r(s_1 - s_2)),$$



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whence, by (*), we get (see [4, Theorem 1]), ignoring the remainder $\mathcal{O}(NL^2 \log^2 q)$:

$$\begin{aligned}
 I_q(N, h) &\ll \frac{1}{q} \sum_{r \leq q} NL^2 \sum_{d \leq \sqrt{2N}} \frac{1}{d} \sum_{|s_1| \leq h} \operatorname{sgn}(s_1) \sum_{\substack{|s_2| \leq h \\ s_2 \equiv s_1(d)}} \operatorname{sgn}(s_2) e_q(r(s_1 - s_2)) \\
 &= NL^2 \sum_{d \leq \sqrt{2N}} \frac{1}{d} \sum_{|s_1| \leq h} \operatorname{sgn}(s_1) \sum_{\substack{|s_2| \leq h \\ s_2 \equiv s_1(d) \\ s_2 \equiv s_1(q)}} \operatorname{sgn}(s_2) \\
 &\ll NL^2 \left(\sum_{\substack{d \leq \frac{h}{L} \\ [d, q] \leq \frac{h}{L}}} \frac{1}{d} h + \sum_{\frac{h}{L} < d \leq \sqrt{2N}} \frac{1}{d} \left(\frac{h^2}{d} + h \right) \right).
 \end{aligned}$$

Thus

$$I_q(N, h) \ll NhL^3 + NL^2 \log^2 q.$$

□

We now prove Theorem 1.3.

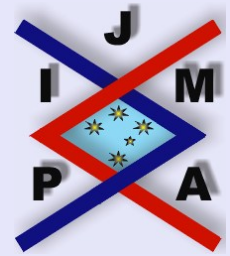
Proof. We first show the second estimate.

First of all, we observe that $\mu^2(n)2^{\Omega(n)} = \mu^2(n)d(n)$, $\forall n \in \mathbb{N}$; here we will apply the flipping property of the divisor function as in [4].

Then, we will try to link our symmetry integral (for $\mu^2 2^\Omega$) with that of $d(n)$.

Writing $\mu^2(n)$ as before

$$\sum_{|n-x| \leq h} \mu^2(n) d(n) \operatorname{sgn}(n-x) = \sum_{d \leq \sqrt{x+h}} \mu(d) \sum_{\substack{|n-x| \leq h \\ n \equiv 0(d^2)}} d(n) \operatorname{sgn}(n-x).$$



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Splitting the range at $D = D(x) \leq \sqrt{x+h}$ (to be chosen later), we treat, say

$$\Sigma_1(x) \stackrel{\text{def}}{=} \sum_{d \leq D} \mu(d) \sum_{\substack{|n-x| \leq h \\ n \equiv 0 \pmod{d^2}}} d(n) \operatorname{sgn}(n-x)$$

by the Cauchy inequality and Theorem 1.4 to get

$$\begin{aligned} \sum_{x \sim N} |\Sigma_1(x)|^2 &\ll D \sum_{d \leq D} \sum_{x \sim N} \left| \sum_{\substack{|n-x| \leq h \\ n \equiv 0 \pmod{d^2}}} d(n) \operatorname{sgn}(n-x) \right|^2 \\ &\ll ND^2 L^3 (h+L) \ll ND^2 h L^3, \end{aligned}$$

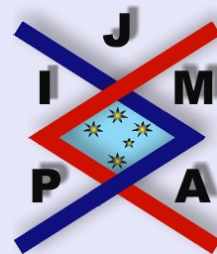
by our hypothesis on h . It remains to bound the mean-square of, say

$$\Sigma_2(x) \stackrel{\text{def}}{=} \sum_{D < d \leq \sqrt{x+h}} \mu(d) \sum_{\substack{|n-x| \leq h \\ n \equiv 0 \pmod{d^2}}} d(n) \operatorname{sgn}(n-x).$$

We split again at $\sqrt{2h}$ (to distinguish non-sporadic and sporadic terms).

Since by the classical estimate $d(n) \ll n^\varepsilon$ (see [7]; here $\varepsilon > 0$ will not be the same at each occurrence) we estimate trivially (the non-sporadic terms)

$$\sum_{D < d \leq \sqrt{2h}} \mu(d) \sum_{\substack{|n-x| \leq h \\ n \equiv 0 \pmod{d^2}}} d(n) \operatorname{sgn}(n-x) \ll \sum_{D < d \leq \sqrt{2h}} \frac{hN^\varepsilon}{d^2} \ll \frac{Nh^2}{D^2} N^\varepsilon$$



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we get, together with (the sporadic terms, treated by Lemma 2.1)

$$\sum_{x \sim N} \left| \sum_{\sqrt{2h} < d \leq \sqrt{x+h}} \mu(d) \sum_{\substack{|n-x| \leq h \\ n \equiv 0 \pmod{d^2}}} d(n) \operatorname{sgn}(n-x) \right|^2 \ll NhN^\varepsilon,$$

that

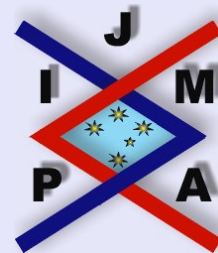
$$\sum_{x \sim N} |\Sigma_2(x)|^2 \ll \left(\frac{Nh^2}{D^2} + Nh \right) N^\varepsilon.$$

Thus, comparing the mean-squares of $\Sigma_1(x)$ and $\Sigma_2(x)$ we make the best choice $D = h^{1/4}$, finally proving the second estimate.

Writing $I_{2\Omega}$ for the symmetry integral of 2^Ω , we apply Theorem 1.1 to this function; then, i) gives us

$$I_{2\Omega}(N, h) \ll L^2 \max_{D \ll J} \sum_{d \sim D} d^2 \frac{N h^{3/2}}{d^2 d^3} N^\varepsilon + \frac{Nh^2}{J^2} N^\varepsilon \ll Nh^{3/2} N^\varepsilon,$$

by the choice $J = \sqrt{h}$. This gives the first estimate, hence finally proving Theorem 1.3. \square



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